THE THEORY OF FUNCTIONS OF REAL VARIABLES

VOLUME II

JAMES PIERPONT

457: Dirichlet's function

494 $\Delta(a\, b) = \dfrac{f(a) - f(b)}{a - b}$

299 Liouville number.

In domain of real nrs i) Every convergent seq. has limit wh. is a no. belonging to the domain and ii) Every no. is the lim. of a properly chosen seq. of nrs. belonging to the domain. These properties i and ii of real nrs are expressed by saying the Agg. of Real nrs is PERFECT (Hobson p 50)

Ot of Rational nrs. has prop. ii but not i ∴ not PERFECT

Dense $a_1 < b_1$ then $a_2 < b_2$ btw $a_1 b_1$ such that $b_2 - a_2 < \epsilon$. Btw $a_2 b_2$ is $a_3 b_3$ etc. An Agg. of real nrs w. this property is connex or everywhere dense

An Agg. wh is connex and perfect = a continuum
The " of real nrs w. these properties = arithmetic continuum
" " of rational nrs. having only one property is not a continuum, but is Discrete

$a \leqq x \leqq b$ = a closed interval.
$a < x < b$ = an open "

LECTURES

ON

THE THEORY OF FUNCTIONS OF REAL VARIABLES

VOLUME II

BY

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TO

ANDREW W. PHILLIPS

THESE LECTURES

ARE INSCRIBED

WITH AFFECTION AND ESTEEM

PREFACE

THE present volume has been written in the same spirit that animated the first. The author has not intended to write a treatise or a manual; he has aimed rather to reproduce his university lectures with necessary modifications, hoping that the freedom in the choice of subjects and in the manner of presentation allowable in a lecture room may prove helpful and stimulating to a larger audience.

A distinctive feature of these Lectures is an attempt to develop the theory of functions with reference to a general domain of definition. The first functions to be considered were simple combinations of the elementary functions. Riemann in his great paper of 1854, "Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe," was the first to consider seriously functions whose singularities ceased to be intuitional. The researches of later mathematicians have brought to light a collection of such functions, whose existence so long unsuspected has revolutionized the older notion of a function and made imperative the creation of finer tools of research. But while minute attention was paid to the singular character of these functions, practically none was accorded to the domain over which a function may be defined. After the epoch-making discoveries inaugurated in 1874 by G. Cantor in the theory of point sets, it was no longer necessary to consider a function of one variable as defined in an interval, a function of two variables as defined over a field bounded by one or more simple curves, etc. The first to make use of this new freedom was C. Jordan in his classic paper of 1892. He has had, however, but few imitators. In the present Lectures the author has endeavored to develop this broader view of Jordan, persuaded that in so doing he is merely carrying a step farther the ideas of Dirichlet and Riemann.

Often such an endeavor leads to nothing new, a mere statement for any n of what is true for $n = 1$, or 2. A similar condition

prevails in the theory of determinants. One may prefer to treat only two and three rowed determinants, but he surely has no ground of complaint if another prefers to state his theorems and demonstrations for general n. On the other hand, the general case may present unexpected and serious problems. For example, Jordan has introduced the notion of functions of a single variable having limited variation. How is this notion to be extended to two or more variables? An answer is far from simple. One was given by the author in Volume I; its serviceableness has since been shown by B. Camp. Another has been essayed by Lebesgue. The reader must be warned, however, against expecting to find the development always extended to the general case. This, in the first place, would be quite impracticable without greatly increasing the size of the present work. Secondly, it would often be quite beyond the author's ability.

Another feature of the present work to which the author would call attention is the novel theory of integration developed in Chapter XVI of Volume I and Chapters I and II of Volume II. It rests on the notion of a cell and the division of space, or in fact any set, into unmixed partial sets. The definition of improper multiple integrals leads to results more general in some respects than yet obtained with Riemann integrals.

Still another feature is a new presentation of the theory of measure. The demonstrations which the author has seen leave much to be desired in the way of completeness, not to say rigor. In attempting to find a general and rigorous treatment, he was at last led to adopt the form given in Chapter XI.

The author also claims as original the theory of Lebesgue integrals developed in Chapter XII. Lebesgue himself considers functions such that the points e at which $a \leq f(x) \leq b$, for all a, b form a measurable set. His integral he defines as

$$\lim_{n=\infty} \sum_1^n l_m e'_m$$

where $l_m \leq f(x) \leq l_{m+1}$ in e_m whose measure is e'_m, and each $l_{m+1} - l_m \doteq 0$, as $n \doteq \infty$. The author has chosen a definition which occurred to him many years ago, and which to him seems far more natural. In Volume I it is shown that if the metric field $\mathfrak{A}$

be divided into a finite number of metric sets $\delta_1, \delta_2 \cdots$ of norm d, then

$$\underline{\int}_{\mathfrak{A}} f = \text{Max}\, \Sigma m_\iota \delta_\iota \quad , \quad \overline{\int}_{\mathfrak{A}} f = \text{Min}\, \Sigma M_\iota \delta_\iota$$

where m_ι, M_ι are the minimum and maximum of f in δ_ι. What then is more natural than to ask what will happen if the cells $\delta_1, \delta_2 \cdots$ are infinite instead of finite in number? From this apparently trivial question results a theory of L-integrals which contains the Lebesgue integrals as a special case, and which, furthermore, has the great advantage that not only is the relation of the new integrals to the ordinary or Riemannian integrals perfectly obvious, but also the form of reasoning employed in Riemann's theory may be taken over to develop the properties of the new integrals.

Finally the author would call attention to the treatment of the area of a curved surface given at the end of this volume. Though the above are the main features of novelty, it is hoped that the experienced reader will discover some minor points, not lacking in originality, but not of sufficient importance to emphasize here.

It is now the author's pleasant duty to acknowledge the invaluable assistance derived from his colleague and former pupil, Dr. W. A. Wilson. He has read the entire manuscript and proof with great care, corrected many errors and oversights in the demonstrations, besides contributing the substance of §§ 372, 373, 401–406, 414–424.

Unstinted praise is also due to the house of Ginn and Company, who have met the author's wishes with unvarying liberality, and have given the utmost care to the press work.

JAMES PIERPONT

NEW HAVEN, December, 1911

CONTENTS

CHAPTER I

POINT SETS AND PROPER INTEGRALS

CHAPTER II

IMPROPER MULTIPLE INTEGRALS

CHAPTER III

SERIES

CHAPTER IV

MULTIPLE SERIES

CHAPTER V

SERIES OF FUNCTIONS

CHAPTER VI

POWER SERIES

CHAPTER VII

INFINITE PRODUCTS

CHAPTER XVII

GEOMETRIC NOTIONS

FUNCTION THEORY OF REAL VARIABLES

CHAPTER I

POINT SETS AND PROPER INTEGRALS

1. In this short chapter we wish to complete our treatment of proper multiple integrals and give a few theorems on point sets which we shall either need now or in the next chapter where we take up the important subject of improper multiple integrals.

In Volume I, 702, we have said that a limited point set whose upper and lower contents are the same is measurable. It seems best to reserve this term for another notion which has come into great prominence of late. We shall therefore in the future call sets whose upper and lower contents are equal, *metric sets*. When a set $\mathfrak{A}$ is metric, either symbol

$$\overline{\mathfrak{A}} \quad \text{or} \quad \underline{\mathfrak{A}}$$

expresses its content. In the following it will be often convenient to denote the content of $\mathfrak{A}$ by

$$\widehat{\mathfrak{A}}.$$

This notation will serve to keep in mind that $\mathfrak{A}$ is metric, when we are reasoning with sets some of which are metric, and some are not.

The frontier of a set as $\mathfrak{A}$, may be denoted by

$$\text{Front } \mathfrak{A}.$$

2. 1. In I, 713 we have introduced the very general notion of cell, division of space into cells, etc. The definition as there

given requires each cell to be metric. For many purposes this is not necessary; it suffices that the cells form an unmixed division of the given set $\mathfrak{A}$. Such divisions we shall call *unmixed divisions of norm* δ. [I, 711.] Under these circumstances we have now theorems analogous to I, 714, 722, 723, viz:

2. *Let $\mathfrak{B}$ contain the limited point set $\mathfrak{A}$. Let Δ denote an unmixed division of $\mathfrak{B}$ of norm δ. Let $\mathfrak{A}_\delta$ denote those cells of $\mathfrak{B}$ containing points of $\mathfrak{A}$. Then*

$$\lim_{\delta=0} \overline{\mathfrak{A}}_\delta = \overline{\mathfrak{A}}.$$

The proof is entirely analogous to I, 714.

3. *Let $\mathfrak{B}$ contain the limited point set $\mathfrak{A}$. Let $f(x_1 \cdots x_m)$ be limited in $\mathfrak{A}$. Let Δ be an unmixed division of $\mathfrak{B}$ of norm δ into cells $\delta_1, \delta_2, \cdots$. Let $\mathfrak{M}_\iota$, $\mathfrak{m}_\iota$ be respectively the maximum and minimum of f in δ_ι. Then*

$$\lim_{\delta=0} \overline{S}_\Delta = \lim_{\delta=0} \Sigma \mathfrak{M}_\iota \overline{\delta}_\iota = \overline{\int_{\mathfrak{A}}} f d\mathfrak{A}, \tag{1}$$

$$\lim_{\delta=0} \underline{S}_\Delta = \lim_{\delta=0} \Sigma \mathfrak{m}_\iota \overline{\delta}_\iota = \underline{\int_{\mathfrak{A}}} f d\mathfrak{A}. \tag{2}$$

Let us prove 1); the relation 2) may be demonstrated in a similar manner. In the first place we show in a manner entirely analogous to I, 722, that

$$\overline{S}_\Delta < \overline{\int_{\mathfrak{A}}} f d\mathfrak{A} + \epsilon, \qquad \delta < \delta_0. \tag{3}$$

The only modifications necessary are to replace δ_ι, δ_ι', $\delta_{\iota\kappa}$, by their upper contents, and to make use of the fact that Δ is unmixed, to establish 5).

To prove the other relation

$$\overline{S}_\Delta > \overline{\int_{\mathfrak{A}}} f d\mathfrak{A} - \epsilon, \qquad \delta < \delta_0, \tag{4}$$

we shall modify the proof as follows. Let E be a cubical division of space of norm $e < e_0$. We may take e_0 so small that

$$\left| \overline{\int_{\mathfrak{A}}} - \overline{S}_E \right| < \frac{\epsilon}{2}. \tag{5}$$

The cells of E containing points of $\mathfrak{A}$ fall into two classes. 1° the cells $e_{\iota\kappa}$ containing points of the cell δ_ι but of no other cell of Δ; 2° the cells e'_ι containing points of two or more cells of Δ. Thus we have

$$\overline{S}_E = \Sigma M_{\iota\kappa} e_{\iota\kappa} + \Sigma M'_\iota e'_\iota,$$

where $M_{\iota\kappa}$, M'_ι are the maxima of f in $e_{\iota\kappa}$, e'_ι. Then as above we have

$$\overline{S}_E < \Sigma \mathfrak{M}_\iota e_{\iota\kappa} + \frac{\epsilon}{4}, \qquad e < e_0, \tag{6}$$

if e_0 is taken sufficiently small.

On the other hand, we have

$$\left|\overline{S}_\Delta - \Sigma \mathfrak{M}_\iota e_{\iota\kappa}\right| \leq F \sum_\iota \left|\overline{\delta}_\iota - \sum_\kappa e_{\iota\kappa}\right|.$$

Now we may suppose δ_0, e_0 are taken so small that

$$\Sigma \overline{\delta}_\iota, \quad \Sigma e_{\iota\kappa}$$

differ from $\overline{\mathfrak{A}}$ by as little as we choose. We have therefore for properly chosen δ_0, e_0,

$$|\overline{S}_\Delta - \Sigma \mathfrak{M}_\iota e_{\iota\kappa}| < \frac{\epsilon}{4}.$$

This with 6) gives

$$\overline{S}_\Delta > \overline{S}_E - \frac{\epsilon}{2},$$

which with 5) proves 4).

4. *Let $f(x_1 \cdots x_m)$ be limited in the limited field $\mathfrak{A}$. Let Δ be an unmixed division of $\mathfrak{A}$ of norm δ, into cells $\delta_1, \delta_2 \cdots$. Let*

$$\underline{S}_\Delta = \Sigma m_\iota \overline{\delta}_\iota, \qquad \overline{S}_\Delta = \Sigma M_\iota \overline{\delta}_\iota,$$

where as usual m_ι, M_ι are the minimum and maximum of f in δ_ι. Then

$$\underline{\int_{\mathfrak{A}}} f d\mathfrak{A} = \text{Max } \underline{S}_\Delta, \qquad \overline{\int_{\mathfrak{A}}} f d\mathfrak{A} = \text{Min } \overline{S}_\Delta.$$

The proof is entirely similar to I, 723, replacing the theorem there used by 2, 3.

5. In connection with 4 and the theorem I, 696, 723 it may be well to caution the reader against an error which students are apt to make. The theorems I, 696, 1, 2 are not necessarily true if f

has both signs in $\mathfrak{A}$. For example, consider a unit square S whose center call C. Let us effect a division E of S into 100 equal squares and let $\mathfrak{A}$ be formed of the lower left-hand square s and of C. Let us define f as follows:

$$\begin{aligned} f &= 1 \text{ within } s \\ &= -100 \text{ at } C. \end{aligned}$$

For the division E,

$$\overline{S}_E = -1 + \tfrac{1}{100} = -\tfrac{99}{100}.$$

Hence,
$$\text{Min } \overline{S}_D \leq -\tfrac{99}{100}.$$

On the other hand,
$$\lim_{d=0} \overline{S}_D = \tfrac{1}{100}.$$

The theorems I, 723, and its analogue 4 are not necessarily true for unmixed divisions of space. The division Δ employed must be unmixed divisions of the field of integration $\mathfrak{A}$. That this is so, is shown by the example just given.

6. In certain cases the field $\mathfrak{A}$ may contain no points at all. In such a case we define
$$\int_{\mathfrak{A}} f = 0.$$

7. From 4 we have at once:

Let Δ be an unmixed division of $\mathfrak{A}$ into cells $\delta_1, \delta_2, \cdots$ Then

$$\overline{\mathfrak{A}} = \text{Min } \Sigma \overline{\delta}_\iota,$$

with respect to the class of all divisions Δ.

8. We also have the following:

Let D be an unmixed division of space. Let $d_1, d_2, \cdots$ denote those cells containing points of $\mathfrak{A}$. Then

$$\overline{\mathfrak{A}} = \text{Min } \Sigma \widehat{d}_\iota,$$

with respect to the class of the divisions D.

For if we denote by δ_ι the points of $\mathfrak{A}$ in d_ι we have obviously $\overline{\delta}_\iota \leq \widehat{d}_\iota$.

Also by I, 696,
$$\overline{\mathfrak{A}} = \text{Min } \Sigma \widehat{e}_\iota$$

with respect to the class of rectangular division of space $E = \{e_\iota\}$. But the class E is a subclass of the class D.

Thus

$$\operatorname*{Min}_{\Delta} \Sigma \overline{\delta}_\iota \leq \operatorname*{Min}_{D} \Sigma \overline{d}_\iota \leq \operatorname*{Min}_{E} \Sigma \overline{e}_\iota .$$

Here the two end terms have the value $\mathfrak{A}$.

3. Let $f(x_1 \cdots x_m)$, $g(x_1 \cdots x_m)$ be limited in the limited field $\mathfrak{A}$. We have then the following theorems:

1. *Let $f = g$ in $\mathfrak{A}$ except possibly at the points of a discrete set $\mathfrak{D}$. Then,*

$$\overline{\underline{\int}}_{\mathfrak{A}} f = \overline{\underline{\int}}_{\mathfrak{A}} g.$$

For let $|f|$, $|g| < M$. Let D be a cubical division of norm d. Let M_ι, N_ι denote the maximum of f, g in the cell d_ι. Let Δ denote the cells containing points of $\mathfrak{D}$, while A may denote the other cells of $\mathfrak{A}_D$.

Then,
$$\sum_{\mathfrak{A}} M_\iota d_\iota = \sum_{A} M_\iota d_\iota + \sum_{\Delta} M_\iota d_\iota ;$$

$$\sum_{\mathfrak{A}} N_\iota d_\iota = \sum_{A} M_\iota d_\iota + \sum_{\Delta} N_\iota d_\iota .$$

Hence,
$$\left| \sum_{\mathfrak{A}} M_\iota d_\iota - \sum_{\mathfrak{A}} N_\iota d_\iota \right| \leq \sum_{\Delta} |M_\iota - N_\iota| \, d_\iota \leq 2 M \sum_{\Delta} d_\iota ,$$

and the term on the right $\doteq 0$ as $d \doteq 0$.

2. *Let $f \geq g$ in $\mathfrak{A}$ except possibly at the points of a discrete set $\mathfrak{D}$. Then*

$$\overline{\underline{\int}}_{\mathfrak{A}} f \geq \overline{\underline{\int}}_{\mathfrak{A}} g.$$

For let
$$\mathfrak{A} = A + \mathfrak{D}.$$

Then
$$\overline{\underline{\int}}_{\mathfrak{A}} f = \overline{\underline{\int}}_{A} f, \qquad \overline{\underline{\int}}_{\mathfrak{A}} g = \overline{\underline{\int}}_{A} g.$$

But in A, $f \geq g$, hence

$$\overline{\underline{\int}}_{A} f \geq \overline{\underline{\int}}_{A} g, \quad \text{by I, 729.}$$

The theorem now follows at once.

3. *If* $c > 0$,

$$\overline{\underline{\int}}_{\mathfrak{A}} cf = c \overline{\underline{\int}}_{\mathfrak{A}} f;$$

but if $c < 0$,

$$\overline{\int}_{\mathfrak{A}} cf = c \underline{\int}_{\mathfrak{A}} f; \qquad \underline{\int}_{\mathfrak{A}} cf = c \overline{\int}_{\mathfrak{A}} f.$$

For in any cell d_ι

$$\text{Max} \cdot cf = c \, \text{Max} \, f; \qquad \text{Min} \cdot cf = c \, \text{Min} \, f$$

when $c > 0$; while

$$\text{Max} \cdot cf = c \, \text{Min} \, f; \qquad \text{Min} \cdot cf = c \, \text{Max} \, f$$

when $c < 0$.

4. *If g is integrable in* $\mathfrak{A}$,

$$\overline{\underline{\int}}_{\mathfrak{A}} (f + g) = \overline{\underline{\int}}_{\mathfrak{A}} f + \int_{\mathfrak{A}} g. \tag{1}$$

For from

$$\text{Max} \, f + \text{Min} \, g \leq \text{Max} \, (f + g) \leq \text{Max} \, f + \text{Max} \, g,$$

we have

$$\overline{\int}_{\mathfrak{A}} f + \underline{\int}_{\mathfrak{A}} g \leq \overline{\int}_{\mathfrak{A}} (f + g) \leq \overline{\int}_{\mathfrak{A}} f + \overline{\int}_{\mathfrak{A}} g. \tag{2}$$

But g being integrable,

$$\underline{\int}_{\mathfrak{A}} g = \overline{\int}_{\mathfrak{A}} g.$$

Hence 2) gives

$$\overline{\int}_{\mathfrak{A}} f + \int_{\mathfrak{A}} g = \overline{\int}_{\mathfrak{A}} (f + g),$$

which is the first half of 1). The other half follows from the relation

$$\text{Min} \, f + \text{Min} \, g \leq \text{Min} \, (f + g) \leq \text{Min} \, f + \text{Max} \, g.$$

5. *The integrands f, g being limited,*

$$\underline{\int}_{\mathfrak{A}} (f + g) \leq \underline{\int}_{\mathfrak{A}} f + \overline{\int}_{\mathfrak{A}} g \leq \overline{\int}_{\mathfrak{A}} (f + g).$$

For in any cell d_ι

$$\text{Min} \, (f + g) \leq \text{Min} \, f + \text{Max} \, g \leq \text{Max} \, (f + g).$$

6. *Let $f = g + h$, $|h| \leq H$ a constant, in $\mathfrak{A}$. Then,*

$$\left| \overline{\int_{\mathfrak{A}}} f - \overline{\int_{\mathfrak{A}}} g \right| \leq H\overline{\mathfrak{A}}.$$

For
$$-H + g \leq f \leq g + H.$$

Then by 2 and 4

$$-\int_{\mathfrak{A}} H + \overline{\int_{\mathfrak{A}}} g \leq \overline{\int_{\mathfrak{A}}} f \leq \overline{\int_{\mathfrak{A}}} g + \int_{\mathfrak{A}} H,$$

or

$$-H\overline{\mathfrak{A}} + \overline{\int_{\mathfrak{A}}} g \leq \overline{\int_{\mathfrak{A}}} f \leq \overline{\int_{\mathfrak{A}}} g + H\overline{\mathfrak{A}}.$$

4. *Let $f(x_1 \cdots x_m)$ be limited in limited $\mathfrak{A}$. Then,*

$$\left| \underline{\int_{\mathfrak{A}}} f \right| \leq \overline{\int_{\mathfrak{A}}} |f|, \qquad (1$$

$$\underline{\int_{\mathfrak{A}}} f \leq \left| \overline{\int_{\mathfrak{A}}} f \right|, \qquad (2$$

$$-\underline{\int_{\mathfrak{A}}} |f| \leq \overline{\int_{\mathfrak{A}}} f \leq \overline{\int_{\mathfrak{A}}} |f|, \qquad (3$$

$$-\overline{\int_{\mathfrak{A}}} |f| \leq \underline{\int_{\mathfrak{A}}} f \leq \underline{\int_{\mathfrak{A}}} |f|. \qquad (4$$

If $|f| \leq M$, we have also,

$$\underline{\overline{\int_{\mathfrak{A}}}} |f| \leq M\overline{\mathfrak{A}}. \qquad (5$$

Let us effect a cubical division of space of norm δ.

To prove 1) let $N_\iota = \text{Max}\,|f|$ in the cell d_ι. Then using the customary notation,

$$-N_\iota \leq m_\iota \leq M_\iota \leq N_\iota.$$

Hence
$$-\Sigma N_\iota d_\iota \leq \Sigma m_\iota d_\iota \leq \Sigma M_\iota d_\iota \leq \Sigma N_\iota d_\iota.$$

Letting $\delta \doteq 0$, this gives

$$-\overline{\int_{\mathfrak{A}}} |f| \leq \underline{\overline{\int_{\mathfrak{A}}}} f \leq \overline{\int_{\mathfrak{A}}} |f|,$$

which is 1).

To prove 3), we use the relation

$$-|f| \leq f \leq |f|.$$

Hence

$$\overline{\int_{\mathfrak{A}}} -|f| \leq \overline{\int_{\mathfrak{A}}} f \leq \overline{\int_{\mathfrak{A}}} |f|,$$

from which 3) follows on using 3, 3.

The demonstration of 4) is similar.

To prove 5), we observe that

$$\Sigma M_\iota d_\iota \leq M \Sigma d_\iota.$$

5. 1. *Let $f \geq 0$ be limited in the limited fields $\mathfrak{B}$, $\mathfrak{C}$. Let $\mathfrak{A}$ be the aggregate formed of the points in either $\mathfrak{B}$ or $\mathfrak{C}$. Then*

$$\underline{\overline{\int_{\mathfrak{A}}}} f \leq \underline{\overline{\int_{\mathfrak{B}}}} f + \underline{\overline{\int_{\mathfrak{C}}}} f. \qquad (1$$

This is obvious since the sums

$$\sum_{\mathfrak{B}} M_\iota d_\iota \quad , \quad \sum_{\mathfrak{C}} M_\iota d_\iota$$

may have terms in common. Such terms are therefore counted twice on the right of 1) and only once on the left, before passing to the limit.

Remark. The relation 1) may not hold when f is not > 0.

Example. Let $\mathfrak{A} = (0, 1)$, $\mathfrak{B}$ = rational points, and $\mathfrak{C}$ = irrational points in $\mathfrak{A}$. Let $f = 1$ in $\mathfrak{B}$, and -1 in $\mathfrak{C}$. Then

$$\overline{\int_{\mathfrak{A}}} f = +1, \qquad \overline{\int_{\mathfrak{B}}} f = 1, \qquad \overline{\int_{\mathfrak{C}}} f = -1,$$

and 1) does not now hold.

2. *Let $\mathfrak{A}$ be an unmixed partial aggregate of the limited field $\mathfrak{B}$. Let $\mathfrak{C} = \mathfrak{B} - \mathfrak{A}$. If*

$$\begin{aligned} g &= f \quad \text{in } \mathfrak{A} \\ &= 0 \quad \text{in } \mathfrak{C}, \end{aligned}$$

then

$$\underline{\overline{\int_{\mathfrak{A}}}} f = \underline{\overline{\int_{\mathfrak{B}}}} g.$$

For
$$\overline{\int}_{\mathfrak{B}} g = \overline{\int}_{\mathfrak{A}} g + \overline{\int}_{\mathfrak{C}} g, \quad \text{by I, 728.}$$

But
$$\overline{\int}_{\mathfrak{A}} g = \overline{\int}_{\mathfrak{A}} f, \qquad \text{by 3, 1,}$$

and obviously
$$\overline{\int}_{\mathfrak{C}} g = 0.$$

3. The reader should note that the above theorem need not be true if $\mathfrak{A}$ is not an unmixed part of $\mathfrak{B}$.

Example. Let $\mathfrak{A}$ denote the rational points in the unit square $\mathfrak{B}$.

Let
$$f = g = -1 \qquad \text{in } \mathfrak{A}.$$

Then
$$\overline{\int}_{\mathfrak{A}} f = -1. \qquad \underline{\int}_{\mathfrak{B}} g = -1. \qquad \overline{\int}_{\mathfrak{B}} g = 0.$$

4. *Let $\mathfrak{A}$ be a part of the limited field $\mathfrak{B}$. Let $f \geq 0$ be limited in $\mathfrak{A}$. Let $g = f$ in $\mathfrak{A}$ and $= 0$ in $\mathfrak{C} = \mathfrak{B} - \mathfrak{A}$. Then*

$$\overline{\int}_{\mathfrak{A}} f = \overline{\int}_{\mathfrak{B}} g ; \tag{1}$$

$$\underline{\int}_{\mathfrak{A}} f \geq \underline{\int}_{\mathfrak{B}} g. \tag{2}$$

For let M_ι, N_ι be the maxima of f, g in the cell d_ι. Then

$$\begin{aligned} \sum_{\mathfrak{B}} N_\iota d_\iota &= \sum_{\mathfrak{A}} N_\iota d_\iota + \sum N_\iota d_\iota \\ &= \sum_{\mathfrak{A}} M_\iota d_\iota . \end{aligned}$$

Passing to the limit we get 1).

To prove 2) we note that in any cell containing a point of $\mathfrak{A}$

$$\operatorname{Min} f \geq \operatorname{Min} g.$$

6. 1. *Let $f(x_1 \cdots x_m)$ be limited in the limited field $\mathfrak{A}$. Let $\mathfrak{B}_u$ be an unmixed part of $\mathfrak{A}$ such that $\overline{\mathfrak{B}}_u \doteq \overline{\mathfrak{A}}$ as $u \doteq 0$. Then*

$$\overline{\int}_{\mathfrak{A}} f = \lim_{u=0} \overline{\int}_{\mathfrak{B}_u} f. \tag{1}$$

For let $|f| \leq M$ in $\mathfrak{A}$. Let $\mathfrak{C}_u = \mathfrak{A} - \mathfrak{B}_u$. Then

$$\overline{\int}_{\mathfrak{A}} f = \overline{\int}_{\mathfrak{B}_u} f + \overline{\int}_{\mathfrak{C}_u} f, \qquad \text{by I, 728.} \tag{2}$$

But

$$\left| \overline{\int}_{\mathfrak{C}_u} f \right| \leq M\overline{\mathfrak{C}}_u, \qquad \text{by 4, 1), 5).}$$

Hence passing to the limit $u = 0$ in 2) we get 1).

2. We note that 1 may be incorrect if the $\mathfrak{B}_u$ are not unmixed. For let $\mathfrak{A}$ be the unit square. Let $\mathfrak{B}_u$ be the rational points in a concentric square whose side is $1 - u$. Let $f = 1$ for the rational points of $\mathfrak{A}$ and $= 2$ for the other points. Then

$$\overline{\int}_{\mathfrak{A}} f = 2, \qquad \lim_{u=0} \overline{\int}_{\mathfrak{B}_u} f = 1.$$

7. In I, 716 we have given a uniform convergence theorem when each $\mathfrak{B}_u \leq \mathfrak{A}$. A similar theorem exists when each $\mathfrak{B}_u \geq \mathfrak{A}$, viz.:

Let $\mathfrak{B}_u \leq \mathfrak{B}_{u'}$ if $u < u'$. Let $\mathfrak{A}$ be a part of each $\mathfrak{B}_u$. Let $\overline{\mathfrak{B}}_u \doteq \overline{\mathfrak{A}}$ as $u \doteq 0$. Then for each $\epsilon > 0$, there exists a pair u_0, d_0 such that

$$\overline{\mathfrak{B}}_{u,D} - \overline{\mathfrak{A}}_D \leq \overline{\mathfrak{B}}_{u,D} - \overline{\mathfrak{A}} < \epsilon, \qquad u < u_0,\ d < d_0,$$

For

$$\overline{\mathfrak{B}}_{u_0} < \overline{\mathfrak{A}} + \frac{\epsilon}{2}, \qquad u_0 \text{ sufficiently small.}$$

Also for any division D of norm $d <$ some d_0.

$$\overline{\mathfrak{B}}_{u_0, D} < \overline{\mathfrak{B}}_{u_0} + \frac{\epsilon}{2}.$$

But

$$\overline{\mathfrak{B}}_{u, D} \leq \overline{\mathfrak{B}}_{u_0, D}, \qquad \text{if } u < u_0.$$

Hence

$$\overline{\mathfrak{B}}_{u, D} < \overline{\mathfrak{A}} + \epsilon \leq \overline{\mathfrak{A}}_D + \epsilon.$$

8. 1. Let $\mathfrak{A}$ be a point set in $m = r + s$ way space. Let us set certain coördinates as $x_{r+1} \cdots x_m = 0$ in each point of $\mathfrak{A}$. The resulting points $\mathfrak{B}$ we call a *projection* of $\mathfrak{A}$. The points of $\mathfrak{A}$

belonging to a given point b of $\mathfrak{B}$, we denote by $\mathfrak{C}_b$ or more shortly by $\mathfrak{C}$. We write

$$\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C},$$

and call $\mathfrak{B}$, $\mathfrak{C}$ *components* of $\mathfrak{A}$.

We note that the fundamental relations of I, 733

$$\underline{\int}_{\mathfrak{A}} f \leq \underline{\int}_{\mathfrak{B}} \underline{\int}_{\mathfrak{C}} f \leq \overline{\int}_{\mathfrak{A}} f$$

hold not only for the components $\mathfrak{x}$, $\mathfrak{P}$, etc., as there given, but also for the general components $\mathfrak{A}$, $\mathfrak{B}$.

In what follows we shall often give a proof for two dimensions for the sake of clearness, but in such cases the form of proof will admit an easy generalization. In such cases $\mathfrak{B}$ will be taken as the x-projection or component of $\mathfrak{A}$.

2. *If* $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ *is limited and* $\mathfrak{B}$ *is discrete*, $\mathfrak{A}$ *is also discrete.*

For let $\mathfrak{A}$ lie within a cube of edge $\frac{1}{2} C > 1$ in $m = r + s$ way space. Then for any $d <$ some d_0,

$$\overline{\mathfrak{B}}_D < \frac{\epsilon}{C^s}.$$

Then $$\overline{\mathfrak{A}}_D < C^s \overline{\mathfrak{B}}_D < \epsilon.$$

3. That the converse of 2 is not necessarily true is shown by the two following examples, which we shall use later:

Example 1. Let $\mathfrak{A}$ denote the points x, y in the unit square determined thus:

For
$$x = \frac{m}{2^n}, \qquad n = 1, 2, 3, \cdots, \qquad m \text{ odd and } < 2^n,$$
let
$$0 \leq y \leq \frac{1}{2^n}.$$

Here $\mathfrak{A}$ is discrete, while $\overline{\mathfrak{B}} = 1$, where $\mathfrak{B}$ denotes the projection of $\mathfrak{A}$ on the x-axis.

4. *Example 2.* Let $\mathfrak{A}$ denote the points x, y in the unit square determined thus:

For

$$x = \frac{m}{n}, \qquad m, n \text{ relatively prime,}$$

let

$$0 \le y \le \frac{1}{n}.$$

Then, $\mathfrak{B}$ denoting the projection of $\mathfrak{A}$ on the x-axis, we have

$$\overline{\mathfrak{A}} = 0, \qquad \overline{\mathfrak{B}} = 1.$$

9. 1. *Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be a limited point set. Then*

$$\underline{\mathfrak{A}} \le \overline{\underline{\int}}_{\mathfrak{B}} \overline{\mathfrak{C}} \le \overline{\mathfrak{A}}. \qquad (1$$

For let $f = 1$ in $\mathfrak{A}$. Let $g = 1$ at each point of $\mathfrak{A}$ and at the other points of a cube $A = B \cdot C$ containing $\mathfrak{A}$, let $g = 0$. Then

$$\underline{\mathfrak{A}} = \underline{\int}_A g, \qquad \overline{\mathfrak{A}} = \overline{\int}_A g.$$

By I, 733,

$$\underline{\int}_A g \le \overline{\underline{\int}}_B \overline{\int}_C g \le \overline{\int}_A g.$$

But by 5, 4,

$$\overline{\int}_B \overline{\int}_C g = \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f,$$

$$\underline{\int}_B \overline{\int}_C g \le \underline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f.$$

Thus

$$\underline{\mathfrak{A}} = \underline{\int}_A g < \underline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f \le \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f = \overline{\int}_B \overline{\int}_C g \le \overline{\int}_A g = \overline{\mathfrak{A}},$$

which gives 1), since

$$\overline{\int}_{\mathfrak{C}} f = \overline{\mathfrak{C}}.$$

2. *In case $\mathfrak{A}$ is metric we have*

$$\widehat{\mathfrak{A}} = \overline{\underline{\int}}_{\mathfrak{B}} \overline{\mathfrak{C}}, \qquad (2$$

and $\overline{\mathfrak{C}}$ is an integrable function over $\mathfrak{B}$.

This follows at once from 1).

3. In this connection we should note, however, that the converse of 2 is not always true, *i.e.* if $\overline{\mathfrak{C}}$ is integrable, then $\mathfrak{A}$ has content and 2, 2) holds. This is shown by the following:

Example. In the unit square we define the points x, y of $\mathfrak{A}$ thus:

For rational x, $$0 \leq y \leq \tfrac{1}{2}.$$

For irrational x, $$\tfrac{1}{2} \leq y \leq 1.$$

Then $\overline{\mathfrak{C}} = \frac{1}{2}$ for every x in $\mathfrak{B}$. Hence

$$\int_{\mathfrak{B}} \overline{\mathfrak{C}} = \tfrac{1}{2}.$$

But $$\underline{\mathfrak{A}} = 0, \qquad \overline{\mathfrak{A}} = 1.$$

10. 1. *Let $f(x_1 \cdots x_m)$ be limited in the limited field $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$.*

If $f \geq 0$,
$$\underline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f \leq \overline{\int}_{\mathfrak{A}} f. \tag{1}$$

If $f \leq 0$,
$$\underline{\int}_{\mathfrak{A}} f \leq \underline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f. \tag{2}$$

Let us first prove 1). Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ lie in the spaces $\mathfrak{R}_m$, $\mathfrak{R}_r$, $\mathfrak{R}_s$, $r + s = m$. Then any cubical division D divides these spaces into cubical cells d_ι, d'_ι, d''_ι of volumes d, d', d'' respectively. Obviously $d = d'd''$. D also divides $\mathfrak{B}$ and *each* $\mathfrak{C}$ into unmixed cells δ', δ''. Let $M_\iota = \text{Max} f$ in one of the cells d_ι, while $M''_\iota = \text{Max} f$ in the corresponding cell δ''_ι. Then by 2, 4,

$$\overline{\int}_{\mathfrak{C}} f \leq \Sigma M''_\iota \delta''_\iota \leq \Sigma M d''_\iota,$$

since M_ι, $M'_\iota \geq 0$. Hence

$$\sum_{\mathfrak{B}} \overline{\delta}'_\kappa \overline{\int}_{\mathfrak{C}} f \leq \sum_{\mathfrak{B}} d'_\kappa \Sigma M_\iota d'' = \Sigma M_\iota d.$$

Letting the norm of D converge to zero, we get 1). We get 2) by similar reasoning or by using 3, 3 and 1).

2. To illustrate the necessity of making $f \geq 0$ in 1), let us take $\mathfrak{A}$ to be the Pringsheim set of I, 740, 2, while f shall $= -1$ in $\mathfrak{A}$. Then

$$\overline{\int}_{\mathfrak{A}} f = -1.$$

On the other hand

$$\overline{\underline{\int}}_{\mathfrak{C}} f = 0.$$

Hence

$$\overline{\underline{\int}}_{\mathfrak{B}} \overline{\underline{\int}}_{\mathfrak{C}} f = 0,$$

and the relation 1) does not hold here.

Iterable Fields

11. 1. There is a large class of limited point sets which do not have content and yet

$$\overline{\mathfrak{A}} = \overline{\underline{\int}}_{\mathfrak{B}} \overline{\mathfrak{C}}. \tag{1}$$

Any limited point set satisfying the relation 1) we call *iterable*, or more specifically *iterable with respect to* $\mathfrak{B}$.

Example 1. Let $\mathfrak{A}$ consist of the rational points in the unit square. Obviously

$$\overline{\mathfrak{A}} = \overline{\underline{\int}}_{\mathfrak{B}} \overline{\mathfrak{C}} = \overline{\underline{\int}}_{\mathfrak{C}} \overline{\mathfrak{B}} = 1,$$

so that $\mathfrak{A}$ is iterable both with respect to $\mathfrak{B}$ and $\mathfrak{C}$.

Example 2. Let $\mathfrak{A}$ consist of the points x, y in the unit square defined thus:

For rational x let $\quad 0 \leq y \leq \frac{1}{2}$.
For irrational x let $\quad 0 \leq y \leq 1$.

Here $\overline{\mathfrak{A}} = 1$.

$$\underline{\int}_{\mathfrak{B}} \overline{\mathfrak{C}} = \tfrac{1}{2}, \qquad \overline{\int}_{\mathfrak{B}} \overline{\mathfrak{C}} = 1.$$

$$\overline{\underline{\int}}_{\mathfrak{C}} \mathfrak{B} = 1.$$

Thus $\mathfrak{A}$ is iterable with respect to $\mathfrak{C}$ but not with respect to $\mathfrak{B}$.

Example 3. Let $\mathfrak{A}$ consist of the points in the unit square defined thus:

$$\text{For rational } x \text{ let } \quad 0 \leq y \leq \tfrac{3}{4}.$$
$$\text{For irrational } x \text{ let } \tfrac{1}{4} \leq y \leq 1.$$

Here $\overline{\mathfrak{A}} = 1$, while

$$\underline{\int_{\mathfrak{B}}} \overline{\mathfrak{C}} = \tfrac{3}{4}; \qquad \underline{\int_{\mathfrak{C}}} \overline{\mathfrak{B}} = 1.$$

Hence $\mathfrak{A}$ is iterable with respect to $\mathfrak{C}$ but not with respect to $\mathfrak{B}$.

Example 4. Let $\mathfrak{A}$ consist of the sides of the unit square and the rational points within the square.

Here $\overline{\mathfrak{A}} = 1$, while

$$\underline{\int_{\mathfrak{B}}} \overline{\mathfrak{C}} = 0, \qquad \overline{\int_{\mathfrak{B}}} \overline{\mathfrak{C}} = 1,$$

and similar relations for $\mathfrak{C}$. Thus $\mathfrak{A}$ is not iterable with respect to either $\mathfrak{B}$ or $\mathfrak{C}$.

Example 5. Let $\mathfrak{A}$ be the Pringsheim set of I, 740, 2.

Here $\overline{\mathfrak{A}} = 1$, while

$$\underline{\overline{\int_{\mathfrak{B}}}} \overline{\mathfrak{C}} = 0, \qquad \underline{\overline{\int_{\mathfrak{C}}}} \overline{\mathfrak{B}} = 0.$$

Hence $\mathfrak{A}$ is not iterable with respect to either $\mathfrak{B}$ or $\mathfrak{C}$.

2. *Every limited metric point set is iterable with respect to any of its projections.*

This follows at once from the definition and 9, 2.

12. 1. Although $\mathfrak{A}$ is not iterable it may become so on removing a properly chosen discrete set $\mathfrak{D}$.

Example. In Example 4 of 11, the points on the sides of the unit square form a discrete set $\mathfrak{D}$; on removing these, the deleted set $\mathfrak{A}^*$ is iterable with respect to either $\mathfrak{B}$ or $\mathfrak{C}$.

2. The reader is cautioned not to fall into the error of supposing that if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are unmixed iterable sets, then $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2$ is also iterable. That this is not so is shown by the Example in 1.

For let $\mathfrak{A}_1 = \mathfrak{A}^*$, $\mathfrak{A}_2 = \mathfrak{D}$ in that example. Then $\mathfrak{D}$ being discrete has content and is thus iterable. But $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2$ is not iterable with respect to either $\mathfrak{B}$ or $\mathfrak{C}$.

13. 1. Let $\mathfrak{A}$ be a limited point set lying in the m dimensional space $\mathfrak{R}_m$. Let $\mathfrak{B}$, $\mathfrak{C}$ be components of $\mathfrak{A}$ in $\mathfrak{R}_r$, $\mathfrak{R}_s$, $r+s=m$. A cubical division D of norm δ divides $\mathfrak{R}_m$ into cells of volume d and $\mathfrak{R}_r$ and $\mathfrak{R}_s$ into cells of volume d_r, d_s, where $d=d_r d_s$. Let b be any point of $\mathfrak{B}$, lying in a cell d_r. Let $\sum\limits_b d_s$ denote the sum of all the cells d_s, containing points of $\mathfrak{A}$ whose projection is b. Let $\sum\limits_{dr} d_s$ denote the sum of all the cells containing points of $\mathfrak{A}$ whose projection falls in d_r, not counting two d_s cells twice.

We have now the following theorem:

If $\mathfrak{A}$ is iterable with respect to $\mathfrak{B}$,

$$\lim_{\delta=0} \sum_{\mathfrak{B}} d_r \{\sum_{d_r} d_s - \sum_b d_s\} = 0. \qquad (1$$

For
$$\overline{\mathfrak{C}}_b \le \sum_b d_s \le \sum_{d_r} d_s.$$

Hence
$$\sum_{\mathfrak{B}} d_r \overline{\mathfrak{C}}_b \le \sum_{\mathfrak{B}} d_r \sum_b d_s \le \sum_{\mathfrak{B}} d_r \sum_{d_r} d_s.$$

Let now $\delta \doteq 0$. The first and third members $\doteq \overline{\mathfrak{A}}$, using I, 699, since $\mathfrak{A}$ is iterable. Thus, the second and third members have the same limit, and this gives 1).

2. *If $\mathfrak{A}$ is iterable with respect to $\mathfrak{B}$,*

$$\lim_{\delta=0} \sum_{\mathfrak{B}} d_r \sum_b d_s = \overline{\mathfrak{A}}.$$

This follows at once from 1).

3. *Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be a limited point set, iterable with respect to $\mathfrak{B}$. Then any unmixed part $\mathfrak{E}$ of $\mathfrak{A}$ is also iterable with respect to the $\mathfrak{B}$-component of $\mathfrak{E}$.*

For let $b=$ a point of $\mathfrak{B}$; $\mathfrak{E}'$ points of $\mathfrak{A}$ not in $\mathfrak{E}$; $C_b=$ points of $\mathfrak{C}_b$ in $\mathfrak{E}$, $C_b' =$ points of $\mathfrak{C}_b$ in $\mathfrak{E}'$. Then for each $\beta > 0$ there exist a pair of points, b_1, b_2, distinct or coincident in any cell d_r such that as b ranges over this cell,

$$\overline{C}_{b_1} = \text{Min}\, \overline{C}_b + \beta', \quad \overline{C}_{b_2} = \text{Max}\, \overline{C}_b + \beta'', \quad |\beta'|, |\beta''| < \beta.$$

Let S denote, as in 13, 1, the cells of $\sum_{d_r} d_s$ which contain points of $\mathfrak{C}'$, and F the cells containing points of both $\mathfrak{C}$, $\mathfrak{C}'$ whose projections fall in d_r. Then from

$$\overline{C}_{b_1} + \overline{C}'_{b_1} \leq \overline{C}_{b_1} + \overline{S} \leq \overline{C}_{b_2} + \overline{S} \leq \sum_{d_r} d_s + F$$

we have

$$\overline{\mathfrak{C}}_{b_1} \leq \overline{C}_{b_1} + \overline{C}'_{b_1} \leq \text{Min}\, \overline{C}_b + \beta' + \overline{S} \leq \text{Max}\, \overline{C}_b + \beta'' + \overline{S} \leq \sum_{d_r} d_s + F$$

Multiplying by d_r and summing over $\mathfrak{B}$ we have,

$$\Sigma\, d_r \overline{\mathfrak{C}}_{b_1} \leq \Sigma\, d_r\, \text{Min}\, \overline{C}_b + \Sigma\, \beta' d_r + \overline{\mathfrak{C}}'_D \leq \Sigma\, d_r\, \text{Max}\, \overline{C}_b + \Sigma\, \beta'' d_r + \mathfrak{C}'_D$$
$$\leq \overline{\mathfrak{A}}_D + \Sigma\, d_r F. \tag{1}$$

Passing to the limit, we have

$$\overline{\mathfrak{A}} \leq \underline{\int}_{\mathfrak{B}} \overline{C} + \eta' + \overline{\mathfrak{C}}' \leq \overline{\int}_{\mathfrak{B}} \overline{C} + \eta'' + \overline{\mathfrak{C}} \leq \overline{\mathfrak{A}}, \tag{2}$$

the limit of the last term vanishing since $\mathfrak{C}$, $\mathfrak{C}'$ are unmixed parts of $\mathfrak{A}$. Here η', η'' are as small as we please on taking β sufficiently small. From 2) we now have

$$\underline{\overline{\int}}_{\mathfrak{B}} \overline{C} = \overline{\mathfrak{A}} - \overline{\mathfrak{C}}' = \overline{\mathfrak{C}}.$$

4. *Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be iterable with respect to $\mathfrak{B}$. Let B be a part of $\mathfrak{B}$ and A all those points of $\mathfrak{A}$ whose projection falls on B. Then A is iterable with respect to B.*

For let D be a cubical division of space of norm d. Then

$$\overline{\mathfrak{A}} = \lim_{d=0} \overline{\mathfrak{A}}_D = \lim_{d=0} \left\{ \overline{A}_D + \sum_{r,s} d_r \cdot d_s \right\}, \tag{1}$$

where the sum on the right extends over those cells containing no point of A. Also

$$\overline{\mathfrak{A}} = \int_{\mathfrak{B}} \overline{\mathfrak{C}} = \lim_{d=0} \left\{ \sum_B d_r \overline{\mathfrak{C}} + \Sigma d_r \overline{\mathfrak{C}} \right\}, \tag{2}$$

where the second sum on the right extends over those cells d_r containing no point of B.

Subtracting 1), 2) gives

$$0 = \lim_{d=0} \left\{ \overline{A}_D - \sum_B d_r \overline{\mathfrak{C}} \right\} + \lim_{d=0} \left\{ \sum_{r,s} d_r d_s - \sum_r d_r \overline{\mathfrak{C}} \right\}.$$

As each of the braces is ≥ 0 we have

$$\bar{A} = \overline{\int_B} \overline{\mathfrak{C}}.$$

14. We can now generalize the fundamental inequalities of I, 733 as follows:

Let $f(x_1 \cdots x_m)$ be limited in the limited field $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$, iterable with respect to $\mathfrak{B}$. Then

$$\underline{\int_{\mathfrak{A}}} f \leq \underline{\int_{\mathfrak{B}}} \underline{\int_{\mathfrak{C}}} f \leq \overline{\int_{\mathfrak{A}}} f. \tag{1}$$

For let us choose the positive constants A, B such that

$$f + A > 0, \qquad f - B < 0, \qquad \text{in } \mathfrak{A}.$$

Let us effect a cubical division of the space of $\mathfrak{R}_m$ of norm δ into cells d. As in 13, this divides $\mathfrak{R}_r$, $\mathfrak{R}_s$ into cells which we denote, as well as their contents, by d_r, d_s. Let b denote any point of $\mathfrak{B}$. As usual let m, M denote the minimum and maximum of f in the cell d containing a point of $\mathfrak{A}$. Let m', M' be the corresponding extremes of f when we consider only those points of $\mathfrak{A}$ in d whose projection is b. Let $|f| \leq F$ in $\mathfrak{A}$.

Then for any b, we have by I, 696,

$$\Sigma(m' - B)d_s \leq \underline{\int_{\mathfrak{C}}} (f - B) = \underline{\int_{\mathfrak{C}}} f - B\overline{\mathfrak{C}},$$

or

$$- B(\Sigma d_s - \overline{\mathfrak{C}}) + \Sigma m d_s \leq \underline{\int_{\mathfrak{C}}} f, \tag{2}$$

since $m \geq m'$.

In a similar manner

$$\overline{\int_{\mathfrak{C}}} f \leq \Sigma M d_s + A(\Sigma d_s - \overline{\mathfrak{C}}). \tag{3}$$

Thus for any b in $\mathfrak{B}$, 2), 3) give

$$- B(\sum_b d_s - \overline{\mathfrak{C}}) + \sum_b m d_s \leq \underline{\overline{\int_{\mathfrak{C}}}} f \leq \sum_b M d_s + A(\sum_b d_s - \overline{\mathfrak{C}}). \tag{4}$$

Let $\beta > 0$ be small at pleasure. There exist two points b_1, b_2 distinct or coincident in the cell d_r, for which

$$\underline{\int_{\mathfrak{C}_1}} f = j + \beta_1, \qquad \overline{\int_{\mathfrak{C}_2}} f = J + \beta_2$$

where $|\beta_1|, |\beta_2| < \beta$ and $\mathfrak{C}_1$, and $\mathfrak{C}_2$ stand for $\mathfrak{C}_{b_1}$, $\mathfrak{C}_{b_2}$, and finally where

$$j = \mathrm{Min}\underline{\overline{\int}}_{\mathfrak{C}} f, \qquad J = \mathrm{Max}\underline{\overline{\int}}_{\mathfrak{C}} f$$

for all points b in d_r.

Let $\mathfrak{c} = \mathrm{Min}\,\overline{\mathfrak{C}}$ in d_r, then 4) gives

$$-B(\sum_1 d_s - \mathfrak{c}) + \sum_1 md_s \leq j + \beta_1 \leq J + \beta_2 \leq \sum_2 Md_s + A(\sum_2 d_s - \mathfrak{c})$$

where the indices 1, 2 indicate that in $\sum_b$ we have replaced b by b_1, b_2.

Multiplying by d_r and summing over all the cells d_r containing points of $\mathfrak{B}$, the last relation gives

$$-B\sum_{\mathfrak{B}} d_r(\sum_1 d_s - \mathfrak{c}) + \sum_{\mathfrak{B}} d_r \sum_1 md_s \leq \sum_{\mathfrak{B}} jd_r + \sum_{\mathfrak{B}} \beta_1 d_r$$

$$\leq \sum_{\mathfrak{B}} Jd_r + \sum_{\mathfrak{B}} \beta_2 d_r \leq \sum_{\mathfrak{B}} d_r \sum_2 Md_s + A\sum_{\mathfrak{B}} d_r(\sum_2 d_s - \mathfrak{c}). \qquad (5$$

Now as $\delta \doteq 0$, $\quad \sum_{\mathfrak{B}} d_r \sum_1 d_s \doteq \overline{\mathfrak{A}}, \qquad \sum_{\mathfrak{B}} d_r \sum_2 d_s \doteq \overline{\mathfrak{A}}, \qquad$ by 13, 2.

$$\sum_{\mathfrak{B}} d_r \mathfrak{c} \doteq \underline{\int} \overline{\mathfrak{C}} = \overline{\mathfrak{A}}, \qquad \text{since } \mathfrak{A} \text{ is iterable.}$$

Thus the first and last sums in 5) are evanescent with δ. On the other hand

$$\left| \sum_{\mathfrak{B}} (d_r \sum_{d_s} d_s m - \sum_1 d_s m) \right| \leq F\sum_{\mathfrak{B}} d_r(\sum_{d_s} d_s - \sum_1 d_s)$$

$$\doteq 0 \text{ as } \delta \doteq 0, \qquad \text{by 13, 1.}$$

Thus

$$\lim_{\delta=0} \sum d_r \sum_1 d_s m = \underline{\int}_{\mathfrak{A}} f. \qquad (6$$

$$\lim_{\delta=0} \sum d_r \sum_2 d_s M = \overline{\int}_{\mathfrak{A}} f. \qquad (7$$

Hence passing to the limit $\delta = 0$ in 5) we get 1), since $\sum \beta_1 d_r$, $\sum \beta_2 d_r$ have limits numerically $< \beta\overline{\mathfrak{C}}$ which may be taken as small as we please as β is arbitrarily small.

2. If $\mathfrak{A}$ is not iterable with respect to $\mathfrak{B}$, let it be so on removing the discrete set $\mathfrak{D}$. Let the resulting field A have the components B, C. Then 1 gives

$$\underline{\int}_{\mathfrak{A}} f \leq \underline{\int}_{B} \overline{\int}_{C} f \leq \overline{\int}_{\mathfrak{A}} f;$$

since

$$\overline{\int}_{\mathfrak{A}} f = \overline{\int}_{A} f.$$

3. The reader should guard against supposing 1) is correct if only $\mathfrak{A}$ is iterable on removing a discrete set $\mathfrak{D}$. For consider the following:

Example. Let the points of $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{D}$ lie in the unit square. Let $\mathfrak{A}_1$ consist of all the points lying on the irrational ordinates. Let $\mathfrak{D}$ lie on the rational ordinates such that, when

$$x = \frac{m}{n}, \qquad m, n \text{ relatively prime,}$$

$$0 \leq y \leq \frac{1}{n}.$$

Let us define f over $\mathfrak{A}$ thus:

$$f = 1 \quad \text{in} \quad \mathfrak{A}_1,$$
$$f = 0 \quad \text{in} \quad \mathfrak{D}.$$

The relation 1) is false in this case. For

$$\overline{\int}_{\mathfrak{A}} f = 1,$$

while

$$\underline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f = 0.$$

15. 1. Let $f(x_1 \cdots x_m)$ be limited in the limited point set $\mathfrak{A}$. Let D denote the rectangular division of norm d. All the points of $\underline{\mathfrak{A}}_D$ except possibly those on its surface are inner points of $\mathfrak{A}$. [I, 702.]

The limits

$$\lim_{d=0} \int_{\underline{\mathfrak{A}}_D} f \quad , \quad \lim_{d=0} \overline{\int}_{\overline{\mathfrak{A}}_D} f \tag{1}$$

exist and will be denoted by

$$\underline{\int}^*_{\mathfrak{A}} f \quad , \quad \overline{\int}^*_{\mathfrak{A}} f \quad , \tag{2}$$

and are called the *inner*, *lower* and *upper integrals* respectively.

To show that 1) exist we need only to show that for each $\epsilon > 0$ there exists a d_0 such that for any rectangular divisions D', D'' of norms $< d_0$

$$\Delta = \left| \overline{\int}_{\underline{\mathfrak{A}}_{D'}} - \overline{\int}_{\underline{\mathfrak{A}}_{D''}} \right| < \epsilon.$$

To this end, we denote by E the division formed by superimposing D'' on D'. Then E is a rectangular division of norm $< d_0$.

Let

$$\underline{\mathfrak{A}}_E - \underline{\mathfrak{A}}_{D'} = A', \qquad \underline{\mathfrak{A}}_E - \underline{\mathfrak{A}}_{D''} = A''.$$

If d_0 is sufficiently small,
$$A',\ A'' < \eta,$$
an arbitrarily small positive number. Then

$$\Delta = \left| \left(\overline{\int}_{\underline{\mathfrak{A}}_D} - \overline{\int}_{\underline{\mathfrak{A}}_E} \right) - \left(\overline{\int}_{\underline{\mathfrak{A}}_{D''}} - \overline{\int}_{\underline{\mathfrak{A}}_E} \right) \right| \leq \left| \overline{\int}_{A'} \right| + \left| \overline{\int}_{A''} \right| < \epsilon$$

if η is taken small enough.

2. The integrals

$$\underline{\int}_{\mathfrak{A}} f, \qquad \overline{\int}_{\mathfrak{A}} f,$$

heretofore considered may be called the *outer*, lower and upper integrals, in contradistinction.

3. *Let f be limited in the limited metric field $\mathfrak{A}$. Then the inner and outer lower (upper) integrals are equal.*

For $\underline{\mathfrak{A}}_D$ is an unmixed part of $\mathfrak{A}$ such that

$$\overline{\text{Cont}}\ \underline{\mathfrak{A}}_D \doteq \overline{\mathfrak{A}}, \qquad \text{as } d \doteq 0.$$

Then by 6, 1,
$$\lim_{d=0} \overline{\int}_{\underline{\mathfrak{A}}_D} f = \overline{\int}_{\underline{\mathfrak{A}}} f.$$

But the limit on the left is by definition

$$\overline{\int}^*_{\underline{\mathfrak{A}}} f.$$

4. *When $\mathfrak{A}$ has no inner points,*

$$\overline{\int}^*_{\underline{\mathfrak{A}}} f = 0.$$

For each $\underline{\mathfrak{A}}_D = 0$, and hence each

$$\overline{\int}_{\underline{\mathfrak{A}}_D} f = 0.$$

Point Sets

16. *Let* $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ *be metric. Then*

$$\widehat{\mathfrak{A}} = \underline{\mathfrak{B}} + \overline{\mathfrak{C}}. \tag{1}$$

For let D be a cubical division of space of norm d. The cells of $\overline{\mathfrak{A}}_D$ fall into three classes: 1°, cells containing only points of $\mathfrak{B}$; these form $\underline{\mathfrak{B}}_D$. 2°, cells containing points of $\mathfrak{C}$; these form $\overline{\mathfrak{C}}_D$. 3°, cells containing frontier points of $\mathfrak{B}$, not already included in 1° or 2°. Call these $\mathfrak{f}_D$. Then

$$\overline{\mathfrak{A}}_D = \underline{\mathfrak{B}}_D + \overline{\mathfrak{C}}_D + \overline{\mathfrak{f}}_D. \tag{2}$$

Let now $d \doteq 0$. As $\mathfrak{A}$ is metric, $\overline{\mathfrak{f}}_D \doteq 0$, since $\mathfrak{f}_D$ is a part of Front $\mathfrak{A}$ and this is discrete. Thus 2) gives 1).

17. 1. Let

$$\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C} \cdots \tag{1}$$

be point sets, limited or not, and finite or infinite in number. The aggregate formed of the points present in at least one of the sets 1) is called their *union*, and may be denoted by

$$U(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C} \cdots),$$

or more shortly by

$$(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C} \cdots).$$

If $\mathfrak{A}$ is a general symbol for the sets 1), the union of these sets may also be denoted by

$$U\{\mathfrak{A}\},$$

or even more briefly by

$$\{\mathfrak{A}\}.$$

If no two of the sets 1) have a point in common, their union may be called their *sum*, and this may be denoted by

$$\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \cdots$$

The set formed of the points common to all the sets 1) we call their *divisor* and denote by

$$Dv(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C} \cdots),$$

or by

$$Dv\{\mathfrak{A}\},$$

if $\mathfrak{A}$ is a general symbol as before.

2. *Examples.*

Let $\mathfrak{A}$ be the interval $(0, 2)$; $\mathfrak{B}$ the interval $(1, \infty)$. Then

$$U(\mathfrak{A}, \mathfrak{B}) = (0, \infty), \qquad Dv(\mathfrak{A}, \mathfrak{B}) = (1, 2).$$

Let

$$\mathfrak{A}_1 = (0, 1), \qquad \mathfrak{A}_2 = (1, 2) \cdots$$

Then

$$U(\mathfrak{A}_1, \mathfrak{A}_2 \cdots) = (0, \infty),$$

$$Dv(\mathfrak{A}_1, \mathfrak{A}_2 \cdots) = 0.$$

Let

$$\mathfrak{A}_2 = (\tfrac{1}{2}, 1), \qquad \mathfrak{A}_2 = (\tfrac{1}{3}, \tfrac{1}{2}), \qquad \mathfrak{A}_3 = (\tfrac{1}{4}, \tfrac{1}{3}) \cdots$$

Then

$$U(\mathfrak{A}_2 . \mathfrak{A}_2 \cdots) = (0^*, 1),$$

$$Dv(\mathfrak{A}_1, \mathfrak{A}_2 \cdots) = 0.$$

Let

$$\mathfrak{A}_1 = (\tfrac{1}{2}, 1\tfrac{1}{2}), \qquad \mathfrak{A}_2 = (\tfrac{1}{4}, 1\tfrac{3}{4}), \qquad \mathfrak{A}_3 = (\tfrac{1}{8}, 1\tfrac{7}{8}) \cdots$$

Then

$$U(\mathfrak{A}_1, \mathfrak{A}_2 \cdots) = (0^*, 2^*),$$

$$Dv(\mathfrak{A}_1, \mathfrak{A}_2 \cdots) = \mathfrak{A}_1.$$

3. Let

$$\mathfrak{A} \geq \mathfrak{A}_1 \geq \mathfrak{A}_2 \geq \mathfrak{A}_3 \geq \cdots \tag{1}$$

Let

$$\mathfrak{D} = Dv(\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2, \cdots).$$

Let

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{C}_1, \ \mathfrak{A}_1 = \mathfrak{A}_2 + \mathfrak{C}_2, \cdots$$

Then

$$\mathfrak{A} = \mathfrak{D} + \mathfrak{C}_1 + \mathfrak{C}_2 + \cdots$$

Let us first exclude the $=$ sign in 1). Then every element of $\mathfrak{A}$ which is not in $\mathfrak{D}$ is in some $\mathfrak{A}_n$ but not in $\mathfrak{A}_{n+1}$. It is therefore in $\mathfrak{C}_{n+1}$ but not in $\mathfrak{C}_{n+2}, \mathfrak{C}_{n+3}, \cdots$ The rest now follows easily.

4. Some writers call the union of two sets $\mathfrak{A}$, $\mathfrak{B}$ their sum, whether $\mathfrak{A}$, $\mathfrak{B}$ have a point in common or not. We have not done this because the associative property of sums, viz. :

$$\alpha + (\beta - \gamma) = (\alpha + \beta) - \gamma$$

does not hold in general for unions.

Example. Let
$$\mathfrak{A} = \text{rectangle } (1\,2\,3\,4),$$
$$\mathfrak{B} = (5\,6\,7\,8),$$
$$\mathfrak{C} = (5\,8\,\alpha\,\beta) = Dv(\mathfrak{A}, \mathfrak{B}).$$

Then
$$U(\mathfrak{A}, (\mathfrak{B} - \mathfrak{C})), \tag{1}$$
and
$$(U(\mathfrak{A}, \mathfrak{B}) - \mathfrak{C}), \tag{2}$$
are different.

Thus if we write + for U, 1), 2) give
$$\mathfrak{A} + (\mathfrak{B} - \mathfrak{C}) \neq (\mathfrak{A} + \mathfrak{B}) - \mathfrak{C}.$$

18. 1. *Let* $\mathfrak{A}_1 \geq \mathfrak{A}_2 \geq \mathfrak{A}_3 \cdots$ *be a set of limited complete point aggregates. Then*
$$\mathfrak{B} = Dv(\mathfrak{A}_1, \mathfrak{A}_2 \cdots) > 0.$$
Moreover $\mathfrak{B}$ *is complete.*

Let a_n be a point of $\mathfrak{A}_n$, $n = 1, 2, \cdots$ and $\mathfrak{A} = a_1, a_2, a_3 \cdots$

Any limiting point α of $\mathfrak{A}$ is in every $\mathfrak{A}_n$. For it is a limiting point of
$$a_m, a_{m+1}, a_{m+2}, \cdots$$
But all these points lie in $\mathfrak{A}_m$, which is complete. Hence α lies in $\mathfrak{A}_m$, and therefore in every $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ Hence α lies in $\mathfrak{B}$, and $\mathfrak{B} > 0$.

$\mathfrak{B}$ *is complete.* For let β be one of its limiting points. Let
$$b_1, b_2, b_3, \cdots \doteq \beta.$$
As each b_m is in each $\mathfrak{A}_n$, and $\mathfrak{A}_n$ is complete, β is in $\mathfrak{A}_n$. Hence β is in $\mathfrak{B}$.

2. *Let* $\mathfrak{A}$ *be a limited point set of the second species. Then*
$$Dv(\mathfrak{A}', \mathfrak{A}'', \mathfrak{A}''', \cdots) > 0,$$
and is complete.

For $\mathfrak{A}^{(n)}$ is complete and > 0. Also $\mathfrak{A}^{(n)} \geq \mathfrak{A}^{(n+1)}$.

19. *Let* $\mathfrak{A}_1, \mathfrak{A}_2 \cdots$ *lie in* $\mathfrak{B}$; *let* $\mathfrak{A} = U\{\mathfrak{A}_n\}$. *Let* A_n *be the complement of* $\mathfrak{A}_n$ *with respect to* $\mathfrak{B}$, *so that* $A_n + \mathfrak{A}_n = \mathfrak{B}$. *Let* $A = Dv\{A_n\}$. *Then* A *and* $\mathfrak{A}$ *are complementary, so that* $A + \mathfrak{A} = \mathfrak{B}$.

For each point b of $\mathfrak{B}$ lies in some $\mathfrak{A}_n$, or it lies in no $\mathfrak{A}_n$, and hence in every A_n. In the first case b lies in $\mathfrak{A}$, in the second in A. Moreover it cannot lie in both A and $\mathfrak{A}$.

20. 1. Let
$$\mathfrak{A}_1 \leq \mathfrak{A}_2 \leq \mathfrak{A}_3 \cdots \tag{1}$$
be an infinite sequence of point sets whose union call $\mathfrak{A}$. This fact may be more briefly indicated by the notation
$$\mathfrak{A} = U(\mathfrak{A}_1 \leq \mathfrak{A}_2 \leq \mathfrak{A}_3 \cdots).$$

Obviously when $\mathfrak{A}$ is limited,
$$\overline{\mathfrak{A}} \geq \lim \overline{\mathfrak{A}}_n. \tag{2}$$

That the inequality may hold as well as the equality in 2) is shown by the following examples.

Example 1. Let $\mathfrak{A}_n =$ the segment $\left(\frac{1}{n}, 1\right)$.

Then
$$\mathfrak{A} = U\{\mathfrak{A}_n\} = (0^*, 1).$$
$$\overline{\mathfrak{A}} = 1. \qquad \overline{\mathfrak{A}}_n = \frac{n-1}{n} \doteq 1.$$

Example 2. Let $\mathfrak{a}_n$ denote the points in the unit interval whose abscissæ are given by
$$x = \frac{m}{n},\ m < n = 1, 2, 3, \cdots\ m, n \text{ relatively prime.}$$

Let
$$\mathfrak{A}_n = \mathfrak{a}_1 + \cdots + \mathfrak{a}_n.$$
Here
$$\mathfrak{A} = U\{\mathfrak{A}_n\}$$
is the totality of rational numbers in $(0^*, 1^*)$.

As
$$\overline{\mathfrak{A}} = 1 \text{ and } \overline{\mathfrak{A}}_n = 0, \text{ we see}$$
$$\overline{\mathfrak{A}} > \lim \overline{\mathfrak{A}}_n.$$

2. Let
$$\mathfrak{B}_1 \geq \mathfrak{B}_2 \geq \cdots \tag{3}$$
Let $\mathfrak{B}$ be their divisor. This we may denote briefly by
$$\mathfrak{B} = Dv(\mathfrak{B}_1 \geq \mathfrak{B}_2 \geq \cdots).$$

Obviously when $\mathfrak{B}_1$ is limited,
$$\overline{\mathfrak{B}} \leq \lim \overline{\mathfrak{B}}_n.$$

Example 1. Let $\mathfrak{B}_n$ = the segment $\left(0, \frac{1}{n}\right)$.

Then
$$\mathfrak{B} = Dv\{\mathfrak{B}_n\} = (0), \text{ the origin.}$$

Here
$$\overline{\mathfrak{B}} = 0. \qquad \lim \overline{\mathfrak{B}}_n = \lim \frac{1}{n} = 0,$$

and
$$\overline{\mathfrak{B}} = \lim \overline{\mathfrak{B}}_n.$$

Example 2. Let $\mathfrak{A}_n$ be as in 1, Example 2. Let $\mathfrak{b}_n = \mathfrak{A} - \mathfrak{A}_n$.

Let
$$\mathfrak{B}_n = (1, 2) + \mathfrak{b}_n.$$

Here
$$\mathfrak{B} = \text{the segment } (1, 2) \text{ and } \overline{\mathfrak{B}}_n = 2.$$

Hence
$$\overline{\mathfrak{B}} < \lim \overline{\mathfrak{B}}_n.$$

3. *Let* $\mathfrak{B}_1 < \mathfrak{B}_2 \leq \cdots$ *be unmixed parts of* $\mathfrak{A}$. *Let* $\overline{\mathfrak{B}}_n \doteq \overline{\mathfrak{A}}$. *Let* $\mathfrak{B} = U\{\mathfrak{B}_n\}$. *Then* $\mathfrak{C} = \mathfrak{A} - \mathfrak{B}$ *is discrete.*

For let $\mathfrak{A} = \mathfrak{B}_n + \mathfrak{C}_n$; then $\mathfrak{C}_n$ is an unmixed part of $\mathfrak{A}$. Hence
$$\overline{\mathfrak{A}} = \overline{\mathfrak{B}}_n + \overline{\mathfrak{C}}_n.$$

Passing to the limit $n = \infty$, this gives
$$\lim \overline{\mathfrak{C}}_n = 0.$$

Hence $\mathfrak{C}$ is discrete by 2.

4. We may obviously apply the terms monotone increasing, monotone decreasing sequences, etc. [Cf. I, 108, 211] to sequences of the type 1), 3).

21. *Let* $\mathfrak{C} = \mathfrak{A} + \mathfrak{B}$. *If* $\mathfrak{A}$, $\mathfrak{B}$ *are complete,*
$$\overline{\mathfrak{C}} = \overline{\mathfrak{A}} + \overline{\mathfrak{B}}. \tag{1}$$

For
$$\delta = \text{Dist}(\mathfrak{A}, \mathfrak{B}) > 0,$$

since $\mathfrak{A}$, $\mathfrak{B}$ are complete and have no point in common. Let D be a cubical division of space of norm d. If d is taken sufficiently small $\mathfrak{A}_D$, $\mathfrak{B}_D$ have no cells in common. Hence
$$\overline{\mathfrak{C}}_D = \overline{\mathfrak{A}}_D + \overline{\mathfrak{B}}_D.$$

Letting $d \doteq 0$ we get 1).

22. 1. *If* $\mathfrak{A}$, $\mathfrak{B}$ *are complete, so are also*

$$\mathfrak{C} = (\mathfrak{A}, \mathfrak{B}), \qquad \mathfrak{D} = Dv(\mathfrak{A}, \mathfrak{B}).$$

Let us first show that $\mathfrak{C}$ is complete. Let c be a limiting point of $\mathfrak{C}$. Let $c_1, c_2, \cdots$ be points of $\mathfrak{C}$ which $\doteq c$. Let us separate the c_n into two classes, according as they belong to $\mathfrak{A}$, or do not. One of these classes must embrace an infinite number of points which $\doteq c$. As both $\mathfrak{A}$ and $\mathfrak{B}$ are complete, c lies in either $\mathfrak{A}$ or $\mathfrak{B}$. Hence it lies in $\mathfrak{C}$.

To show that $\mathfrak{D}$ *is complete.* Let $d_1, d_2, \cdots$ be points of $\mathfrak{D}$ which $\doteq d$. As each d_n is in both $\mathfrak{A}$ and $\mathfrak{B}$, their limiting point d is in $\mathfrak{A}$ and $\mathfrak{B}$, since these are complete. Hence d is in $\mathfrak{D}$.

2. *If* $\mathfrak{A}$, $\mathfrak{B}$ *are metric so are*

$$\mathfrak{C} = (\mathfrak{A}, \mathfrak{B}) \qquad \mathfrak{D} = Dv(\mathfrak{A}, \mathfrak{B}).$$

For the points of Front $\mathfrak{C}$ lie either in Front $\mathfrak{A}$ or in Front $\mathfrak{B}$, while the points of Front $\mathfrak{D} \leq$ Front $\mathfrak{A}$ and also $\leq$ Front $\mathfrak{B}$. But Front $\mathfrak{A}$ and Front $\mathfrak{B}$ are discrete since $\mathfrak{A}$, $\mathfrak{B}$ are metric.

23. *Let the complete set* $\mathfrak{A}$ *have a complete part* $\mathfrak{B}$. *Then however small* $\epsilon > 0$ *is taken, there exists a complete set* $\mathfrak{C}$ *in* $\mathfrak{A}$, *having no point in common with* $\mathfrak{B}$ *such that*

$$\overline{\mathfrak{C}} > \overline{\mathfrak{A}} - \overline{\mathfrak{B}} - \epsilon. \tag{1}$$

Moreover there exists no complete set $\mathfrak{C}$, *having no point in common with* $\mathfrak{B}$ *such that*

$$\overline{\mathfrak{C}} > \overline{\mathfrak{A}} - \overline{\mathfrak{B}}.$$

The second part of the theorem follows from 21. To prove 1) let D be a cubical division such that

$$\overline{\mathfrak{A}}_D = \overline{\mathfrak{A}} + \epsilon', \qquad \overline{\mathfrak{B}}_D = \overline{\mathfrak{B}} + \epsilon'', \qquad 0 \leq \epsilon', \epsilon'' < \epsilon. \tag{2}$$

Since $\mathfrak{B}$ is complete, no point of $\mathfrak{B}$ lies on the frontier of $\overline{\mathfrak{B}}_D$. Let $\mathfrak{C}$ denote the points of $\mathfrak{A}$ lying in cells containing no point of $\mathfrak{B}$. Since $\mathfrak{A}$ is complete so is $\mathfrak{C}$, and $\mathfrak{B}$, $\mathfrak{C}$ have no point in common.

Thus

$$\overline{\mathfrak{A}}_D = \overline{\mathfrak{B}}_D + \overline{\mathfrak{C}}_D. \tag{3}$$

But the cells of $\overline{\mathfrak{C}}_D$ may be subdivided, forming a new division Δ, which does not change the cells of $\overline{\mathfrak{B}}_D$, so that $\overline{\mathfrak{B}}_D = \overline{\mathfrak{B}}_\Delta$, but so that

$$\overline{\mathfrak{C}}_\Delta = \overline{\mathfrak{C}} + \epsilon''', \qquad 0 \leq \epsilon''' < \epsilon. \tag{4}$$

Thus 2), 3), 4) give

$$\overline{\mathfrak{A}} + \epsilon' = \overline{\mathfrak{B}} + \epsilon'' + \overline{\mathfrak{C}} + \epsilon''',$$

or

$$\overline{\mathfrak{C}} = \overline{\mathfrak{A}} - \overline{\mathfrak{B}} - (\epsilon'' + \epsilon''' - \epsilon'),$$

$$> \overline{\mathfrak{A}} - \overline{\mathfrak{B}} - \epsilon.$$

24. *Let* $\mathfrak{A}$, $\mathfrak{B}$ *be complete. Let*

$$\mathfrak{U} = (\mathfrak{A}, \mathfrak{B}), \qquad \mathfrak{D} = Dv(\mathfrak{A}, \mathfrak{B}).$$

Then

$$\overline{\mathfrak{A}} + \overline{\mathfrak{B}} = \overline{\mathfrak{U}} + \overline{\mathfrak{D}}. \tag{1}$$

For let

$$\mathfrak{U} = \mathfrak{A} + A.$$

Then A contains complete sets C, such that

$$\overline{C} > \overline{\mathfrak{U}} - \overline{\mathfrak{A}} - \epsilon, \tag{2}$$

but no complete set such that

$$\overline{C} > \overline{\mathfrak{U}} - \overline{\mathfrak{A}}, \tag{3}$$

by 23. On the other hand,

$$\mathfrak{B} = A + \mathfrak{D}.$$

Hence A contains complete sets C, such that

$$\overline{C} > \overline{\mathfrak{B}} - \overline{\mathfrak{D}} - \epsilon, \tag{4}$$

but no complete set such that

$$\overline{C} > \overline{\mathfrak{B}} - \overline{\mathfrak{D}}. \tag{5}$$

From 2), 3), and 4), 5) we have 1), since ϵ is arbitrarily small.

25. *Let*

$$\mathfrak{D} = Dv(\mathfrak{A}_1 \geq \mathfrak{A}_2 \geq \mathfrak{A}_3 \geq \cdots),$$

each $\mathfrak{A}_n$ *being complete and such that* $\overline{\mathfrak{A}}_n \geq$ *some constant* k.

Then

$$\overline{\mathfrak{D}} \geq k.$$

For suppose $$l = k - \overline{\mathfrak{D}} > 0.$$

Let $$l = \epsilon + \eta; \qquad \epsilon, \eta > 0.$$

Then by 23 there exists in $\mathfrak{A}_1$ a complete set $\mathfrak{C}_1$, having no point in common with $\mathfrak{D}$ such that

$$\overline{\mathfrak{C}}_1 > \overline{\mathfrak{A}}_1 - \overline{\mathfrak{D}} - \epsilon;$$

or as $\overline{\mathfrak{A}}_1 \geq k$, such that $$\overline{\mathfrak{C}}_1 \geq \eta.$$

Let $$\mathfrak{C}_2 = Dv(\mathfrak{A}_2, \mathfrak{C}_1), \qquad \mathfrak{U} = (\mathfrak{A}_2, \mathfrak{C}_1).$$

Then by 24, $$\overline{\mathfrak{A}}_2 + \overline{\mathfrak{C}}_1 = \overline{\mathfrak{U}} + \overline{\mathfrak{C}}_2.$$

Thus
$$\begin{aligned} \overline{\mathfrak{C}}_2 &= \overline{\mathfrak{A}}_2 + \overline{\mathfrak{C}}_1 - \overline{\mathfrak{U}} \\ &\geq \overline{\mathfrak{A}}_2 + \overline{\mathfrak{C}}_1 - \overline{\mathfrak{A}}_1 \\ &\geq \overline{\mathfrak{A}}_2 + (\overline{\mathfrak{A}}_1 - \overline{\mathfrak{D}} - \epsilon) - \overline{\mathfrak{A}}_1 = \overline{\mathfrak{A}}_2 - \overline{\mathfrak{D}} - \epsilon \\ &\geq k - \overline{\mathfrak{D}} - \epsilon \\ &\geq \eta. \end{aligned}$$

Thus $\mathfrak{A}_2$ contains the non-vanishing complete set $\mathfrak{C}_2$ having no point in common with $\mathfrak{D}$. In this way we may continue. Thus $\mathfrak{A}_1, \mathfrak{A}_2, \cdots$ contain a non-vanishing complete component not in $\mathfrak{D}$, which is absurd.

Corollary. Let $\mathfrak{A} = (\mathfrak{A}_1 < \mathfrak{A}_2 < \cdots)$ *be complete. Then* $\overline{\mathfrak{A}}_n \doteq \overline{\mathfrak{A}}$.

This follows easily from 23, 25.

CHAPTER II

IMPROPER MULTIPLE INTEGRALS

26. Up to the present we have considered only proper multiple integrals. We take up now the case when the integrand $f(x_1 \cdots x_m)$ is not limited. Such integrals are called improper. When $m = 1$, we get the integrals treated in Vol. I, Chapter 14. An important application of the theory we are now to develop is the inversion of the order of integration in iterated improper integrals. The treatment of this question given in Vol. I may be simplified and generalized by making use of the properties of improper multiple integrals.

27. Let $\mathfrak{A}$ be a limited point set in m-way space $\mathfrak{R}_m$. At each point of $\mathfrak{A}$ let $f(x_1 \cdots x_m)$ have a definite value assigned to it. The points of infinite discontinuity of f which lie in $\mathfrak{A}$ we shall denote by $\mathfrak{J}$. In general $\mathfrak{J}$ is discrete, and this case is by far the most important. But it is not necessary. We shall call $\mathfrak{J}$ the *singular points*.

Example. Let $\mathfrak{A}$ be the unit square. At the point $x = \frac{m}{n}$, $y = \frac{r}{s}$, these fractions being irreducible, let $f = ns$. At the other points of $\mathfrak{A}$ let $f = 1$. Here every point of $\mathfrak{A}$ is a point of infinite discontinuity and hence $\mathfrak{J} = \mathfrak{A}$.

Several types of definition of improper integrals have been proposed. We shall mention only three.

28. *Type I.* Let us effect a division Δ of norm δ of $\mathfrak{R}_m$ into cells, such that each cell is complete. Such divisions may be called *complete*. Let $\mathfrak{A}_\delta$ denote the cells containing points of $\mathfrak{A}$, but no point of $\mathfrak{J}$, while $\mathfrak{A}'_\delta$ may denote the cells containing a point of $\mathfrak{J}$. Since Δ is complete, f is limited in $\mathfrak{A}_\delta$. Hence f admits an upper and a lower proper integral in $\mathfrak{A}_\delta$. The limits, when they exist,

$$\lim_{\delta=0} \underline{\int}_{\mathfrak{A}_\delta} f, \qquad \lim_{\delta=0} \overline{\int}_{\mathfrak{A}_\delta} f, \tag{1}$$

for all possible complete divisions Δ of norm δ, are called the *lower* and *upper integrals* of f in $\mathfrak{A}$, and are denoted by

$$\underline{\int}_{\mathfrak{A}} fd\mathfrak{A}, \qquad \overline{\int}_{\mathfrak{A}} fd\mathfrak{A}, \tag{2}$$

or more shortly by

$$\underline{\int}_{\mathfrak{A}} f, \qquad \overline{\int}_{\mathfrak{A}} f.$$

When the limits 1) are finite, the corresponding integrals 2) are *convergent.* We also say *f admits a lower or an upper improper integral in* $\mathfrak{A}$. When the two integrals 2) are equal, we say that f is integrable in $\mathfrak{A}$ and denote their common value by

$$\int_{\mathfrak{A}} fd\mathfrak{A} \qquad \text{or by} \qquad \int_{\mathfrak{A}} f. \tag{3}$$

We call 3) the *improper integral of f in* $\mathfrak{A}$; we also say that *f admits an improper integral in* $\mathfrak{A}$ and that the integral 3) is *convergent.*

The definition of an improper integral just given is an extension of that given in Vol. I, Chapter 14. It is the natural development of the idea of an improper integral which goes back to the beginnings of the calculus.

It is convenient to speak of the *symbols* 2) as upper and lower integrals, even when the limits 1) do not exist. A similar remark applies to the symbol 3).

Let us replace f by $|f|$ in one of the symbols 2), 3). The resulting symbol is called the *adjoint* of the integral in question. We write

$$\overline{\int}_{\mathfrak{A}} |f| = \text{Adj} \overline{\int}_{\mathfrak{A}} f, \text{ etc.} \tag{4}$$

When the adjoint of one of the integrals 2), 3) is convergent, the first integral is said to be *absolutely convergent.* Thus if 4) is convergent, the second integral in 2) is absolutely convergent, etc.

29. *Type II.* Let $\lambda, \mu \geq 0$. We introduce a *truncated* function $f_{\lambda\mu}$ defined as follows:

$$\begin{aligned} f_{\lambda\mu} &= f(x_1 \cdots x_m) && \text{when } -\lambda \leq f \leq \mu \\ &= -\lambda && \text{when } f < -\lambda \\ &= \mu && \text{when } f > \mu. \end{aligned}$$

We define now the lower integral as

$$\underline{\int}_{\mathfrak{A}} f = \lim_{\lambda, \mu = \infty} \underline{\int}_{\mathfrak{A}} f_{\lambda\mu}.$$

A similar definition holds for the upper integral. The other terms introduced in 28 apply here without change.

This definition of an improper integral is due to *de la Vallée Poussin*. It has been employed by him and *R. G. D. Richardson* with great success.

30. *Type III.* Let $\alpha, \beta \geq 0$. Let $\mathfrak{A}_{\alpha\beta}$ denote the points of $\mathfrak{A}$ at which

$$-\alpha \leq f(x_1 \cdots x_m) \leq \beta.$$

We define now

$$\underline{\int}_{\mathfrak{A}} f = \lim_{\alpha, \beta = \infty} \underline{\int}_{\mathfrak{A}_{\alpha\beta}} f \quad ; \quad \overline{\int}_{\mathfrak{A}} f = \lim_{\alpha, \beta = \infty} \overline{\int}_{\mathfrak{A}_{\alpha\beta}} f. \tag{1}$$

The other terms introduced in 28 apply here without change. This type of definition originated with the author and has been developed in his lectures.

31. When the points of infinite discontinuity $\mathfrak{J}$ are discrete and the upper integrals are absolutely convergent, all three definitions lead to the same result, as we shall show.

When this condition is not satisfied, the results may be quite different.

Example. Let $\mathfrak{A}$ be the unit square. Let $\mathfrak{A}_1, \mathfrak{A}_2$ denote respectively the upper and lower halves. At the rational* points $\mathfrak{B}$, $x = \frac{m}{n}$, $y = \frac{r}{s}$, in $\mathfrak{A}_1$, let $f = ns$. At the other points $\mathfrak{C}$ of $\mathfrak{A}_1$, let $f = -2$. In $\mathfrak{A}_2$ let $f = 0$.

1° *Definition.* Here $\mathfrak{J} = \mathfrak{A}_1$.

Hence

$$\overline{\int}_{\underline{\mathfrak{A}}} f = 0.$$

2° *Definition.* Here

$$\underline{\int}_{\mathfrak{A}} f = -1, \qquad \overline{\int}_{\mathfrak{A}} f = +\infty.$$

* Here as in all following examples of this sort, fractions are supposed to be irreducible.

3° *Definition.* Here $\mathfrak{A}_{\alpha\beta}$ embraces all the points of $\mathfrak{A}_2$, $\mathfrak{C}$ and a finite number of points of $\mathfrak{B}$ for $\alpha \geq 2$, β arbitrarily large. Hence

$$\underline{\int_{\mathfrak{A}}} f = -1, \qquad \overline{\int_{\mathfrak{A}}} f = -1,$$

and thus

$$\int_{\mathfrak{A}} f = -1.$$

32. In the following we shall adopt the third type of definition, as it seems to lead to more general results when treating the important subject of inversion of the order of integration in iterated integrals.

We note that if f is limited in $\mathfrak{A}$,

$$\lim_{\alpha, \beta = \infty} \overline{\underline{\int_{\mathfrak{A}}}} f = \text{the proper integral } \overline{\underline{\int_{\mathfrak{A}}}} f.$$

For α, β being sufficiently large, $\mathfrak{A}_{\alpha\beta} = \mathfrak{A}$.

Also, if $\mathfrak{A}$ is discrete,

$$\underline{\int_{\mathfrak{A}}} f = \int_{\mathfrak{A}} f = 0.$$

For $\mathfrak{A}_{\alpha\beta}$ is discrete, and hence

$$\overline{\underline{\int_{\mathfrak{A}_{\alpha\beta}}}} f = 0.$$

Hence the limit of these integrals is 0.

33. *Let* $\qquad m = |\operatorname{Min} f| \quad , \quad M = |\operatorname{Max} f| \quad$ in $\mathfrak{A}$.

Then

$$\lim_{\alpha, \beta = \infty} \overline{\underline{\int_{\mathfrak{A}_{\alpha\beta}}}} f = \lim_{\beta = \infty} \overline{\underline{\int_{\mathfrak{A}_{m, \beta}}}} f, \qquad m \text{ finite.}$$

$$\lim_{\alpha, \beta = \infty} \overline{\underline{\int_{\mathfrak{A}_{\alpha\beta}}}} f = \lim_{\alpha = \infty} \overline{\underline{\int_{\mathfrak{A}_{\alpha, M}}}} f, \qquad M \text{ finite.}$$

For these limits depend only on large values of α, β, and when m is finite.

$$\mathfrak{A}_{m, \beta} = \mathfrak{A}_{\alpha, \beta} \quad , \quad \text{for all } \alpha \geq m.$$

Similarly, when M is finite

$$\mathfrak{A}_{\alpha, \beta} = \mathfrak{A}_{\alpha, M} \quad , \quad \text{for all } \beta \geq M.$$

Thus in these cases we may simplify our notation by replacing

$$\mathfrak{A}_{\alpha, M} \quad , \quad \mathfrak{A}_{m\beta}$$

by

$$\mathfrak{A}_{-\alpha} \quad , \quad \mathfrak{A}_{\beta} \quad ,$$

respectively.

2. Thus we have:

$$\overline{\int_{\mathfrak{A}}} f = \lim_{\beta=\infty} \overline{\int_{\mathfrak{A}_\beta}} f, \qquad \textit{when } \mathrm{Min} f \textit{ is finite.}$$

$$\underline{\int_{\mathfrak{A}}} f = \lim_{\alpha=\infty} \underline{\int_{\mathfrak{A}_{-\alpha}}} f\ , \qquad \textit{when } \mathrm{Max} f \textit{ is finite.}$$

3. Sometimes we have to deal with several functions $f, g, \cdots$ In this case the notation $\mathfrak{A}_{\alpha\beta}$ is ambiguous. To make it clear we let $\mathfrak{A}_{f, \alpha, \beta}$ denote the points of $\mathfrak{A}$ where

$$-\alpha \leq f \leq \beta.$$

Similarly, $\mathfrak{A}_{g, \alpha, \beta}$ denotes the points where

$$-\alpha \leq g \leq \beta, \text{ etc.}$$

34. $\underline{\int_{\mathfrak{A}_{\alpha\beta}}} f$ *is a monotone decreasing function of* α *for each* β.

$\overline{\int_{\mathfrak{A}_{\alpha\beta}}} f$ *is a monotone increasing function of* β *for each* α.

If Max f *is finite*

$$\overline{\underline{\int_{\mathfrak{A}_{-\alpha}}}} f \quad \textit{are monotone decreasing functions of } \alpha.$$

If Min f *is finite*

$$\overline{\underline{\int_{\mathfrak{A}_{\beta}}}} f \quad \textit{are monotone increasing functions of } \beta.$$

Let us prove the first statement. Let $\alpha' > \alpha$.
Let D be a cubical division of space of norm d.
Then β being fixed,

$$\underline{\int_{\mathfrak{A}_{\alpha\beta}}} f = \lim_{d=0} \sum_{\mathfrak{A}_{\alpha\beta}} m_\iota d_\iota, \tag{1}$$

$$\underline{\int_{\mathfrak{A}_{\alpha'\beta}}} f = \lim_{d=0} \sum_{\mathfrak{A}_{\alpha'\beta}} m'_\iota d'_\iota, \tag{2}$$

using the notation so often employed before.

But each cell d_ι of $\mathfrak{A}_{\alpha\beta}$ lies among the cells d_ι' of $\mathfrak{A}_{\alpha'\beta}$. Thus we can break up the sum 2), getting

$$\sum_{\mathfrak{A}_{\alpha'\beta}} m_\iota' d_\iota' = \sum_{\mathfrak{A}_{\alpha\beta}} m_\iota' d_\iota' + \Sigma m_\iota'' d_\iota''.$$

Here the second term on the right is summed over those cells not containing points of $\mathfrak{A}_{\alpha\beta}$. It is thus ≤ 0. In the first term on the right $m_\iota' \leq m_\iota$. It is thus less than the sum in 1). Hence

$$\sum_{\mathfrak{A}_{\alpha'\beta}} m_\iota' d_\iota' \leq \sum_{\mathfrak{A}_{\alpha\beta}} m_\iota d_\iota.$$

Thus

$$\underline{\int}_{\mathfrak{A}_{\alpha'\beta}} \leq \underline{\int}_{\mathfrak{A}_{\alpha\beta}}, \qquad \alpha' > \alpha.$$

In a similar manner we may prove the second statement; let us turn to the third.

We need only to show that

$$\overline{\int}_{\mathfrak{A}_{-\alpha}} f \text{ is monotone decreasing.}$$

Let $\alpha' > \alpha$. Then

$$\overline{\int}_{\mathfrak{A}_{-\alpha}} = \lim_{d=0} \sum_{\mathfrak{A}_{-\alpha}} M_\iota d_\iota. \tag{3}$$

$$\overline{\int}_{\mathfrak{A}_{-\alpha'}} = \lim_{d=0} \sum_{\mathfrak{A}_{-\alpha'}} M_\iota' d_\iota'. \tag{4}$$

As before

$$\sum_{\mathfrak{A}_{-\alpha'}} M_\iota' d_\iota' = \sum_{\mathfrak{A}_{-\alpha}} M_\iota' d_\iota' + \Sigma M_\iota'' d_\iota''. \tag{5}$$

But in the cells d_ι, $M_\iota' = M_\iota$. Hence the first term of 5) is the same as Σ in 3). The second term of 5) is ≤ 0. The proof follows now as before.

35. *If* Max f *is finite and* $\underline{\int}_{\mathfrak{A}_{-\alpha}} f$ *are limited,* $\underline{\int}_{\mathfrak{A}} f$ *is convergent and*

$$\underline{\int}_{\mathfrak{A}} f \leq \underline{\int}_{\mathfrak{A}_{-\alpha}} f.$$

If Min f *is finite and* $\overline{\int}_{\mathfrak{A}_\beta}$ *are limited,* $\overline{\int}_{\mathfrak{A}} f$ *is convergent and*

$$\overline{\int}_{\mathfrak{A}_\beta} f \leq \overline{\int}_{\mathfrak{A}} f.$$

For by 34

$$\underline{\int}_{\mathfrak{A}_{-\alpha}} f \quad , \quad \overline{\int}_{\mathfrak{A}_\beta} f$$

are limited monotone functions. Their limits exist by I, 277, 8.

36. *If* $M = \mathrm{Max}\, f$ *is finite, and* $\underline{\int}_{\mathfrak{A}} f$ *is convergent, the corresponding upper integral is convergent and*

$$\underline{\int}_{\mathfrak{A}} f \le \overline{\int}_{\mathfrak{A}} f \le M \lim_{\alpha=\infty} \overline{\mathfrak{A}}_{-\alpha},$$

where $f \ge -\alpha$ *in* $\mathfrak{A}_{-\alpha}$.

Similarly, if $m = \mathrm{Min}\, f$ *is finite and* $\overline{\int}_{\mathfrak{A}} f$ *is convergent, the corresponding lower integral is convergent and*

$$m \lim_{\beta=\infty} \overline{\mathfrak{A}}_\beta \le \underline{\int}_{\mathfrak{A}} f \le \overline{\int}_{\mathfrak{A}} f \quad , \quad f \le \beta \text{ in } \mathfrak{A}_\beta.$$

Let us prove the first half of the theorem.

We have

$$\overline{\int}_{\mathfrak{A}} f = \lim_{\alpha=\infty} \overline{\int}_{\mathfrak{A}_{-\alpha}} \cdot$$

Now

$$\underline{\int}_{\mathfrak{A}} f \le \underline{\int}_{\mathfrak{A}_{-\alpha}} \le \overline{\int}_{\mathfrak{A}_{-\alpha}} \le M \overline{\mathfrak{A}}_{-\alpha}.$$

We have now only to pass to the limit.

37. *If* $\underline{\int}_{\mathfrak{A}} f$ *is convergent, and* $\mathfrak{B} < \mathfrak{A}$,

$$\underline{\int}_{\mathfrak{B}} f$$

does not need to converge. Similarly

$$\overline{\int}_{\mathfrak{B}} f$$

does not need to converge, although $\overline{\int}_{\mathfrak{A}} f$ *does.*

Example. Let $\mathfrak{A}$ be the unit square; let $\mathfrak{B}$ denote the points for which x is rational.

Let

$$f = 1 \quad \text{when } x \text{ is irrational}$$

$$= \frac{1}{y} \quad \text{when } x \text{ is rational.}$$

Then

$$\underline{\int}_{\mathfrak{A}_\beta} f = 1 \quad ; \quad \text{hence } \underline{\int}_{\mathfrak{A}} f = 1.$$

On the other hand,

$$\underline{\int}_{\mathfrak{B}_\beta} = \int_0^1 dx \int_{\frac{1}{\beta}}^1 \frac{dy}{y} = \log \beta.$$

Hence

$$\underline{\int}_{\mathfrak{B}} = \lim_{\beta=\infty} \underline{\int}_{\mathfrak{B}\beta} = \lim \log \beta = +\infty$$

is divergent.

38. 1. In the future it will be convenient to let $\mathfrak{P}$ denote the points of $\mathfrak{A}$ where $f \geq 0$, and $\mathfrak{N}$ the points where $f \leq 0$. We may call them the *positive* and *negative components of* $\mathfrak{A}$.

2. *If* $\overline{\int}_{\mathfrak{A}} f$ *converges, so do* $\underline{\overline{\int}}_{\mathfrak{P}} f$.

If $\underline{\int}_{\mathfrak{A}} f$ *converges, so do* $\underline{\overline{\int}}_{\mathfrak{N}} f$.

For let us effect a cubical division of space of norm d. Let $\beta' > \beta$. Let e denote those cells containing a point of $\mathfrak{P}_\beta$; e' those cells containing a point of $\mathfrak{P}_{\beta'}$ but no point of $\mathfrak{P}_\beta$; δ those cells containing a point of $\mathfrak{A}_{\alpha\beta}$ but none of $\mathfrak{P}_{\beta'}$.

Then

$$\overline{\int}_{\mathfrak{A}_{\alpha\beta'}} = \lim_{d=0} \{\Sigma M_e' \cdot e + \Sigma M_{e'}' \cdot e' + \Sigma M' \cdot \delta\}.$$

$$\overline{\int}_{\mathfrak{A}_{\alpha\beta}} = \lim_{d=0} \{\Sigma M_e \cdot e + \Sigma M_{e'} \cdot e' + \Sigma M_\delta \cdot \delta\}.$$

Obviously

$$M_e' \geq M_e \quad , \quad M_\delta' = M_\delta \quad , \quad M_{e'} < 0.$$

Hence

$$\overline{\int}_{\mathfrak{A}_{\alpha\beta'}} - \overline{\int}_{\mathfrak{A}_{\alpha\beta}} = \lim_{d=0} \{\Sigma(M_e' - M_e)e + \Sigma M_{e'}' e' - \Sigma M_{e'} \cdot e'\}.$$

We find similarly

$$\overline{\int}_{\mathfrak{P}_{\beta'}} - \overline{\int}_{\mathfrak{P}_\beta} = \lim_{d=0}\{\Sigma(M'_e - M_e)e + \Sigma M'_{e'}e'\}. \qquad (1$$

Now

$$\left|\overline{\int}_{\mathfrak{A}_{\alpha\beta'}} - \overline{\int}_{\mathfrak{A}_{\alpha\beta}}\right| < \epsilon$$

for a sufficiently large α, and for any $\beta, \beta' > \beta_0$.
Hence the same is true of the left side of 1).

As corollaries we have:

3. *If the upper integral of f is convergent in $\mathfrak{A}$, then*

$$\overline{\int}_P f \leq \overline{\int}_{\mathfrak{P}} f \qquad P < \mathfrak{P}.$$

If the lower integral of f is convergent in $\mathfrak{A}$,

$$\underline{\int}_N f \geq \underline{\int}_{\mathfrak{N}} f \qquad N < \mathfrak{N}.$$

For

$$\overline{\int}_{P_\beta} \leq \overline{\int}_{\mathfrak{P}_\beta} \leq \overline{\int}_{\mathfrak{P}} \qquad \text{etc.}$$

4. *If $f \geq 0$ and $\overline{\int}_{\mathfrak{A}} f$ is convergent, so is*

$$\overline{\int}_{\mathfrak{B}} f \quad , \quad \mathfrak{B} < \mathfrak{A}.$$

Moreover the second integral is $\leq$ the first.

This follows at once from 3, as $\mathfrak{A} = \mathfrak{P}$.

39. *If $\overline{\int}_{\mathfrak{P}} f$ and $\underline{\int}_{\mathfrak{N}} f$ converge, so do $\overline{\underline{\int}}_{\mathfrak{A}} f$.*

We show that $\overline{\int}_{\mathfrak{A}} f$ converges; a similar proof holds for $\underline{\int}_{\mathfrak{A}}$. To this end we have only to show that

$$\epsilon > 0\,;\ \alpha, \beta > 0\,;\ \left|\overline{\int}_{\mathfrak{A}_{\alpha'\beta'}} - \overline{\int}_{\mathfrak{A}_{\alpha''\beta''}}\right| < \epsilon\,;\ \alpha < \alpha' < \alpha'',\ \beta < \beta' \leq \beta''. \qquad (1$$

Let D be a cubical division of space of norm d. Let $\mathfrak{P}_{\beta'}$, $\mathfrak{P}_{\beta''}$ denote cells containing at least one point of $\mathfrak{A}_{\alpha'\beta'}$, $\mathfrak{A}_{\alpha''\beta''}$ at which $f \geq 0$. Let $\mathfrak{n}_{\alpha'}$, $\mathfrak{n}_{\alpha''}$ denote cells containing only points of $\mathfrak{A}_{\alpha'\beta'}$, $\mathfrak{A}_{\alpha''\beta''}$ at which $f < 0$. We have

$$\sum_{\mathfrak{A}_{\alpha'\beta'}} M_\iota d_\iota = \sum_{\mathfrak{P}_{\beta'}} + \sum_{\mathfrak{n}_{\alpha'}} \quad ; \quad \sum_{\mathfrak{A}_{\alpha''\beta''}} M_\iota d_\iota = \sum_{\mathfrak{P}_{\beta''}} + \sum_{\mathfrak{n}_{\alpha''}}$$

Subtracting,

$$\Big| \sum_{\mathfrak{A}_{\alpha'\beta'}} M_\iota d_\iota - \sum_{\mathfrak{A}_{\alpha''\beta''}} M_\iota d_\iota \Big| \leq \Big| \sum_{\mathfrak{P}_{\beta'}} M_\iota d_\iota - \sum_{\mathfrak{P}_{\beta''}} M_\iota d_\iota \Big| + \Big| \sum_{\mathfrak{n}_{\alpha'}} M_\iota d_\iota - \sum_{\mathfrak{n}_{\alpha''}} M_\iota d_\iota \Big|. \tag{2}$$

Let $M'_\iota = \operatorname{Max} f$ for points of $\mathfrak{N}$ in d_ι. Then since f has one sign in $\mathfrak{N}$,

$$\Big| \sum_{\mathfrak{n}_{\alpha'}} M_\iota d_\iota - \sum_{\mathfrak{n}_{\alpha''}} M_\iota d_\iota \Big| \leq \Big| \sum_{\mathfrak{N}_{\alpha'}} M'_\iota d_\iota - \sum_{\mathfrak{N}_{\alpha''}} M'_\iota d_\iota \Big|. \tag{3}$$

Letting $d \doteq 0$, 2) and 3) give

$$\left| \overline{\int}_{\mathfrak{A}_{\alpha'\beta'}} - \overline{\int}_{\mathfrak{A}_{\alpha''\beta''}} \right| \leq \left| \overline{\int}_{\mathfrak{P}_{\beta'}} - \overline{\int}_{\mathfrak{P}_{\beta''}} \right| + \left| \overline{\int}_{\mathfrak{N}_{\alpha'}} - \overline{\int}_{\mathfrak{N}_{\alpha''}} \right|. \tag{4}$$

Now if β is taken sufficiently large, the first term on the right is $< \epsilon/2$. On the other hand, since $\underline{\int}_{\mathfrak{N}} f$ is convergent, so is $\overline{\int}_{\mathfrak{N}} f$ by 36. Hence for α sufficiently large, the last term on the right is $< \epsilon/2$. Thus 4) gives 1).

40. *If f is integrable in $\mathfrak{A}$, it is in any $\mathfrak{B} < \mathfrak{A}$.*

Let us first show it is integrable in any $\mathfrak{A}_{\alpha\beta}$.

Let
$$A_{\alpha\beta} = \overline{\int}_{\mathfrak{A}_{\alpha\beta}} - \underline{\int}_{\mathfrak{A}_{\alpha\beta}}.$$

Let D be a cubical division of space of norm d.

Then
$$A_{\alpha\beta} = \lim_{d=0} \sum_{\mathfrak{A}_{\alpha\beta}} \omega_\iota d_\iota \quad , \quad \omega_\iota = \operatorname{Osc} f \text{ in } d_\iota.$$

Let $\alpha' \geq \alpha$, $\beta' \geq \beta$. Then

$$A_{\alpha'\beta'} - A_{\alpha\beta} = \lim_{d=0} \Big\{ \sum_{\mathfrak{A}_{\alpha'\beta'}} \omega'_\iota d'_\iota - \sum_{\mathfrak{A}_{\alpha\beta}} \omega_\iota d_\iota \Big\}.$$

Now any cell d_ι of $\mathfrak{A}_{\alpha\beta}$ is a cell of $\mathfrak{A}_{\alpha'\beta'}$, and in d_ι, $\omega'_\iota \geq \omega_\iota$. Hence $A_{\alpha'\beta'} \geq A_{\alpha\beta}$. Thus $A_{\alpha\beta}$ is a monotone increasing function of α, β. On the other hand

$$\lim A_{\alpha\beta} = 0,$$

by hypothesis. Hence $A_{\alpha\beta} = 0$ and thus f is integrable in $\mathfrak{A}_{\alpha\beta}$.

Next let f be limited in $\mathfrak{B}$, then $|f| <$ some γ in $\mathfrak{B}$. Then $\mathfrak{B} \leq \mathfrak{A}_{\gamma,\gamma}$. But f being integrable in $\mathfrak{A}_{\gamma,\gamma}$, it is in $\mathfrak{B}$ by I, 700, 3.

Let us now consider the general case. Since f is integrable in $\mathfrak{A}$

$$\overline{\int_{\mathfrak{P}}} f \quad , \quad \underline{\int_{\mathfrak{N}}} f,$$

both converge by 38. Let now P, N be the points of $\mathfrak{P}$, $\mathfrak{N}$ lying in $\mathfrak{B}$. Then

$$\overline{\int_P} f \leq \overline{\int_{\mathfrak{P}}} f \quad , \quad \left|\underline{\int_N} f\right| \leq \left|\underline{\int_{\mathfrak{N}}} f\right|.$$

Thus

$$\overline{\int_P} f \quad , \quad \underline{\int_N} f$$

both converge. Hence by 39,

$$\overline{\underline{\int_{\mathfrak{B}}}} f$$

both converge. But if $\mathfrak{B}_{a,b}$ denote the points of $\mathfrak{B}$ at which $-a \leq f \leq b$,

$$\overline{\underline{\int_{\mathfrak{B}}}} f = \lim_{a,\, b=\infty} \overline{\underline{\int_{\mathfrak{B}_{ab}}}} f,$$

by definition.

But as just seen,

$$\underline{\int_{\mathfrak{B}_{ab}}} = \overline{\int_{\mathfrak{B}_{ab}}}.$$

Hence

$$\underline{\int_{\mathfrak{B}}} f = \overline{\int_{\mathfrak{B}}} f,$$

and f is integrable in $\mathfrak{B}$.

41. As a corollary of 40 we have:

1. *If f is integrable in $\mathfrak{A}$, it admits a proper integral in any part of $\mathfrak{A}$ in which f is limited.*

2. *If f is integrable in any part of $\mathfrak{A}$ in which f is limited, and if either the lower or upper integral of f in $\mathfrak{A}$ is convergent, f is integrable in $\mathfrak{A}$.*

For let

$$\overline{\int_{\mathfrak{A}}} f = \lim_{\alpha, \beta = \infty} \overline{\int_{\mathfrak{A}_{\alpha\beta}}} f \tag{1}$$

exist. Since

$$\overline{\int_{\mathfrak{A}_{\alpha\beta}}} = \underline{\int_{\mathfrak{A}_{\alpha\beta}}},$$

necessarily

$$\underline{\int_{\mathfrak{A}}} f = \lim \underline{\int_{\mathfrak{A}_{\alpha\beta}}} f \tag{2}$$

exists and 1), 2) are equal.

42. 1. In studying the function f it is sometimes convenient to introduce two auxiliary functions defined as follows:

$$\begin{aligned} g &= f \qquad && \text{where } f \geq 0, \\ &= 0 && \text{where } f \leq 0. \\ h &= -f && \text{where } f \leq 0, \\ &= 0 && \text{where } f \geq 0. \end{aligned}$$

Thus g, h are both ≥ 0 and

$$f = g - h,$$
$$|f| = g + h.$$

We call them the *associated non-negative functions.*

2. *As usual let $\mathfrak{A}_{\alpha\beta}$ denote the points of $\mathfrak{A}$ at which $-\alpha \leq f \leq \beta$. Let $\mathfrak{A}_\beta$ denote the points where $g \leq \beta$, and $\mathfrak{A}_\alpha$ the points where $h \leq \alpha$. Then*

$$\overline{\int_{\mathfrak{A}}} g = \lim_{\alpha, \beta = \infty} \overline{\int_{\mathfrak{A}_{\alpha\beta}}} g, \tag{1}$$

$$\overline{\int_{\mathfrak{A}}} h = \lim_{\alpha, \beta = \infty} \overline{\int_{\mathfrak{A}_{\alpha\beta}}} h. \tag{2}$$

For

$$\overline{\int_{\mathfrak{A}_\beta}} g = \overline{\int_{\mathfrak{A}_{\alpha\beta}}} g, \qquad \text{by 5, 4.}$$

Letting α, $\beta \doteq \infty$, this last gives 1).

A similar demonstration establishes 2).

3. We cannot say always

$$\underline{\int_{\mathfrak{A}}} g = \lim_{\alpha, \beta = \infty} \underline{\int_{\mathfrak{A}_{\alpha\beta}}} g \quad ; \quad \underline{\int_{\mathfrak{A}}} h = \lim_{\alpha, \beta = \infty} \underline{\int_{\mathfrak{A}_{\alpha\beta}}} h,$$

as the following example shows.

Let $\quad f = 1$ at the irrational points in $\mathfrak{A} = (0, 1)$,

$$= -n, \text{ for } x = \frac{m}{n} \text{ in } \mathfrak{A}.$$

Then

$$\underline{\int}_{\mathfrak{A}_\beta} g = 0 \quad , \quad \underline{\int}_{\mathfrak{A}_{\alpha\beta}} g = 1.$$

Again let $f = -1$ for the irrational points in $\mathfrak{A}$,

$$= n \text{ for the rational points } x = \frac{m}{n}.$$

Then

$$\underline{\int}_{\mathfrak{A}_\alpha} h = 0. \qquad \underline{\int}_{\mathfrak{A}_{\alpha\beta}} h = 1.$$

43. 1.

$$1) \quad \overline{\int}_{\mathfrak{A}} g = \overline{\int}_{\mathfrak{P}} f; \qquad \underline{\int}_{\mathfrak{A}} g \leq \underline{\int}_{\mathfrak{P}} f; \qquad (2$$

$$3) \quad \overline{\int}_{\mathfrak{A}} h = -\underline{\int}_{\mathfrak{N}} f; \qquad \underline{\int}_{\mathfrak{A}} h \leq -\overline{\int}_{\mathfrak{N}} f; \qquad (4$$

provided the integral on either side of the equations converges, or provided the integrals on the right side of the inequalities converge.

Let us prove 1); the others are similarly established. Effecting a cubical division of space of norm d, we have for a fixed β,

$$\overline{\int}_{\mathfrak{A}_\beta} g = \lim_{d=0} \{\Sigma_{\mathfrak{P}_\beta} M_\iota d_\iota + \Sigma' 0 \cdot d_\iota\}$$

$$= \lim_{d=0} \Sigma_{\mathfrak{P}_\beta} M_\iota d_\iota = \overline{\int}_{\mathfrak{P}_\beta} f. \qquad (5$$

Thus if either integral in 1) is convergent, the passage to the limit $\beta = \infty$ in 5), gives 1).

2. *If* $\overline{\int}_{\mathfrak{A}} f$ *is convergent,* $\overline{\underline{\int}}_{\mathfrak{A}} g$ *converge.*

If $\underline{\int}_{\mathfrak{A}} f$ *is convergent,* $\overline{\underline{\int}}_{\mathfrak{A}} h$ *converge.*

This follows from 1 and from 38.

3. If $\overline{\int}_{\mathfrak{A}} f$ is convergent, we cannot say that $\overline{\int}_{\mathfrak{R}} f$ is always convergent. A similar remark holds for the lower integral.

For let $\quad f = 1$ at the rational points of $\mathfrak{A} = (0, 1)$

$$= -\frac{1}{x} \text{ at the irrational points.}$$

Then
$$\overline{\int}_{\mathfrak{A}} f = 1 \quad , \quad \overline{\int}_{\mathfrak{R}} f = -\infty.$$

4. That the inequality sign in 2) or 4) may be necessary is shown thus:

Let
$$f = \frac{1}{\sqrt{x}} \text{ for rational } x \text{ in } \mathfrak{A} = (0, 1)$$

$$= -\frac{1}{\sqrt{x}} \text{ for irrational } x.$$

Then
$$\underline{\int}_{\mathfrak{A}} g = 0 \quad , \quad \underline{\int}_{\mathfrak{P}} f = 2.$$

44. 1.
$$\overline{\int}_{\mathfrak{A}} f = \overline{\int}_{\mathfrak{A}} g - \lim_{\alpha, \beta = \infty} \underline{\int}_{\mathfrak{A}_{\alpha\beta}} h, \tag{1}$$

$$\underline{\int}_{\mathfrak{A}} f = \lim_{\alpha, \beta = \infty} \underline{\int}_{\mathfrak{A}_{\alpha\beta}} g - \overline{\int}_{\mathfrak{A}} h, \tag{2}$$

provided, 1° *the integral on the left exists*, *or* 2° *the integral and the limit on the right exist.*

For let us effect a cubical division of norm d. The cells containing points of $\mathfrak{A}$ fall into two classes:

a) those in which f is always ≤ 0,

b) those in which f is > 0 for at least one point.

In the cells *a*), since $f = g - h$,

$$\text{Max} f = \text{Max}\,(g - h) = \text{Max}\, g - \text{Min}\, h, \tag{3}$$

as $\text{Max}\, g = 0$. In the cells *b*) this relation also holds as $\text{Min}\, h = 0$. Thus 3) gives

$$\overline{\int}_{\mathfrak{A}_{\alpha\beta}} f = \overline{\int}_{\mathfrak{A}_{\alpha\beta}} g - \underline{\int}_{\mathfrak{A}_{\alpha\beta}} h. \tag{4}$$

Let now $\alpha, \beta \doteq \infty$. If the integral on the left of 1) is convergent, the integral on the right of 1) is convergent by 43, 2. Hence the limit on the right of 1) exists. Using now 42, 2, we get 1).

Let us now look at the 2° hypothesis. By 42, 2,

$$\lim_{\alpha, \beta = \infty} \overline{\int}_{\mathfrak{A}_{\alpha\beta}} g = \overline{\int}_{\mathfrak{A}} g.$$

Thus passing to the limit in 4), we get 1).

2. A relation of the type

$$\overline{\int}_{\mathfrak{A}} f = \overline{\int}_{\mathfrak{A}} g - \underline{\int}_{\mathfrak{A}} h$$

does not always hold as the following shows.

Example. Let $f = n$ at the points $x = \frac{m}{2^{2n}}$

$$= -n \text{ for } x = \frac{m}{2^{2n+1}}$$

$$= -1 \text{ at the other points of } \mathfrak{A} = (0, 1).$$

Then $\overline{\int}_{\mathfrak{A}} f = -1 \qquad \overline{\int}_{\mathfrak{A}} g = 0 \qquad \underline{\int}_{\mathfrak{A}} h = 0.$

45. *If* $\underline{\overline{\int}}_{\mathfrak{A}} f$ *is convergent, it is in any unmixed part* $\mathfrak{B}$ *of* $\mathfrak{A}$.

Let us consider the upper integral first. By 43, 2,

$$\overline{\int}_{\mathfrak{A}} g$$

exists. Hence *a fortiori*,

$$\overline{\int}_{\mathfrak{B}} g \tag{1}$$

exists. Since $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ is an unmixed division,

$$\underline{\int}_{\mathfrak{A}_{\alpha\beta}} h = \underline{\int}_{\mathfrak{B}_{\alpha\beta}} h + \underline{\int}_{\mathfrak{C}_{\alpha\beta}} h.$$

Hence

$$\underline{\int}_{\mathfrak{B}_{\alpha\beta}} h \le \underline{\int}_{\mathfrak{A}_{\alpha\beta}} h.$$

As the limit of the right side exists, that of the left exists also. From this fact, and because 1) exists,

$$\overline{\int_{\mathfrak{B}}} f$$

exists by 44, 1.

A similar demonstration holds for the lower integral over $\mathfrak{B}$.

46. *If* $\mathfrak{A}_1, \mathfrak{A}_2 \cdots \mathfrak{A}_m$ *form an unmixed division of* $\mathfrak{A}$, *then*

$$\underline{\overline{\int_{\mathfrak{A}}}} f = \underline{\overline{\int_{\mathfrak{A}_1}}} f + \cdots + \underline{\overline{\int_{\mathfrak{A}_m}}} f, \tag{1}$$

provided the integral on the left exists or all the integrals on the right exist.

For if $\mathfrak{A}_{m,\alpha\beta}$ denote the points of $\mathfrak{A}_{\alpha\beta}$ in $\mathfrak{A}_m$, we have

$$\underline{\overline{\int_{\mathfrak{A}_{\alpha\beta}}}} = \underline{\overline{\int_{\mathfrak{A}_{1\alpha\beta}}}} + \cdots + \underline{\overline{\int_{\mathfrak{A}_{m\alpha\beta}}}}. \tag{2}$$

Now if the integral on the left of 1) is convergent, the integrals on the right of 1) all converge by 45. Passing to the limit in 2) gives 1). On the other hypothesis, the integrals on the right of 1) existing, a passage to the limit in 2) shows that 1) holds in this case also.

47. *If* $\overline{\int_{\mathfrak{P}}} f$ *and* $\underline{\int_{\mathfrak{N}}} f$ *converge, so does* $\overline{\int_{\mathfrak{A}}} |f|$, *and*

$$\overline{\int_{\mathfrak{A}}} |f| \leq \overline{\int_{\mathfrak{P}}} f - \underline{\int_{\mathfrak{N}}} f \tag{1}$$

$$\leq \overline{\int_{\mathfrak{A}}} g + \overline{\int_{\mathfrak{A}}} h. \tag{2}$$

For let A_β denote the points of $\mathfrak{A}$ where

$$0 \leq |f| \leq \beta.$$

Then since

$$|f| = g + h,$$

$$\int_{A_\beta} |f| = \int_{A_\beta} (g+h) \leq \int_{A_\beta} g + \int_{A_\beta} h$$

$$\leq \overline{\int_{\mathfrak{A}}} g + \overline{\int_{\mathfrak{A}}} h \tag{3}$$

$$\leq \overline{\int_{\mathfrak{P}}} f - \underline{\int_{\mathfrak{N}}} f \qquad \text{by 43, 1.} \tag{4}$$

Passing to the limit in 3), 4), we get 1), 2).

48. 1. *If* $\overline{\int}_{\mathfrak{A}} |f|$ *converges, both* $\overline{\underline{\int}}_{\mathfrak{A}} f$ *converge.*

For as usual let $\mathfrak{P}$ denote the points of $\mathfrak{A}$ where $f \geq 0$. Then

$$\overline{\int}_{\mathfrak{P}} f = \overline{\int}_{\mathfrak{P}} |f|$$

is convergent by 38, 3, since $\overline{\int}_{\mathfrak{A}} |f|$ is convergent.

Similarly,

$$\overline{\int}_{\mathfrak{N}} (-f) = -\underline{\int}_{\mathfrak{N}} f$$

is convergent. The theorem follows now by 39.

2. *If* $\overline{\int}_{\mathfrak{A}} |f|$ *converges, so do*

$$\overline{\underline{\int}}_{\mathfrak{A}} g \quad , \quad \overline{\underline{\int}}_{\mathfrak{A}} h. \tag{1}$$

For by 1,

$$\overline{\underline{\int}}_{\mathfrak{A}} f$$

both converge. The theorem now follows by 43, 2.

3. *For*

$$\overline{\underline{\int}}_{\mathfrak{A}} f \tag{2}$$

both to converge it is necessary and sufficient that

$$\overline{\int}_{\mathfrak{A}} |f| \tag{3}$$

is convergent.

For if 3) converges, the integrals 2) both converge by 1. On the other hand if both the integrals 2) converge,

$$\overline{\int}_{\mathfrak{P}} f \quad , \quad \underline{\int}_{\mathfrak{N}} f$$

converge by 38, 2. Hence 3) converges by 47.

4. *If f is integrable in $\mathfrak{A}$, so is $|f|$.*

For let A_β denote the points of $\mathfrak{A}$ where $0 \leq |f| \leq \beta$. Then

$$\underline{\int}_{\mathfrak{A}} |f| = \lim_{\beta=\infty} \underline{\int}_{A_\beta} |f|,$$

and the limit on the right exists by 3.

But by 41, 1, f is integrable in A_β. Hence $|f|$ is integrable in A_β by I, 720. Thus

$$\overline{\int_{\mathfrak{A}}}|f| = \lim_{\beta=\infty}\int_{A_\beta}|f|.$$

49. From the above it follows that if both integrals

$$\overline{\underline{\int_{\mathfrak{A}}}} f$$

converge, they converge absolutely. Thus, in particular, if

$$\int_{\mathfrak{A}} f$$

converges, it is absolutely convergent.

We must, however, guard the reader against the error of supposing that *only absolutely* convergent upper and lower integrals exist.

Example. At the rational points of $\mathfrak{A}=(0, 1)$ let

$$f(x)=\frac{1}{2\sqrt{x}}.$$

At the irrational points let

$$f(x)=-\frac{1}{x}.$$

Here

$$\overline{\int_{\mathfrak{A}}} f = 1 \qquad \underline{\int_{\mathfrak{A}}} f = -\infty.$$

Thus, f admits an upper, but not a lower integral. On the other hand the upper integral of f does not converge absolutely. For obviously

$$\overline{\int_{\mathfrak{A}}}|f| = +\infty.$$

50. We have just noted that if

$$\int_{\mathfrak{A}} f(x_1 \cdots x_m)$$

is convergent, it is absolutely convergent. For $m=1$, this result apparently stands in contradiction with the theory developed in Vol. I, where we often dealt with convergent integrals which do not converge absolutely.

Let us consider, for example,

$$J = \int_0^1 \frac{\sin \frac{1}{x}}{x} dx = \int_{\mathfrak{A}} f dx.$$

If we set $x = \frac{1}{u}$, we get

$$J = \int_1^\infty \frac{\sin u}{u} du,$$

which converges by I, 667, but is not absolutely convergent by I, 646.

This apparent discrepancy at once disappears when we observe that according to the definition laid down in Vol. I,

$$J = R \lim_{a=0} \int_a^1 f dx,$$

while in the present chapter

$$J = \lim_{\alpha,\, \beta = \infty} \int_{\mathfrak{A}_{\alpha\beta}} f dx.$$

Now it is easy to see that, taking α large at pleasure but fixed,

$$\int_{\mathfrak{A}_{\alpha\beta}} f dx \doteq \infty \qquad \text{as } \beta \doteq \infty,$$

so that J does not converge according to our present definition.

In the theory of integration as ordinarily developed in works on the calculus a similar phenomenon occurs, viz. only *absolutely* convergent integrals exist when $m > 1$.

51. 1. *If* $\overline{\int}_{\mathfrak{A}} |f|$ *is convergent,*

$$\left| \underline{\overline{\int}}_{\mathfrak{A}} f \right| \leq \overline{\int}_{\mathfrak{A}} |f|. \tag{1}$$

For $\mathfrak{A}_{\alpha\beta}$ denoting as usual the points of $\mathfrak{A}$ where $-\alpha \leq f \leq \beta$ we have

$$\left| \underline{\overline{\int}}_{\mathfrak{A}_{\alpha\beta}} f \right| \leq \overline{\int}_{\mathfrak{A}_{\alpha\beta}} |f| \leq \overline{\int}_{\mathfrak{A}} |f|.$$

Passing to the limit, we get 1).

2. *If* $\overline{\int}_{\mathfrak{A}}|f|$ *is convergent,* $\overline{\underline{\int}}_{\mathfrak{B}} f$ *are convergent for any* $\mathfrak{B} \leq \mathfrak{A}$.

For $\overline{\int}_{\mathfrak{B}}|f|$ is convergent by 38, 4.

Hence

$$\overline{\underline{\int}}_{\mathfrak{B}} f$$

converge by 48, 3.

3. *If*, 1°, $\overline{\int}_{\mathfrak{A}}|f|$ *is convergent and* Min f *is finite, or if*, 2°, $\underline{\int}_{\mathfrak{A}} f$ *is convergent and* Max f *is finite, then*

$$\overline{\int}_{\mathfrak{A}}|f|$$

is convergent.

This follows by 36 and 48, 3.

52. *Let* $f \geq 0$ *in* $\mathfrak{A}$. *Let the integral*

$$\overline{\underline{\int}}_{\mathfrak{A}} f$$

converge. If

$$\overline{\underline{\int}}_{\mathfrak{A}} f = \overline{\underline{\int}}_{\mathfrak{A}_\beta} f + \alpha, \tag{1}$$

then for any unmixed part $\mathfrak{B} < \mathfrak{A}$,

$$\overline{\underline{\int}}_{\mathfrak{B}} f = \overline{\underline{\int}}_{\mathfrak{B}_\beta} f + \alpha', \tag{2}$$

where

$$0 \leq \alpha' \leq \alpha. \tag{3}$$

For let $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$. Then $\mathfrak{A}_\beta = \mathfrak{B}_\beta + \mathfrak{C}_\beta$ is an unmixed division. Also

$$\overline{\underline{\int}}_{\mathfrak{A}} f = \overline{\underline{\int}}_{\mathfrak{B}} + \overline{\underline{\int}}_{\mathfrak{C}}$$

$$= \overline{\underline{\int}}_{\mathfrak{A}_\beta} + \alpha \qquad \text{by 1)}$$

$$= \overline{\underline{\int}}_{\mathfrak{B}_\beta} + \overline{\underline{\int}}_{\mathfrak{C}_\beta} + \alpha.$$

Hence
$$\overline{\int_{\mathfrak{B}}} + \overline{\int_{\mathfrak{C}}} = \overline{\int_{\mathfrak{B}_\beta}} + \overline{\int_{\mathfrak{C}_\beta}} + \alpha. \tag{4}$$

From 2)
$$\begin{aligned} \alpha' &= \overline{\int_{\mathfrak{B}}} - \overline{\int_{\mathfrak{B}_\beta}} \\ &= \alpha - \left\{ \overline{\int_{\mathfrak{C}}} - \overline{\int_{\mathfrak{C}_\beta}} \right\} \qquad \text{by 4)} \\ &\leq \alpha, \end{aligned}$$
which establishes 3).

53. *If the integral*
$$\overline{\int_{\mathfrak{A}}} |f| \tag{1}$$
converges, then
$$\epsilon > 0, \qquad \sigma > 0, \qquad \left| \overline{\int_{\mathfrak{B}}} f \right| \leq \epsilon \tag{2}$$
for any $\mathfrak{B} < \mathfrak{A}$ *such that*
$$\mathfrak{B} > \sigma. \tag{3}$$

Let us suppose first that $f \geq 0$. If the theorem is not true, there exists, however small $\sigma > 0$ is taken, a $\mathfrak{B}$ satisfying 3) such that
$$\overline{\int_{\mathfrak{B}}} f > \epsilon. \tag{4}$$

Then there exists a cubical division of space such that those points of $\mathfrak{A}$, call them $\mathfrak{C}$, which lie in cells containing a point of $\mathfrak{B}$, are such that $\overline{\mathfrak{C}} < \sigma$ also. Moreover $\mathfrak{C}$ is an unmixed part of $\mathfrak{A}$. Then from 4) follows, as $f \geq 0$, that
$$\overline{\int_{\mathfrak{C}}} f > \epsilon \tag{5}$$
also.

Let us now take β so that
$$\overline{\int_{\mathfrak{A}}} = \overline{\int_{\mathfrak{A}_\beta}} + \alpha.$$

Then
$$\overline{\int_{\mathfrak{C}}} = \overline{\int_{\mathfrak{C}_\beta}} + \alpha',$$
and
$$0 \leq \alpha' \leq \alpha$$
by 52. But
$$\overline{\int_{\mathfrak{C}_\beta}} \leq \beta \overline{\mathfrak{C}}_\beta \leq \beta \overline{\mathfrak{C}}.$$

Let now $\beta\sigma < \epsilon$, then

$$\overline{\int}_{\mathfrak{E}_\beta} < \epsilon,$$

which contradicts 5).

Let us now make no restrictions on the sign of f. We have

$$\left|\overline{\int}_{\mathfrak{B}} f\right| < \overline{\int}_{\mathfrak{B}} |f|.$$

But since 1) converges, the present case is reduced to the preceding.

54. 1. *Let* $\overline{\int}_{\mathfrak{A}} |f|$ *converge.*

Let as usual $\mathfrak{A}_{\alpha\beta}$ *denote the points of* $\mathfrak{A}$ *at which* $-\alpha \leq f \leq \beta$. *Let* A_{ab} *be such that each* $\mathfrak{A}_{\alpha\beta}$ *lies in some* A_{ab} *in which latter* f *is limited. Let* $\mathfrak{D}_{\alpha\beta} = A_{ab} - \mathfrak{A}_{\alpha\beta}$ *and let* $a, b \doteq \infty$ *with* α, β. *Then*

$$\lim_{\alpha,\beta=\infty} \overline{\mathfrak{D}}_{\alpha\beta} = 0.$$

For if not, let

$$\overline{\lim_{\alpha,\beta=\infty}} \overline{\mathfrak{D}}_{\alpha\beta} = l, \qquad l > 0.$$

Then for any $0 < \lambda < l$, there exists a monotone sequence $\{\alpha_n, \beta_n\}$ such that

$$\overline{\mathfrak{D}}_{\alpha_n\beta_n} > \lambda \qquad \text{for } n > \text{some } m.$$

Let $\mu_n = \text{Min}(\alpha_n, \beta_n)$, then $|f| \geq \mu_n$ in $\mathfrak{D}_{\alpha_n\beta_n}$, and $\mu_n \doteq \infty$. Hence

$$\int_{\mathfrak{D}_{\alpha_n\beta_n}} |f| \geq \mu_n\lambda \doteq \infty. \tag{1}$$

But $\mathfrak{D}_{\alpha_n\beta_n}$ being a part of $\mathfrak{A}$

$$\overline{\int}_{\mathfrak{D}_{\alpha_n\beta_n}} |f| \leq \overline{\int}_{\mathfrak{A}} |f|$$

by 38, 3. This contradicts 1).

2. *Definition.* We say $A_{a,b}$ is *conjugate* to $\mathfrak{A}_{\alpha\beta}$ with respect to f.

55. 1. *As usual let* $-\alpha \leq f \leq \beta$ *in* $\mathfrak{A}_{\alpha\beta}$. *Let* $0 \leq f \leq \beta$ *in* $\mathfrak{A}_\beta$. *Let* A_{ab} *be conjugate to* $\mathfrak{A}_{\alpha\beta}$ *with reference to* f; *and* A_b *conjugate to* $\mathfrak{A}_\beta$ *with respect to* $|f|$.

If, 1°,
$$\lim_{\beta=\infty} \overline{\int}_{\mathfrak{A}_\beta} |f|,$$
or if, 2°,
$$\lim_{b=\infty} \overline{\int}_{A_b} |f|$$
exist, then,
$$\lim_{a,\, b=\infty} \underline{\overline{\int}}_{A_{a,b}} f = \underline{\overline{\int}}_{\mathfrak{A}} f. \tag{1}$$

For, if 2° holds, 1° holds also, since
$$\overline{\int}_{\mathfrak{A}_\beta} |f| \leq \overline{\int}_{A_b} |f|, \qquad \text{as } \mathfrak{A}_\beta \leq A_b.$$

Thus case 2° is reduced to 1°. Let then the 1° limit exist. We have
$$\overline{\int}_{\mathfrak{A}_{\alpha\beta}} f = \overline{\int}_{\mathfrak{A}_{\alpha\beta}} g - \underline{\int}_{\mathfrak{A}_{\alpha\beta}} h, \tag{2}$$
as 4) in 44, 1 shows. Let now
$$\mathfrak{D}_{\alpha\beta} = A_{ab} - \mathfrak{A}_{\alpha\beta}.$$

Then,
$$\overline{\int}_{\mathfrak{A}_{\alpha\beta}} g \leq \overline{\int}_{A_{ab}} g \leq \overline{\int}_{\mathfrak{A}_{\alpha\beta}} g + \overline{\int}_{\mathfrak{D}_{\alpha\beta}} g. \tag{3}$$

But $\overline{\mathfrak{D}}_{\alpha\beta} \doteq 0$, as $\alpha, \beta \doteq \infty$, by 54. Let us now pass to the limit $\alpha, \beta = \infty$ in 3). Since the limit of the last term is 0 by 53, 54, we get
$$\lim_{\alpha,\, \beta=\infty} \overline{\int}_{\mathfrak{A}_{\alpha\beta}} g = \lim_{a,\, b=\infty} \overline{\int}_{A_{a,b}} g. \tag{4}$$

Similarly,
$$\lim_{\alpha,\, \beta=\infty} \underline{\int}_{\mathfrak{A}_{\alpha\beta}} h = \lim_{a,\, b=\infty} \underline{\int}_{A_{ab}} h. \tag{5}$$

Passing to the limit in 2), we get, using 4), 5),
$$\overline{\int}_{\mathfrak{A}_{\alpha\beta}} f = \lim_{a,\, b=\infty} \left\{ \overline{\int}_{A_{ab}} g - \underline{\int}_{A_{ab}} h \right\}$$
$$= \lim_{a,\, b=\infty} \overline{\int}_{A_{ab}} f$$

In a similar manner we may establish 1) for the lower integrals.

2. The following example is instructive as showing that when the conditions imposed in 1 are not fulfilled, the relation 1) may not hold.

Example. Since

$$\int_0^1 \frac{dx}{x} = +\infty,$$

there exists, for any $b_n > 0$, a $0 < b_{n+1} < b_n$, such that if we set

$$\int_{b_{n+1}}^{b_n} \frac{dx}{x} = G_{n+1},$$

then

$$G_1 < G_2 < \cdots \doteq \infty,$$

as $b_n \doteq 0$. Let now

$$f = 1 \text{ for the rational points in } \mathfrak{A} = (0, 1),$$

$$= \frac{1}{x} \text{ for the irrational.}$$

Then

$$\underline{\int}_{\mathfrak{A}_\beta} f = 1 - \frac{1}{\beta}.$$

Let

$$\int_{b_1}^1 \frac{dx}{x} = G_1, \qquad 0 < b_1 < 1.$$

Let A_n denote the points of $\mathfrak{A}$ in $(b_n, 1)$ and the irrational points in (b_{n+1}, b_n).

Then

$$\underline{\int}_{A_n} > G_{n+1} \doteq +\infty.$$

But obviously the set A_n is conjugate to $\mathfrak{A}_\beta$. On the other hand,

$$\underline{\int}_{\mathfrak{A}} f = 1,$$

while

$$\lim_{n=\infty} \underline{\int}_{A_n} f = +\infty.$$

56. *If the integral*

$$\overline{\underline{\int}}_{\mathfrak{A}} f \qquad (1$$

converges, then

$$\epsilon > 0, \qquad \sigma > 0, \qquad \left| \overline{\underline{\int}}_{\mathfrak{B}} f \right| < \epsilon \qquad (2$$

for any unmixed part $\mathfrak{B}$ *of* $\mathfrak{A}$ *such that*

$$\overline{\mathfrak{B}} \leq \sigma.$$

Let us establish the theorem for the upper integral; similar reasoning may be used for the lower. Since 1) is convergent,

$$\overline{\int_{\mathfrak{A}}} g \tag{3}$$

and

$$\lambda = \lim_{\alpha,\,\beta=\infty} \underline{\int_{\mathfrak{A}_{\alpha\beta}}} h \tag{4}$$

exist by 44, 1. Since 3) exists, we have by 53,

$$0 \leq \overline{\int_{\mathfrak{B}}} g < \frac{\epsilon}{4}, \tag{5}$$

for any $\mathfrak{B} < \mathfrak{A}$ such that $\overline{\mathfrak{B}} <$ some σ'.

Since 4) exists, there exists a pair of values a, b such that

$$\lambda = \underline{\int_{\mathfrak{A}_{ab}}} h + \eta \quad , \quad 0 < \eta < \frac{\epsilon}{4}, \tag{6}$$

since the integral on the right side of 4) is a monotone increasing function of a, b.

Since $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ is an unmixed division of $\mathfrak{A}$,

$$\underline{\int_{\mathfrak{A}_{\alpha\beta}}} h = \underline{\int_{\mathfrak{B}_{\alpha\beta}}} h + \underline{\int_{\mathfrak{C}_{\alpha\beta}}} h.$$

Since $h \geq 0$, and the limit 4) exists, the above shows that

$$\mu = \lim_{\alpha,\,\beta=\infty} \underline{\int_{\mathfrak{B}_{\alpha\beta}}} h \quad , \quad \nu = \lim_{\alpha,\,\beta=\infty} \underline{\int_{\mathfrak{C}_{\alpha\beta}}} h$$

exist and that

$$\lambda = \mu + \nu. \tag{7}$$

Then a, b being the same as in 6),

$$\mu = \underline{\int_{\mathfrak{B}_{ab}}} h + \eta', \tag{8}$$

and we show that

$$0 \leq \eta' \leq \eta \tag{9}$$

as in 52. Let now $c > a, b$; then

$$\underline{\int_{\mathfrak{B}_{ab}}} h \leq c\overline{\mathfrak{B}} < \frac{\epsilon}{4} \tag{10}$$

if we take

$$\overline{\mathfrak{B}} < \frac{\epsilon}{4\,c} = \sigma''.$$

Thus,

$$\mu < \frac{\epsilon}{4}. \tag{11}$$

But

$$\overline{\int_{\mathfrak{B}}} f = \overline{\int_{\mathfrak{B}}} g - \mu$$

by 44, 1. Thus 2) follows on using 5), 11) and taking $\sigma < \sigma', \sigma''$.

57. *If the integral* $\overline{\int_{\mathfrak{A}}} f$ *converges and* $\mathfrak{B}_u$ *is an unmixed part of* $\mathfrak{A}$ *such that* $\overline{\mathfrak{B}}_u \doteq \overline{\mathfrak{A}}$ *as* $u \doteq 0$, *then*

$$\lim_{u=0} \overline{\int_{\mathfrak{B}_u}} f = \overline{\int_{\mathfrak{A}}} f. \tag{1}$$

For if we set $\mathfrak{A} = \mathfrak{B}_u + \mathfrak{C}_u$, the last set is an unmixed part of $\mathfrak{A}$ and $\overline{\mathfrak{C}}_u \doteq 0$. Now

$$\overline{\int_{\mathfrak{A}}} f = \overline{\int_{\mathfrak{B}_u}} + \overline{\int_{\mathfrak{C}_u}}.$$

Passing to the limit, we get 1) on using 56.

58. 1. *Let* $\mathfrak{D}_{\alpha\beta} = Dv(\mathfrak{A}_{f\alpha\beta}, \mathfrak{A}_{g\alpha\beta}, \mathfrak{A}_{f+g,\,\alpha\beta})$.

If, 1°, *the upper contents of*

$$\mathfrak{f}_{\alpha\beta} = \mathfrak{A}_{f\alpha\beta} - \mathfrak{D}_{\alpha\beta} \quad , \quad \mathfrak{g}_{\alpha\beta} = \mathfrak{A}_{g\,\alpha\beta} - \mathfrak{D}_{\alpha\beta} \quad , \quad \mathfrak{h}_{\alpha\beta} = \mathfrak{A}_{f+g,\,\alpha\beta} - \mathfrak{D}_{\alpha\beta} \tag{1}$$
$$\doteq 0 \text{ as } \alpha, \beta \doteq \infty,$$

and if, 2°, *the upper integrals of* $f, g, f+g$ *are convergent, then*

$$\overline{\int_{\mathfrak{A}}} (f+g) \leq \overline{\int_{\mathfrak{A}}} f + \overline{\int_{\mathfrak{A}}} g. \tag{2}$$

If 1° *holds, and if*, 3°, *the lower integrals of* $f, g, f+g$ *are convergent, then*

$$\underline{\int_{\mathfrak{A}}} f + \underline{\int_{\mathfrak{A}}} g \leq \underline{\int_{\mathfrak{A}}} (f+g). \tag{3}$$

Let us prove 2); the relation 3) is similarly established. Let $D_{\alpha,\beta}$ be a cubical division of space. Let $\mathfrak{E}_{\alpha\beta}$ denote the points of $\mathfrak{D}_{\alpha\beta}$ lying in cells of $D_{\alpha\beta}$, containing no point of the sets 1). Let

$$\mathfrak{F}_{\alpha\beta} = \mathfrak{A}_{f,\,\alpha\beta} - \mathfrak{E}_{\alpha\beta}.$$

Then $D_{\alpha\beta}$ may be chosen so that $\overline{\mathfrak{F}}_{\alpha\beta} \doteq 0$.

Now

$$\overline{\int}_{\mathfrak{A}_{f,\,\alpha\beta}} f = \overline{\int}_{\mathfrak{E}_{\alpha\beta}} + \overline{\int}_{\mathfrak{F}_{\alpha\beta}}$$

since the fields are unmixed. By 56, the second integral on the right $\doteq 0$ as $\alpha, \beta \doteq \infty$. Hence

$$\lim_{\alpha,\,\beta=\infty} \overline{\int}_{\mathfrak{A}_{f,\,\alpha\beta}} f = \lim_{\alpha,\,\beta=\infty} \overline{\int}_{\mathfrak{E}_{\alpha\beta}} f.$$

Similar reasoning applies to g and $f+g$.

Again,

$$\overline{\int}_{\mathfrak{E}_{\alpha\beta}} (f+g) \leq \overline{\int}_{\mathfrak{E}_{\alpha\beta}} f + \overline{\int}_{\mathfrak{E}_{\alpha\beta}} g.$$

Thus, letting $\alpha, \beta \doteq \infty$ we get 2).

2. *When the singular points of f, g are discrete, the condition* 1° *holds.*

3. *If g is integrable and the conditions* 1°, 2°, 3° *are satisfied,*

$$\underline{\overline{\int}}_{\mathfrak{A}} (f+g) = \underline{\overline{\int}}_{\mathfrak{A}} f + \int_{\mathfrak{A}} g.$$

4. *If f, g are integrable and condition* 1° *is satisfied, $f+g$ is integrable and*

$$\int_{\mathfrak{A}} (f+g) = \int_{\mathfrak{A}} f + \int_{\mathfrak{A}} g.$$

5.
$$\underline{\overline{\int}}_{\mathfrak{A}} (f+C) = \underline{\overline{\int}}_{\mathfrak{A}} f + C \lim_{\alpha,\,\beta=\infty} \overline{\mathfrak{A}}_{\alpha\beta}$$

provided the integral of f in question converges or is definitely infinite.

For

$$\underline{\overline{\int}}_{\mathfrak{D}_{\alpha\beta}} (f+C) = \underline{\overline{\int}}_{\mathfrak{D}_{\alpha\beta}} f + \int_{\mathfrak{D}_{\alpha\beta}} C.$$

Also

$$\lim \overline{\mathfrak{D}}_{\alpha\beta} = \lim \overline{\mathfrak{A}}_{\alpha\beta}$$

where $\mathfrak{A}_{\alpha\beta}$ refers to f.

6. When condition 1° is not satisfied, the relations 2) or 3) may not hold.

Example. Let $\mathfrak{A}$ consist of the rational points in $(0, 1)$.

Let
$$f = 1 + n \quad , \quad g = 1 - n$$
at the point $x = \frac{m}{n}$. Then
$$f + g = 2 \qquad \text{in } \mathfrak{A}.$$
Now
$$\mathfrak{A}_{f,\,\alpha\beta} \quad , \quad \mathfrak{A}_{g,\,\alpha\beta}$$
embrace only a finite number of points for a given α, β. On the other hand,
$$\mathfrak{A}_{f+g,\,\alpha\beta} = \mathfrak{A} \qquad \text{for } \beta > 2.$$

Thus the upper content of the last set in 1) does not $\doteq 0$ as $\alpha, \beta \doteq \infty$ and condition 1° is not fulfilled. Also relation 2) does not hold in this case. For
$$\int_{\mathfrak{A}} (f+g) = 2 \quad , \quad \int_{\mathfrak{A}} f = 0 \quad , \quad \int_{\mathfrak{A}} g = 0.$$

59. *If $c > 0$, then* $\overline{\int_{\mathfrak{A}}} cf = c \overline{\int_{\mathfrak{A}}} f,$ (1

if $c < 0$, then $\overline{\int_{\mathfrak{A}}} cf = c \underline{\int_{\mathfrak{A}}} f,$ (2

provided the integral on either side is convergent.

For
$$\overline{\int}_{\mathfrak{A}_{cf,\,\alpha\beta}} cf = c \overline{\int}_{\mathfrak{A}_{cf,\,\alpha\beta}} f \qquad \text{if } c > 0 \tag{3}$$
$$= c \underline{\int}_{\mathfrak{A}_{cf,\,\alpha\beta}} f \qquad \text{if } c < 0. \tag{4}$$

Let $c > 0$. Since
$$-\alpha \le cf \le \beta \qquad \text{in } \mathfrak{A}_{cf,\,\alpha\beta},$$
therefore
$$-\frac{\alpha}{c} \le f \le \frac{\beta}{c} \qquad \text{in this set.}$$

Hence any point of $\mathfrak{A}_{cf,\,\alpha\beta}$, is a point of $\mathfrak{A}_{f,\,\frac{\alpha}{c},\frac{\beta}{c}}$ and conversely.

Thus
$$\mathfrak{A}_{cf,\,\alpha\beta} = \mathfrak{A}_{f,\,\frac{\alpha}{c},\frac{\beta}{c}} \qquad \text{when } c > 0.$$

Similarly
$$\mathfrak{A}_{cf,\,\alpha\beta} = \mathfrak{A}_{f,\,\frac{\beta}{c},\,\frac{\alpha}{c}} \qquad \text{when } c < 0.$$

Thus 3), 4) give

$$\overline{\int}_{\mathfrak{A}_{cf,\,\alpha\beta}} cf = c\overline{\int}_{\mathfrak{A}_{f,\,\frac{\alpha}{c},\,\frac{\beta}{c}}} f \qquad c > 0$$

$$= c\underline{\int}_{\mathfrak{A}_{f,\,\frac{\beta}{c},\,\frac{\alpha}{c}}} f \qquad c < 0.$$

We now need only to pass to the limit $\alpha, \beta = \infty$.

60. *Let one of the integrals*

$$\overline{\int}_{\mathfrak{A}} f \quad , \quad \overline{\int}_{\mathfrak{A}} g \tag{1}$$

converge. If $f = g$, except at a discrete set $\mathfrak{D}$ in $\mathfrak{A}$, both integrals converge and are equal. A similar theorem holds for the lower integrals.

For let us suppose the first integral in 1) converges. Let

$$\mathfrak{A} = A + \mathfrak{D};$$

then

$$\overline{\int}_{\mathfrak{A}} f = \overline{\int}_{A} f + \overline{\int}_{\mathfrak{D}} f = \overline{\int}_{A} f. \tag{2}$$

Now

$$\overline{\int}_{\mathfrak{A}} g = \lim_{\alpha,\,\beta=\infty} \int_{\mathfrak{A}_{g,\,\alpha\beta}} g = \lim \int_{A_{g,\,\alpha\beta}} g$$

$$= \lim \overline{\int}_{A_{f,\,\alpha\beta}} f = \overline{\int}_{A} f. \tag{3}$$

Thus the second integral in 1) converges, and 2), 3) show that the integrals in 1) are equal.

61. 1. *Let*
$$\underline{\int}_{\mathfrak{A}} f, \quad \underline{\int}_{\mathfrak{A}} g \tag{1}$$

converge. Let $f \geq g$ except possibly at a discrete set. Let

$$\mathfrak{D}_{\alpha\beta} = Dv(\mathfrak{A}_{f,\,\alpha\beta}\mathfrak{A}_{g,\,\alpha\beta}) \quad ; \quad \mathfrak{f} = \mathfrak{A}_{f,\,\alpha\beta} - \mathfrak{D}_{\alpha\beta} \quad ; \quad \mathfrak{g}_{\alpha\beta} = \mathfrak{A}_{g,\,\alpha\beta} - \mathfrak{D}_{\alpha\beta}.$$

If
$$\overline{\mathfrak{f}}_{\alpha\beta} \doteq 0, \quad \overline{\mathfrak{g}}_{\alpha\beta} \doteq 0, \quad \text{as } \alpha, \beta \doteq \infty,$$

then

$$\underline{\int}_{\mathfrak{A}} f \geq \underline{\int}_{\mathfrak{A}} g. \tag{2}$$

For let $\mathfrak{E}_{\alpha\beta}$ be defined as in 58, 1. Then

$$\overline{\int}_{\mathfrak{E}_{\alpha\beta}} f \geq \overline{\int}_{\mathfrak{E}_{\alpha\beta}} g.$$

Let $\alpha, \beta \doteq \infty$, we get 2) by the same style of reasoning as in 58.

2. *If the integrals* 1) *converge, and their singular points are discrete, the relation* 2) *holds.*

This follows by 58, 2.

3. If the conditions of 1 do not hold, the relation 2) may not be true.

Example. Let $\mathfrak{A}$ denote the rational points in $(0^*, 1^*)$. Let

$$f = n \qquad \text{at } x = \frac{m}{n} \text{ in } \mathfrak{A}.$$

$$g = 1 \qquad \text{in } \mathfrak{A}.$$

Then
$$f \geq g \qquad \text{in } \mathfrak{A}.$$

But
$$\int_{\mathfrak{A}} f = 0 \qquad \int_{\mathfrak{A}} g = 1.$$

Relation between the Integrals of Types I, II, III

62. Let us denote these integrals over the limited field $\mathfrak{A}$ by

$$C_{\mathfrak{A}} \quad , \quad V_{\mathfrak{A}} \quad , \quad P_{\mathfrak{A}}$$

respectively. The upper and lower integrals may be denoted by putting a dash above and below them. When no ambiguity arises, we may omit the subscript $\mathfrak{A}$. The singular points of the integrand f, we denote as usual by $\mathfrak{J}$.

63. *If one of the integrals* $\underline{P}$ *is convergent, and* $\mathfrak{J}$ *is discrete, the corresponding C integral converges, and both are equal.*

For
$$\overline{P}_{\mathfrak{A}} = \overline{P}_{\mathfrak{A}_\delta} + \overline{P}_{\mathfrak{A}'_\delta}, \qquad \text{using the notation of 28,}$$

$$= \overline{C}_{\mathfrak{A}_\delta} + \overline{P}_{\mathfrak{A}'_\delta}.$$

Now
$$\overline{P}_{\mathfrak{A}'_\delta} \doteq 0 \qquad \text{as } \delta \doteq 0 \qquad \text{by 56.}$$

Hence
$$\overline{P}_{\mathfrak{A}} = \lim_{\delta=0} \overline{C}_{\mathfrak{A}_\delta}$$
$$= \overline{C}_{\mathfrak{A}}, \qquad \text{by definition.}$$

64. If $\overline{C}$ is convergent, we cannot say that $\overline{P}$ converges. A similar remark holds for the lower integrals.

Example. For the rational points in $\mathfrak{A} = (0, 1)$ let
$$f(x) = \frac{1}{2\sqrt{x}};$$
for the irrational points let
$$f(x) = -\frac{1}{x}.$$

Then
$$\overline{C}_{\mathfrak{A}} = \lim_{a=0} \overline{\int_a^1} f(x)\,dx = \lim \left[\sqrt{x}\right]_a^1 = 1.$$

On the other hand,
$$\overline{P}_{\mathfrak{A}} = \lim_{\alpha,\beta=\infty} \overline{\int_{\mathfrak{A}_{\alpha\beta}}} f$$
does not exist. For however large β is taken and then fixed,
$$\overline{\int_{\mathfrak{A}_{\alpha\beta}}} f \doteq -\infty \qquad \text{as } \alpha \doteq \infty.$$

65. *If $\overline{C}$ is absolutely convergent and $\mathfrak{J}$ is discrete, then both $\overline{\underline{P}}$ converge and are equal to the corresponding C integrals.*

For let D be any complete division of $\mathfrak{A}$ of norm δ. Then
$$\overline{\int_{\mathfrak{A}_{\alpha\beta}}} = \overline{\int_{\mathfrak{A}_{\alpha\beta,\delta}}} + \overline{\int_{\mathfrak{A}'_{\alpha\beta,\delta}}} \tag{1}$$
using the notation of 28. Now since
$$\overline{C}_{\mathfrak{A}}|f| \text{ converges,} \qquad \overline{C}_{\mathfrak{A}'_\delta}|f| \doteq 0 \qquad \text{as } \delta \doteq 0.$$

But
$$\left|\overline{\int_{\mathfrak{A}'_{\alpha\beta,\delta}}} f\right| \leq \overline{\int_{\mathfrak{A}'_{\alpha\beta,\delta}}} |f| = \overline{C}_{\mathfrak{A}'_{\alpha\beta,\delta}}|f| < \overline{C}_{\mathfrak{A}'_\delta}|f| \doteq 0. \tag{2}$$

Again, D being fixed, if $\alpha_0\beta_0$ are sufficiently large,
$$\overline{\int_{\mathfrak{A}_{\alpha\beta,\delta}}} f = \overline{C}_{\mathfrak{A}_\delta} f \qquad \alpha > \alpha_0, \qquad \beta > \beta_0.$$

Hence 1), 2) give

$$\overline{\int}_{\mathfrak{A}_{\alpha\beta}} f = \overline{C}_{\mathfrak{A}_\delta} + \epsilon' \qquad |\epsilon'| < \frac{\epsilon}{2} \qquad \text{for any } \delta < \text{some } \delta_0.$$

On the other hand, if δ_0 is sufficiently small,

$$\overline{C}_{\mathfrak{A}} = \overline{C}_{\mathfrak{A}_\delta} + \epsilon'' \qquad |\epsilon''| < \frac{\epsilon}{2} \qquad \text{for } \delta < \delta_0.$$

Hence
$$\overline{\int}_{\mathfrak{A}_{\alpha\beta}} f = \overline{C}_{\mathfrak{A}} + \epsilon''' \qquad |\epsilon'''| < \epsilon.$$

Passing to the limit $\alpha, \beta = \infty$, we get

$$\overline{P} = \overline{C}.$$

66. *If $V_{\mathfrak{A}} f$ is absolutely convergent, the singular points $\mathfrak{J}$ are discrete.*

For suppose $\overline{\mathfrak{J}} > 0$. Let $\mathfrak{B}$ denote the points of $\mathfrak{A}$ where $|f| \geq \beta$. Then $\overline{\mathfrak{B}} \geq \overline{\mathfrak{J}}$ for any β. Hence

$$\overline{\int}_{\mathfrak{A}} |f|_\beta \geq \overline{\int}_{\mathfrak{B}} |f|_\beta = \beta \overline{\mathfrak{B}} \geq \beta \overline{\mathfrak{J}} \doteq \infty,$$

as $\beta \doteq \infty$ unless $\overline{\mathfrak{J}} = 0$.

67. *If $\overline{V}_{\mathfrak{A}} f$ is absolutely convergent, so is $\overline{C}$.*

For let D be a cubical division of space of norm d.
Then

$$|f| \leq \text{some } \beta \text{ in } \mathfrak{A}_d.$$

Hence
$$\overline{\int}_{\mathfrak{A}_d} |f| \leq \overline{\int}_{\mathfrak{A}} |f|_\beta \leq \overline{V}_{\mathfrak{A}} |f|.$$

Hence $\overline{C}$ is absolutely convergent.

68. *Let $f \geq 0$. If $\overline{V}_{\mathfrak{A}} f$ is convergent, there exists for each $\epsilon > 0$, a $\sigma > 0$ such that*

$$\overline{V}_{\mathfrak{B}} f < \epsilon \tag{1}$$

for any $\mathfrak{B}$ such that

$$\overline{\mathfrak{B}} < \sigma. \tag{2}$$

For
$$\overline{V}_{\mathfrak{A}} f = \overline{\int}_{\mathfrak{A}} f_\lambda + \epsilon' \quad , \quad 0 \leq \epsilon' < \frac{\epsilon}{2},$$

for λ sufficiently large. Let λ be so taken, then

$$\overline{V}_{\mathfrak{B}} f = \overline{\int}_{\mathfrak{B}} f_\lambda + \epsilon'' \quad , \quad 0 \leq \epsilon'' < \frac{\epsilon}{2}. \tag{3}$$

Also,
$$\overline{\int}_{\mathfrak{B}} f_\lambda \leq \lambda \overline{\mathfrak{B}} < \frac{\epsilon}{2}, \tag{4}$$

if σ is taken sufficiently small in 2).

From 3), 4) follows 1).

69. *If $\overline{V}_{\mathfrak{A}} f$ is absolutely convergent, both $\underline{\overline{C}}$ converge and are equal to the corresponding V integrals.*

For by 67, $\overline{C}$ is absolutely convergent. Hence $\underline{\overline{C}}$ converge by 65.

Thus
$$\underline{\overline{C}}_{\mathfrak{A}} f = \underline{\overline{\int}}_{\mathfrak{A}_d} f + \alpha \quad , \quad |\alpha| < \frac{\epsilon}{3} \quad \text{for some } d.$$

Also
$$\underline{\overline{V}}_{\mathfrak{A}} f = \underline{\overline{\int}}_{\mathfrak{A}} f_{\lambda\mu} + \beta \quad , \quad |\beta| < \frac{\epsilon}{3} \quad \text{for some } \lambda, \mu.$$

Hence
$$\eta = \underline{\overline{C}}_{\mathfrak{A}} f - \underline{\overline{V}}_{\mathfrak{A}} f = \underline{\overline{\int}}_{\mathfrak{A}_d} f - \underline{\overline{\int}}_{\mathfrak{A}} f_{\lambda\mu} + (\alpha - \beta). \tag{1}$$

Now
$$\underline{\overline{\int}}_{\mathfrak{A}} f_{\lambda\mu} = \underline{\overline{\int}}_{\mathfrak{A}_d} f_{\lambda\mu} + \underline{\overline{\int}}_{\mathfrak{A}'_d} f_{\lambda\mu}.$$

But
$$\left| \underline{\overline{\int}}_{\mathfrak{A}'_d} f_{\lambda\mu} \right| \leq \overline{\int}_{\mathfrak{A}'_d} |f_{\lambda\mu}| \leq \overline{V}_{\mathfrak{A}'_d} |f| = \gamma,$$

and $\gamma < \frac{\epsilon}{3}$ if d is sufficiently small, and for any λ, μ, by 68.

Taking a division of space having this norm, we then take λ, μ so large that
$$f_{\lambda\mu} = f \quad \text{in } \mathfrak{A}_d.$$

Then
$$\eta = \alpha - \beta - \gamma,$$

and hence
$$|\eta| < \epsilon.$$

From this and 1) the theorem now follows at once.

Iterated Integrals

70. 1. We consider now the relations which exist between the integrals

$$\overline{\int_{\mathfrak{A}}} f, \tag{1}$$

and

$$\overline{\int_{\mathfrak{B}}}\overline{\int_{\mathfrak{C}}} f, \tag{2}$$

where $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ lies in a space $\mathfrak{R}_m$, $m = p + q$, and $\mathfrak{B}$ is a projection of $\mathfrak{A}$ in the space $\mathfrak{R}_p$.

It is sometimes convenient to denote the last q coördinates of a point $x = (x_1 \cdots x_p \; x_{p+1} \cdots x_{p+q})$ by $y_1 \cdots y_q$. Thus the coördinates $x_1 \cdots x_p$ refer to $\mathfrak{B}$ and $y_1 \cdots y_q$ to $\mathfrak{C}$. The section of $\mathfrak{A}$ corresponding to the point x in $\mathfrak{B}$ may be denoted by $\mathfrak{C}_x$ when it is desirable to indicate which of the sections $\mathfrak{C}$ is meant.

2. Let us set

$$\phi(x_1 \cdots x_p) = \overline{\int_{\mathfrak{C}}} f; \tag{3}$$

then the integral 2) is

$$\overline{\int_{\mathfrak{B}}} \phi. \tag{4}$$

It is important to note at once that although the integrand f is defined for each point in $\mathfrak{A}$, the integrand ϕ in 4) may not be.

Example. Let $\mathfrak{A}$ consist of the points (x, y) in the unit square:

$$x = \frac{m}{n} \quad , \quad 0 \leq y \leq \frac{1}{n}.$$

Then $\mathfrak{A}$ is discrete. At the point (x, y) in $\mathfrak{A}$, let

$$f = \frac{1}{y}.$$

Then
$$\int_{\mathfrak{A}} f = 0 \qquad \text{by 32.}$$

On the other hand
$$\phi(x) = \int_{\mathfrak{C}} f = +\infty,$$

for each point of $\mathfrak{B}$. Thus the integrals 2) are not defined.

To provide for the case that ϕ may not be defined for certain points of $\mathfrak{B}$ we give the symbol 2) the following definition.

$$\overline{\underline{\int}}_{\mathfrak{B}}\overline{\underline{\int}}_{\mathfrak{C}} f = \lim_{\alpha,\beta=\infty} \overline{\underline{\int}}_{\mathfrak{B}_{\alpha\beta}}\overline{\underline{\int}}_{\Gamma} f, \tag{5}$$

where $\Gamma = \mathfrak{C}$ when the integral 3) is convergent, or in the contrary case Γ is such a part of $\mathfrak{C}$ that

$$-\alpha \leq \overline{\underline{\int}}_{\Gamma} f \leq \beta, \tag{6}$$

and such that the integral in 6) is numerically as large as 6) will permit.

Sometimes it is convenient to denote Γ more specifically by $\Gamma_{\alpha\beta}$.

The points $\mathfrak{B}_{\alpha\beta}$ are the points of $\mathfrak{B}$ at which 6) holds. It will be noticed that each $\mathfrak{B}_{\alpha\beta}$ in 5) contains all the points of $\mathfrak{B}$ where the integral 3) is *not* convergent. Thus

$$\mathfrak{B} = U\{\mathfrak{B}_{\alpha\beta}\}.$$

Hence when $\mathfrak{B}$ is complete or metric,

$$\lim_{\alpha,\beta=\infty} \overline{\mathfrak{B}}_{\alpha\beta} = \overline{\mathfrak{B}}. \tag{7}$$

Before going farther it will aid the reader to consider a few examples.

71. *Example 1.* Let $\mathfrak{A}$ be as in the example in 70, 2, while $f = n^2$ at $x = \frac{m}{n}$. We see that

$$\int_{\mathfrak{A}} f = 0. \tag{1}$$

On the other hand $\mathfrak{B}_{\alpha\beta}$ contains but a finite number of points for any α, β. Thus

$$\int_{\mathfrak{B}}\int_{\mathfrak{C}} f = 0. \tag{2}$$

Thus the two integrals 1), 2) exist and are equal.

Example 2. The fact that the integrals in Ex. 1 vanish may lead the reader to depreciate the value of an example of this kind. This would be unfortunate, as it is easy to modify the function so that these integrals do not vanish.

Let $\mathfrak{A}$ denote all the points of the unit square. Let us denote the discrete point set used in Ex. 1 by $\mathfrak{D}$. We define f now as follows: f shall have in $\mathfrak{D}$ the values assigned to it at these points in Ex. 1. At the other points $A = \mathfrak{A} - \mathfrak{D}$, f shall have the value 1. Then

$$\int_{\mathfrak{A}} = \int_A + \int_{\mathfrak{D}} = \int_A = 1. \qquad (3$$

On the other hand $\mathfrak{B}_{\alpha\beta}$ consists of the irrational points in $\mathfrak{B}$ and a finite number of other points. Thus

$$\int_{\mathfrak{B}}\int_{\mathfrak{C}} f = 1. \qquad (4$$

Hence again the two 3), 4) exist and are equal.

Let us look at the results we get if we use integrals of types I and II. We will denote them by C and V as in 62.

We see at once that

$$C_{\mathfrak{A}} = V_{\mathfrak{A}} = P_{\mathfrak{A}} = 1.$$

Let us now calculate the iterated integrals

$$C_{\mathfrak{B}} C_{\mathfrak{C}}, \qquad (5$$

and

$$V_{\mathfrak{B}} V_{\mathfrak{C}}. \qquad (6$$

We observe that

$$C_{\mathfrak{C}} = 1 \quad \text{for } x \text{ irrational}$$

$$= +\infty \quad \text{for } x \text{ rational.}$$

Thus the integral 5) either is not defined at all since the field $\mathfrak{B}_\delta$ does not exist, or if we interpret the definition as liberally as possible, its value is 0. In neither case is

$$C_{\mathfrak{A}} = C_{\mathfrak{B}} C_{\mathfrak{C}}.$$

Let us now look at the integral 6). We see at once that

$$V_{\mathfrak{B}} V_{\mathfrak{C}}$$

does not exist, as $V_{\mathfrak{C}} = 1$ for rational x, and $= +\infty$ for irrational x. On the other hand

$$\underline{V}_{\mathfrak{B}} V_{\mathfrak{C}} = 1 \quad , \quad \overline{V}_{\mathfrak{B}} V_{\mathfrak{C}} = +\infty.$$

Hence in this case

$$V_{\mathfrak{A}} = \underline{V}_{\mathfrak{B}} V_{\mathfrak{C}}.$$

Example 3. Let $\mathfrak{A}$ be the unit square.

Let

$$f = n \quad \text{for } x = \frac{m}{n} \qquad n \text{ even}$$

$$= -n \quad \text{for } x = \frac{m}{n} \qquad n \text{ odd.}$$

At the other points of $\mathfrak{A}$ let $f = 1$.

Then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f = 1.$$

Here every point of $\mathfrak{A}$ is a point of infinite discontinuity and thus $\mathfrak{J} = \mathfrak{A}$.

Here $C_{\mathfrak{A}}$ is not defined, as $\mathfrak{A}_\delta$ does not exist; or giving the definition its most liberal interpretation,

$$C_{\mathfrak{A}} = 0.$$

The same remarks hold for $C_{\mathfrak{B}} C_{\mathfrak{C}}$.

On the other hand

$$\overline{V}_{\mathfrak{A}} = +\infty,$$

while

$$V_{\mathfrak{B}} V_{\mathfrak{C}}$$

does not exist, since

$$V_{\mathfrak{C}} = \pm n \quad \text{for } x = \frac{m}{n}$$

$$= 1 \text{ for irrational } x.$$

Moreover

$$\underline{V}_{\mathfrak{B}} V_{\mathfrak{C}} = -\infty \quad , \quad \overline{V}_{\mathfrak{B}} V_{\mathfrak{C}} = +\infty.$$

Example 4. Let $\mathfrak{A}$ denote the unit square. Let

$$f = n^2 \quad \text{for } x = \frac{m}{n}, \quad n \text{ even}, \ 0 \leq y \leq \frac{1}{n}$$

$$= -n^2 \quad \text{for } x = \frac{m}{n}, \quad n \text{ odd}, \ 0 \leq y \leq \frac{1}{n}.$$

At the other points of $\mathfrak{A}$ let $f = 1$.

Then

$$\int_{\mathfrak{A}} f = 1 = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f.$$

Let us look at the corresponding C and V integrals.

We see at once that

$$C_{\mathfrak{A}} = V_{\mathfrak{A}} = 1.$$

Again the integral $C_{\mathfrak{B}} C_{\mathfrak{C}}$ does not exist, or on a liberal interpretation it has the value 0. Also in this example

$$\underline{C}_{\mathfrak{B}} C_{\mathfrak{C}} \text{ and } \overline{C}_{\mathfrak{B}} C_{\mathfrak{C}}$$

do not exist or on a liberal interpretation, they $= 0$.

Turning to the V integrals we see that

$$\underline{V}_{\mathfrak{B}} V_{\mathfrak{C}} = -\infty \quad , \quad \overline{V}_{\mathfrak{B}} V_{\mathfrak{C}} = +\infty,$$

while $V_{\mathfrak{B}} V_{\mathfrak{C}}$ does not exist finite or infinite.

Example 5. Let our field of integration $\mathfrak{A}$ consist of the unit square considered in Ex. 4, let us call it $\mathfrak{E}$, and another similar square $\mathfrak{F}$, lying to its right. Let f be defined over $\mathfrak{E}$ as it was defined in Ex. 4, and let $f = 1$ in $\mathfrak{F}$.

Then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}} = 2.$$

Also

$$C_{\mathfrak{A}} = V_{\mathfrak{A}} = 2.$$

Then

$$C_{\mathfrak{B}} C_{\mathfrak{C}} = 1,$$

while

$$V_{\mathfrak{B}} V_{\mathfrak{C}} \text{ does not exist,}$$

and

$$\underline{V}_{\mathfrak{B}} V_{\mathfrak{C}} = -\infty \quad , \quad \overline{V}_{\mathfrak{B}} V_{\mathfrak{C}} = +\infty.$$

72. 1. In the following sections we shall restrict ourselves as follows:

1° $\mathfrak{A}$ shall be limited and iterable with respect to $\mathfrak{B}$.

2° $\mathfrak{B}$ shall be complete or metric.

3° The singular points $\mathfrak{J}$ of the integrand f shall be discrete.

2. Let us effect a sequence of superposed cubical divisions of space

$$D_1, \ D_2, \cdots$$

whose norms $d_n \doteq 0$.

Let $\mathfrak{A}_n = \mathfrak{B}_n \cdot \mathfrak{C}_n$ denote the points of $\mathfrak{A}$ lying in cells of D_n which contain no point of $\mathfrak{J}$. We observe that *we may always take without loss of generality*

$$\mathfrak{B}_n = \mathfrak{B}.$$

For let us adjoin to $\mathfrak{A}$ a discrete set $\mathfrak{D}$ lying at some distance from $\mathfrak{A}$ such that the projection of $\mathfrak{D}$ on $\Re_p$ is precisely $\mathfrak{B}$.

Let
$$A = \mathfrak{A} + \mathfrak{D} = \mathfrak{B} \cdot C \quad , \quad C = \mathfrak{C} + \mathfrak{c} \quad , \quad \bar{\mathfrak{c}} = 0.$$

We now set
$$\phi = f \text{ in } \mathfrak{A}$$
$$= 0 \text{ in } \mathfrak{D}.$$

Then
$$\overline{\int_A}\phi = \overline{\int_{\mathfrak{A}}}\phi + \overline{\int_{\mathfrak{D}}}\phi = \overline{\int_{\mathfrak{A}}}\phi$$
$$= \int_{\mathfrak{A}} f.$$

Similarly
$$\overline{\int_C}\phi = \overline{\int_{\mathfrak{C}}}\phi + \overline{\int_{\mathfrak{c}}}\phi = \overline{\int_{\mathfrak{C}}}\phi = \overline{\int_{\mathfrak{C}}} f.$$

Hence
$$\overline{\int_{\mathfrak{B}}}\overline{\int_C}\phi = \overline{\int_{\mathfrak{B}}}\overline{\int_{\mathfrak{C}}} f.$$

3. The set $\mathfrak{C}_n$ being as in 2, we shall write

$$\mathfrak{C} = \mathfrak{C}_n + \mathfrak{c}_n.$$

73. *Let $B_{\sigma, n}$ denote the points of $\mathfrak{B}$ at which $\bar{\mathfrak{c}}_n > \sigma$. Then if $\mathfrak{A}$ is iterable, with respect to $\mathfrak{B}$,*

$$\lim_{n=\infty} \bar{B}_{\sigma, n} = 0. \qquad (1$$

For since $\mathfrak{A}$ is iterable,

$$\bar{\mathfrak{A}} = \int_{\mathfrak{B}} \bar{\mathfrak{C}} \qquad \text{by definition.}$$

Hence $\mathfrak{C}$ considered as a function of x is an integrable function in $\mathfrak{B}$.

Similarly
$$\bar{\mathfrak{A}}_n = \int_{\mathfrak{B}} \bar{\mathfrak{C}}_n$$

and $\bar{\mathfrak{C}}_n$ is an integrable function in $\mathfrak{B}$.

We have now
$$\overline{\mathfrak{C}} = \overline{\mathfrak{C}}_n + \bar{\mathfrak{c}}_n \quad , \quad \bar{\mathfrak{c}}_n \geq 0$$

as $\mathfrak{C}_n$, $\mathfrak{c}_n$ are unmixed. Hence $\bar{\mathfrak{c}}_n$ is an integrable function in $\mathfrak{B}$. But
$$\overline{\mathfrak{A}} - \overline{\mathfrak{A}}_n = \int_{\mathfrak{B}} (\overline{\mathfrak{C}}_n + \bar{\mathfrak{c}}_n) - \int_{\mathfrak{B}} \overline{\mathfrak{C}}_n$$
$$= \int_{\mathfrak{B}} \bar{\mathfrak{c}}_n.$$

As the left side $\doteq 0$ as $n \doteq \infty$,
$$\lim \int_{\mathfrak{B}} \bar{\mathfrak{c}}_n = 0. \qquad (2$$

But
$$\int_{\mathfrak{B}} \bar{\mathfrak{c}}_n \geq \int_{B_{\sigma n}} \bar{\mathfrak{c}}_n \geq \sigma \overline{B}_{\sigma n}.$$

As the left side $\doteq 0$, we have for a given σ
$$\lim \overline{B}_{\sigma n} = 0,$$
which is 1).

74. *Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be iterable. Let the integral*
$$\overline{\underline{\int}}_{\mathfrak{C}} f \quad , \quad f \geq 0 \qquad (1$$
be convergent and limited in complete $\mathfrak{B}$. Let $\mathfrak{E}_n$ denote the points of $\mathfrak{B}$ at which
$$\overline{\underline{\int}}_{\mathfrak{c}_n} f \leq \epsilon. \qquad (2$$

Then
$$\lim_{n=\infty} \overline{\mathfrak{E}}_n = \overline{\mathfrak{B}}. \qquad (3$$

For let
$$\sigma_1 > \sigma_2 > \cdots \doteq 0.$$

Since $B_{\sigma,n} \doteq 0$ as $n \doteq \infty$ by 48, we may take ν_1 so large, and then a cubical division of $\mathfrak{R}_p$ of norm so small that those cells containing points of $B_{\sigma_1 \nu_1}$ have a content $< \eta/2$. Let the points of $\mathfrak{B}$ lying in these cells be called B_1, and let $\mathfrak{B}_1 = \mathfrak{B} - B_1$. Then B_1, $\mathfrak{B}_1$ form an unmixed division of $\mathfrak{B}$ and
$$\overline{\mathfrak{B}}_1 = \overline{\mathfrak{B}} - \overline{B}_1 > \overline{\mathfrak{B}} - \eta/2.$$

$\mathfrak{B}_1$ is complete since $\mathfrak{B}$ is.

We may now reason on $\mathfrak{B}_1$ as we did on $\mathfrak{B}$, replacing $\eta/2$ by $\eta/2^2$. We get a complete set $\mathfrak{B}_2 \leq \mathfrak{B}_1$ such that

$$\overline{\mathfrak{B}}_2 > \overline{\mathfrak{B}}_1 - \eta/2^2.$$

Continuing we get
$$\overline{\mathfrak{B}}_n > \mathfrak{B}_{n-1} - \eta/2^n.$$

Thus
$$\overline{\mathfrak{B}}_n > \overline{\mathfrak{B}} - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n}\right)\eta$$
$$> \overline{\mathfrak{B}} - \eta.$$

Let now
$$\mathfrak{b} = Dv\{\mathfrak{B}_n\}.$$

Then
$$\overline{\mathfrak{b}} \geq \mathfrak{B} - \eta \tag{4}$$
by 25.

Let $\mathfrak{b}_n$ denote those points of $\mathfrak{b}$ for which 2) does hold. Then $\mathfrak{b} = \{\mathfrak{b}_n\}$. For let b be any point of $\mathfrak{b}$. Since 1) is convergent, there exists a σ_ι such that

$$\text{at } b, \qquad \overline{\int_{\mathfrak{c}}} f \leq \epsilon,$$

for any $\mathfrak{c}$ such that $\bar{\mathfrak{c}} \leq \sigma_\iota$. Thus b is a point of $\mathfrak{b}_{\nu_\iota}$ and hence of $\{\mathfrak{b}_n\}$. Thus $\overline{\mathfrak{b}}_n \doteq \overline{\mathfrak{b}}$ as $\mathfrak{b}$ is complete. But $\mathfrak{C}_n \geq \mathfrak{b}_n$.

Hence
$$\lim \overline{\mathfrak{C}}_n \geq \overline{\mathfrak{b}},$$
which with 4), gives 3).

75. *Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be iterable. Let the integral*

$$\overline{\int_{\mathfrak{C}}} f \qquad f \geq 0$$

be convergent and limited in complete $\mathfrak{B}$.

Then
$$\lim_{n=\infty} \underline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}_n}} f = 0. \tag{1}$$

For let D be a cubical division of $\mathfrak{R}_p$ of norm d.

Then
$$\underline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}_n}} f = \lim_{d=0} \Sigma d_\iota \operatorname*{Min}_\iota \overline{\int_{\mathfrak{C}_n}} f = \lim_{d=0} S_d.$$

Let d_ι' denote those cells of D containing a point of $\mathfrak{C}_n$ where $\mathfrak{C}_n$ is defined as in 74.

Let d_ι'' denote the other cells containing points of $\mathfrak{B}$. Then

$$S_d \overline{<} \Sigma d_\iota' \epsilon + \Sigma d_\iota'' M,$$

where

$$0 \leq \overline{\int_{\mathfrak{C}}} f \leq M.$$

Hence

$$S_d \leq \epsilon \overline{\mathfrak{B}}_D + M(\overline{\mathfrak{B}}_D - \mathfrak{C}_{n,D}).$$

Letting $d \doteq 0$, we get

$$\int_{\mathfrak{B}} \overline{\int_{\mathfrak{C}_n}} f \leq \epsilon \overline{\mathfrak{B}} + M(\overline{\mathfrak{B}} - \overline{\mathfrak{C}}_n).$$

Letting now $n \doteq \infty$ and using 3) of 74, we get 1), since ϵ is small at pleasure.

76. *Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be iterable with respect to $\mathfrak{B}$, which last is complete or metric. Let the singular points $\mathfrak{J}$ of f be discrete. Then*

$$\text{if,} \quad f \geq -G \quad , \quad \underline{\int_{\mathfrak{A}}} f \leq \underline{\int_{\mathfrak{B}}} \underline{\overline{\int_{\mathfrak{C}}}} f \leq \int_{\mathfrak{A}} f. \tag{1}$$

$$\text{if,} \quad f \leq G \quad , \quad \underline{\int_{\mathfrak{A}}} f \leq \overline{\int_{\mathfrak{B}}} \underline{\overline{\int_{\mathfrak{C}}}} f \leq \overline{\int_{\mathfrak{A}}} f. \tag{2}$$

Here any one of the members in 1) *may be infinite. Then all that follow are also infinite. A similar remark applies to* 2).

Let us *first* suppose:

$$f \geq 0 \quad , \quad \mathfrak{B} \text{ is complete} \quad , \quad \underline{\int_{\mathfrak{B}}} \underline{\overline{\int_{\mathfrak{C}}}} f \text{ is convergent.}$$

We have by 14,

$$\underline{\int_{\mathfrak{A}_n}} f \leq \underline{\overline{\int_{\mathfrak{B}}}} \underline{\overline{\int_{\mathfrak{C}_n}}} f \leq \overline{\int_{\mathfrak{A}_n}} f.$$

Passing to the limit gives

$$\underline{\int_{\mathfrak{A}}} f \leq \lim \underline{\int_{\mathfrak{B}}} \overline{\int_{\mathfrak{C}_n}} f. \tag{3}$$

and also

$$\lim \overline{\int_{\mathfrak{B}}} \underline{\overline{\int_{\mathfrak{C}_n}}} f \leq \overline{\int_{\mathfrak{A}}} f \quad , \quad \text{finite or infinite.} \tag{4}$$

Now $\epsilon > 0$ being small at pleasure, there exists a G_0 such that

$$-\epsilon + \underline{\int_{\mathfrak{B}}} \underline{\overline{\int_{\mathfrak{C}}}} f < \underline{\int_{\mathfrak{B}_G}} \underline{\overline{\int_{\Gamma}}} f \qquad 0 \leq G_0 \leq G. \tag{5}$$

But for a fixed n

$$\overline{\int}_{\mathfrak{C}} \quad \text{is limited in } \mathfrak{B}.$$

Hence for G_0 sufficiently large,

$$\overline{\int}_{\mathfrak{C}_n} f \leq \overline{\int}_{\Gamma} f \quad , \text{ at each point of } \mathfrak{B}, \qquad G_0 < G. \tag{6}$$

Then

$$\underline{\int}_{\mathfrak{B}_G} \overline{\int}_{\mathfrak{C}_n} \leq \underline{\int}_{\mathfrak{B}_G} \overline{\int}_{\Gamma} = \underline{\int}_{\mathfrak{B}_G} \left\{ \overline{\int}_{\Gamma_n} + \overline{\int}_{\gamma_n} \right\}, \tag{7}$$

where Γ_n, γ_n are points of Γ in $\mathfrak{C}_n$, $\mathfrak{c}_n$.

Hence

$$\underline{\int}_{\mathfrak{B}_G} \overline{\int}_{\Gamma} \leq \underline{\int}_{\mathfrak{B}_G} \overline{\int}_{\mathfrak{C}_n} + \underline{\int}_{\mathfrak{B}_G} \overline{\int}_{\gamma_n}. \tag{8}$$

Now $\mathfrak{B}_G$ may not be complete; if not let B_G be completed $\mathfrak{B}_G$. As $\mathfrak{B}$ is complete,

$$\underline{\int}_{\mathfrak{B}_G} \overline{\int}_{\gamma_n} f = \underline{\int}_{B_G} \overline{\int}_{\gamma_n} f.$$

We may therefore write 8), using 5)

$$-\epsilon + \underline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} \leq \overline{\int}_{\mathfrak{B}_G} \overline{\int}_{\mathfrak{C}_n} + \underline{\int}_{B_G} \overline{\int}_{\gamma_n} \leq \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}_n} + \underline{\int}_{B_G} \overline{\int}_{\gamma_n}.$$

By 75, the last term on the right $\doteq 0$ as $n \doteq \infty$. Thus passing to the limit,

$$\underline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f \leq \lim_{n=\infty} \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}_n} f, \tag{9}$$

since $\epsilon > 0$ is small at pleasure.

On the other hand, passing to the limit $G = \infty$ in 7), and then $n = \infty$, we get

$$\lim_{n=\infty} \underline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}_n} \leq \underline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}}. \tag{10}$$

Thus 3), 10), 9), and 4) give 1).

Let us now suppose that the middle term of 1) is divergent. We have as before

$$\underline{\int}_{\mathfrak{B}_G} \underline{\int}_{\Gamma} \leq \lim_{n=\infty} \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}_n} \leq \overline{\int}_{\mathfrak{A}} f.$$

Hence the integral on the right of 1) is divergent.

Let us now suppose $\mathfrak{B}$ is metric. We effect a cubical division of $\mathfrak{R}_p$ of norm d, and denote by B_d those cells containing only points of $\mathfrak{B}$. Then B_d is complete and

$$\lim_{d=0} \overline{B}_d = \overline{\mathfrak{B}}.$$

Let A_d denote those points of $\mathfrak{A}$ whose projections fall on B_d. Then A_d is iterable with respect to B_d by 13, 3, and we have as in the preceding case

$$\underline{\int}_{A_d} \leq \underline{\int}_{B_d} \overline{\int}_{\mathfrak{C}} \leq \overline{\int}_{A_d}. \tag{11}$$

If the middle integral in 11) is divergent, $\overline{\int}_{\mathfrak{A}}$ is divergent and 1) holds, also if the last integral in 11) is divergent, 1) holds. Suppose then that the two last integrals in 11) are convergent. Then by 57

$$\lim_{d=0} \underline{\int}_{Bd} \overline{\int}_{\mathfrak{C}} = \underline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}}.$$

$$\lim_{d=0} \overline{\int}_{Ad} = \overline{\int}_{\mathfrak{A}}.$$

Thus passing to the limit $d = 0$ in 11) we get 1).

Let us now suppose $f \geq -G$, $G > 0$.

Then

$$g = f + G \geq 0,$$

and we can apply 1) to the new function g.

Thus

$$\underline{\int}_{\mathfrak{A}} g = \underline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} g \leq \overline{\int}_{\mathfrak{A}} g. \tag{12}$$

Now

$$\overline{\underline{\int}}_{\mathfrak{A}} g = \overline{\underline{\int}}_{\mathfrak{A}} f + G\overline{\mathfrak{A}}, \tag{13}$$

by 58, 5, since $\mathfrak{Z}$ is discrete. Also by the same theorem,

$$\overline{\underline{\int}}_{\mathfrak{C}} g = \overline{\underline{\int}}_{\mathfrak{C}} f + G \lim_{\gamma=\infty} \overline{\mathfrak{C}}_\gamma = \overline{\underline{\int}}_{\mathfrak{C}} f + G\Gamma, \tag{14}$$

denoting by $\mathfrak{C}_\gamma$ the points of $\mathfrak{C}$ where

$$-G \leq f \leq \gamma$$

and setting

$$\Gamma = \lim_{\gamma=\infty} \overline{\mathfrak{C}}_\gamma.$$

Now for any n

$$\underline{\int}_{\mathfrak{A}_n} G \leq \overline{\int}_{\mathfrak{B}} \int_{\mathfrak{C}_n} G \leq \overline{\int}_{\mathfrak{A}_n} G.$$

Hence
$$G\overline{\mathfrak{A}} = \lim_{n=\infty} \overline{\int}_{\mathfrak{B}} \int_{\mathfrak{C}_n} G = G \lim_{n=\infty} \overline{\int}_{\mathfrak{B}} \overline{\mathfrak{C}}_n,$$

or
$$\overline{\mathfrak{A}} = \lim_{n=\infty} \overline{\int}_{\mathfrak{B}} \overline{\mathfrak{C}}_n. \tag{15}$$

Now for a fixed n, γ may be taken so large that for all points of $\mathfrak{B}$,

$$\overline{\mathfrak{C}}_\gamma \geq \overline{\mathfrak{C}}_n.$$

Hence
$$\overline{\mathfrak{C}} \geq \lim_{\gamma=\infty} \overline{\mathfrak{C}}_\gamma > \overline{\mathfrak{C}}_n.$$

Hence
$$\overline{\mathfrak{A}} = \overline{\int}_{\mathfrak{B}} \overline{\mathfrak{C}} \gtrless \overline{\int}_{\mathfrak{B}} \Gamma \gtrless \overline{\int}_{\mathfrak{B}} \overline{\mathfrak{C}}_n \doteq \overline{\mathfrak{A}}.$$

Hence
$$\overline{\mathfrak{A}} = \overline{\int}_{\mathfrak{B}} \Gamma, \tag{16}$$

and thus Γ is integrable in $\mathfrak{B}$.

This result in 14) gives, on using 58, 3,

$$\underline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} g = \underline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f + G\overline{\mathfrak{A}}. \tag{17}$$

From 12), 13), and 17) follows 1).

77. As corollaries of the last theorem we have, *supposing* $\mathfrak{A}$ *to be as in* 76,

1. *If f is integrable in $\mathfrak{A}$ and $f \geq -G$, then*

$$\int_{\mathfrak{A}} f = \underline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f.$$

If $f \leq G$, then
$$\int_{\mathfrak{A}} f = \overline{\int}_{\mathfrak{B}} \overline{\int}_{\mathfrak{C}} f.$$

2. *If $f \geq -G$ and $\underline{\int}_{\mathfrak{A}}$ is divergent, then*

$$\overline{\underline{\int}}_{\mathfrak{B}} \overline{\underline{\int}}_{\mathfrak{C}} f$$

are divergent.

3. *If $f \geq -G$ and one of the integrals* $\underline{\int_{\mathfrak{B}}} \underline{\int_{\mathfrak{C}}} f$ *is convergent, then*

$$\underline{\int_{\mathfrak{A}}} f$$

is convergent.

78. *Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be iterable with respect to $\mathfrak{B}$, which last is complete or metric. Let the singular points $\mathfrak{J}$ be discrete. If*

$$\int_{\mathfrak{A}} f, \tag{1}$$

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}} f, \tag{2}$$

both converge, they are equal.

For let $D_1, D_2 \cdots$ be a sequence of superimposed cubical divisions as in 72, 2. We may suppose as before that each $\mathfrak{B}_n = \mathfrak{B}$.

Since 1) is convergent

$$\epsilon > 0, \qquad n_0, \qquad \left| \int_{\mathfrak{A}} f - \int_{\mathfrak{A}_n} f \right| < \frac{\epsilon}{2} \qquad n < n_0. \tag{3}$$

Since f is limited in $\mathfrak{A}_n$, which latter is iterable,

$$\int_{\mathfrak{A}_n} f = \overline{\underline{\int_{\mathfrak{B}}}} \int_{\mathfrak{C}_n} f. \tag{4}$$

This shows that

$$\int_{\mathfrak{C}_n} f \tag{5}$$

is an integrable function in $\mathfrak{B}$, and hence in any part of $\mathfrak{B}$.

From 3), 4) we have

$$\left| \int_{\mathfrak{A}} - \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} \right| \leq \frac{\epsilon}{2} \qquad n > n_0. \tag{6}$$

We wish now to show that

$$\left| \int_{\mathfrak{B}} \int_{\mathfrak{C}} - \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} \right| < \frac{\epsilon}{2} \qquad n > n_0. \tag{7}$$

When this is done, 6) and 7) prove the theorem.

To establish 7) we begin by observing that

$$\int_{\mathfrak{B}} \int_{\mathfrak{C}} = \lim_{\alpha, \beta = \infty} \int_{\mathfrak{B}_{\alpha\beta}} \int_{\Gamma}.$$

Now for a fixed n, α, β may be taken so that Γ shall embrace all the points of $\mathfrak{C}_n$ for every point of $\mathfrak{B}$. Let us set

$$\Gamma = \mathfrak{C}_n + \gamma_n.$$

Then

$$\int_{\mathfrak{B}_{\alpha\beta}}\int_{\Gamma} = \int_{\mathfrak{B}_{\alpha\beta}}\int_{\mathfrak{C}_n} + \int_{\mathfrak{B}_{\alpha\beta}}\int_{\gamma_n}. \tag{8}$$

As

$$\overline{\mathfrak{B}}_{\alpha\beta} \doteq \overline{\mathfrak{B}},$$

$$\lim_{\alpha,\beta=\infty}\int_{\mathfrak{B}_{\alpha\beta}}\int_{\mathfrak{C}_n} = \int_{\mathfrak{B}}\int_{\mathfrak{C}_n} \qquad \text{by I, 724.}$$

On the other hand,

$$\left|\int_{\mathfrak{B}_{\alpha\beta}}\int_{\gamma_n} f\right| \leq \int_{\mathfrak{B}_{\alpha\beta}}\left|\int_{\gamma_n} f\right| \leq \underline{\int}_{\mathfrak{B}_{\alpha\beta}}\int_{\gamma_n}|f| \leq \underline{\int}_{\mathfrak{B}}\int_{c_n}|f|.$$

Thus 7) is established when we show that

$$\underline{\int}_{\mathfrak{B}}\overline{\int}_{c_n}|f| < \frac{\epsilon}{2} \qquad n > n_0. \tag{9}$$

To this end we note that $|f|$ is integrable in $\mathfrak{A}$ by 48, 4. Hence by 77, 1,

$$\int_{\mathfrak{A}}|f| = \underline{\int}_{\mathfrak{B}}\overline{\int}_{\mathfrak{C}}|f|. \tag{10}$$

Also by I, 734,

$$\int_{\mathfrak{A}_n}|f| = \int_{\mathfrak{B}}\int_{\mathfrak{C}_n}|f|. \tag{11}$$

From 10), 11) we have for $n > n_0$,

$$\int_{\mathfrak{A}}|f| - \int_{\mathfrak{A}_n}|f| = \underline{\int}_{\mathfrak{B}}\overline{\int}_{\mathfrak{C}}|f| - \int_{\mathfrak{B}}\int_{\mathfrak{C}_n}|f| < \frac{\epsilon}{2}, \tag{12}$$

since the left side $\doteq 0$.

But as in 8)

$$\underline{\int}_{\mathfrak{B}_G}\overline{\int}_{\Gamma'}|f| = \int_{\mathfrak{B}_G}\overline{\int}_{\mathfrak{C}_n}|f| + \underline{\int}_{\mathfrak{B}_G}\overline{\int}_{\gamma_{n'}}|f|.$$

Passing to the limit $G = \infty$ gives

$$\underline{\int}_{\mathfrak{B}}\overline{\int}_{\mathfrak{C}}|f| = \int_{\mathfrak{B}}\int_{\mathfrak{C}_n}|f| + \underline{\int}_{\mathfrak{B}}\overline{\int}_{c_n}|f|.$$

This in 12) gives 9).

CHAPTER III

SERIES

Preliminary Definitions and Theorems

79. Let $a_1, a_2, a_3 \cdots$ be an infinite sequence of numbers.

The symbol
$$A = a_1 + a_2 + a_3 + \cdots \qquad (1$$
is called an *infinite series*. Let
$$A_n = a_1 + a_2 + \cdots + a_n. \qquad (2$$
If
$$\lim_{n=\infty} A_n \qquad (3$$
is finite, we say the series 1) is *convergent*. If the limit 3) is infinite or does not exist, we say 1) is *divergent*. When 1) is convergent, the limit 3) is called the *sum* of the series. It is customary to represent a series and its sum by the same letter, when no confusion will arise. Whenever practicable we shall adopt the following uniform notation. The terms of a series will be designated by small *Roman* letters, the series and its sum will be denoted by the corresponding capital letter. The *sum of the first n terms* of a series as A will be denoted by A_n. The infinite series formed by removing the first n terms, as for example,
$$a_{n+1} + a_{n+2} + a_{n+3} + \cdots \qquad (4$$
will be denoted by $\bar{A}_n$, and will be called the *remainder after n terms.*

The series formed by replacing each term of a series by its numerical value is called the *adjoint series*. We shall designate it by replacing the *Roman* letters by the corresponding Greek or German letters. Thus the adjoint of 1) would be denoted by
$$\mathfrak{A} = \alpha_1 + \alpha_2 + \alpha_3 + \cdots = \text{Adj } A \qquad (5$$
where
$$\alpha_n = |a_n|.$$

If all the terms of of a series are $\gtreqless 0$, it is identical with its adjoint.

A sum of p consecutive terms as

$$a_{n+1} + a_{n+2} + \cdots + {}_{n+p}$$

we denote by $A_{n,p}$.

Let

$$B = a_{\iota_1} + a_{\iota_2} + a_{\iota_3} + \cdots \quad , \quad \iota_1 < \iota_2 < \cdots$$

be the series obtained from A by omitting all its terms that vanish. Then A and B converge or diverge simultaneously, and when convergent they have the same sum.

For

$$B_n = A_{\iota_n}.$$

Thus if the limit on either side exists, the limit of the other side exists and both are equal.

This shows that in an infinite series we may omit its zero terms without affecting its character or value. We shall suppose this done unless the contrary is stated.

A series whose terms are all > 0 we shall call a *positive term series;* similarly if its terms are all < 0, we call it a *negative term series.* If $a_n > 0$, $n > m$ we shall say the series is *essentially a positive term series.* Similarly if $a_n < 0$, $n > m$ we call it an *essentially negative term series.*

If A is an essentially positive term series and divergent, $\lim A_n = +\infty$; if it is an essentially negative term series and divergent, $\lim A_n = -\infty$.

When $\lim A_n = \pm\infty$, we sometimes say A is $\pm\infty$.

80. 1. *For A to converge, it is necessary and sufficient that*

$$\epsilon > 0, \;\; m, \;\; |A_{n,p}| < \epsilon, \;\; n > m, \;\; p = 1, 2, \cdots \tag{1}$$

For the necessary and sufficient condition that

$$\lim_{n=\infty} A_n$$

exists is

$$\epsilon > 0, \;\; m, \;\; |A_\nu - A_n| < \epsilon, \;\; \nu, \;\; n > m. \tag{2}$$

But if $\nu = n + p$

$$A_\nu - A_n = A_{n,p} = a_{n+1} + a_{n+2} + \cdots + a_{n+p}.$$

Thus 2) is identical with 1).

2. *The two series A, $\bar{A}_s$ converge and diverge simultaneously. When convergent,*

$$A = A_s + \bar{A}_s. \tag{3}$$

For obviously if either series satisfies theorem 1, the other must, since the first terms of a series do not enter the relation 1). On the other hand,

$$A_{s+p} = A_s + A_{s,p}.$$

Letting $p \doteq \infty$ we get 3).

3. *If A is convergent, $\bar{A}_n \doteq 0$.*

For

$$\begin{aligned}\lim \bar{A}_n &= \lim (A - A_n) \\ &= A - \lim A_n = A - A \\ &= 0.\end{aligned}$$

For A to converge it is necessary that $a_n \doteq 0$.

For in 1) take $p = 1$; it becomes

$$|a_{n+1}| < \epsilon \qquad n > m$$

We cannot infer conversely because $a_n \doteq 0$, therefore A is convergent. For as we shall see in 81, 2,

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots$$

is divergent, yet $\lim a_n = 0$.

4. *The positive term series A is convergent if A_n is limited.*

For then $\lim A_n$ exists by I, 109.

5. *A series whose adjoint converges is convergent.*

For the adjoint A of A being convergent,

$$\epsilon > 0, \quad m, \quad |\mathrm{A}_{n,p}| < \epsilon, \quad n > m, \quad p = 1, 2, 3 \cdots$$

But

$$\mathrm{A}_{n,p} = \alpha_{n+1} + \alpha_{n+2} + \cdots + \alpha_{n+p} \geq |a_{n+1} + a_{n+2} + \cdots + a_{n+p}| = |A_{n,p}|.$$

Thus

$$|A_{n,p}| < \epsilon$$

and A is convergent.

Definition. A series whose adjoint is convergent is called *absolutely convergent.*

Series which do not converge absolutely may be called, when necessary to emphasize this fact, *simply convergent.*

6. *Let* $$A = a_1 + a_2 + \cdots$$
be absolutely convergent.

Let $$B = a_{\iota_1} + a_{\iota_2} + \cdots \quad ; \quad \iota_1 < \iota_2 < \cdots$$
be any series whose terms are taken from A, *preserving their relative order. Then* B *is absolutely convergent and*

$$|B| \leq \mathrm{A}.$$

For
$$|B_m| \leq \mathrm{B}_m \leq \mathrm{A}_n < \mathrm{A}, \tag{1}$$

choosing n so large that A_n contains every term in B_m. Moreover for $m \geq$ some m', $\mathrm{A}_n - \mathrm{B}_m \geq$ some term of A. Thus passing to the limit in 1), the theorem is proved.

7. *Let* $A = a_1 + a_2 + \cdots$ *The series* $B = ka_1 + ka_2 + \cdots$, $k \neq 0$, *converges or diverges simultaneously with* A. *When convergent,*

$$B = kA.$$

For
$$B_n = kA_n.$$

We have now only to pass to the limit.

From this we see that a negative or an essentially negative term series can be converted into a positive or an essentially positive term series by multiplying its terms by $k = -1$.

8. *If* A *is simply convergent, the series* B *formed of its positive terms taken in the order they occur in* A, *and the series* C *formed of the negative terms, also taken in the order they occur in* A, *are both divergent.*

If B and C are convergent, so are B, Γ. Now

$$\mathrm{A}_n = \mathrm{B}_{n_1} + \Gamma_{n_2}, \qquad n = n_1 + n_2.$$

Hence A would be convergent, which is contrary to hypothesis. If only one of the series B, C is convergent, the relation

$$A_n = B_{n_1} + C_{n_2}$$

shows that A would be divergent, which is contrary to hypothesis.

9. The following theorem often affords a convenient means of estimating the remainder of an absolutely convergent series.

Let $A = a_1 + a_2 + \cdots$ *be an absolutely convergent series. Let* $B = b_1 + b_2 + \cdots$ *be a positive term convergent series whose sum is known either exactly or approximately. Then if* $|a_n| \leq b_n$, $n \geq m$

$$|\bar{A}_n| \leq \bar{B}_n < B.$$

For

$$\begin{aligned} |A_{n,p}| &\leq \alpha_{n+1} + \cdots + \alpha_{n+p} \\ &\leq b_{n+1} + \cdots + b_{n+p} \\ &< B_n < B. \end{aligned}$$

Letting $p \doteq \infty$ gives the theorem.

EXAMPLES

81. 1. *The geometric series* is defined by

$$G = 1 + g + g^2 + g^3 + \cdots \tag{1}$$

The geometric series is absolutely convergent when $|g| < 1$ *and divergent when* $|g| \overline{>} 1$. *When convergent,*

$$G = \frac{1}{1-g}. \tag{2}$$

When $g \neq 1$,

$$\frac{1}{1-g} = 1 + g + g^2 + \cdots + g^{n-1} + \frac{g^n}{1-g}.$$

Hence

$$G_n = \frac{1}{1-g} - \frac{g^n}{1-g}.$$

When $|g| < 1$, $\lim g^n = 0$, and then

$$\lim G_n = \frac{1}{1-g}.$$

When $|g| \geq 1$, $\lim g^n$ is not 0, and hence by 80, 3, G is not convergent.

2. The series $$H = 1 + \frac{1}{2^\mu} + \frac{1}{3^\mu} + \frac{1}{4^\mu} + \cdots \tag{3}$$

is called the *general harmonic series of exponent* μ. When $\mu=1$, it becomes

$$J=1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\cdots \tag{4}$$

the *harmonic series.* We show now that

The general harmonic series is convergent when $\mu>1$ *and is divergent for* $\mu\leq 1$.

Let $\mu>1$. Then

$$\frac{1}{2^\mu}+\frac{1}{3^\mu}<\frac{1}{2^\mu}+\frac{1}{2^\mu}<\frac{2}{2^\mu}=\frac{1}{2^{\mu-1}}=g \quad ; \quad g<1.$$

$$\frac{1}{4^\mu}+\frac{1}{5^\mu}+\frac{1}{6^\mu}+\frac{1}{7^\mu}<\frac{1}{4^\mu}+\frac{1}{4^\mu}+\frac{1}{4^\mu}+\frac{1}{4^\mu}=\frac{4}{4^\mu}=g^2.$$

$$\frac{1}{8^\mu}+\cdots+\frac{1}{15^\mu}<\frac{1}{8^\mu}+\frac{1}{8^\mu}+\cdots+\frac{1}{8^\mu}=\frac{8}{8^\mu}=g^3, \text{ etc.}$$

Let $n\leq 2^\nu$. Then

$$H_n<1+g+\cdots+g^\nu<\frac{1}{1-g}.$$

Thus $\lim H_n$ exists, by I, 109, and

$$H<\frac{1}{1-\dfrac{1}{2^{\mu-1}}}. \tag{5}$$

Let $\mu<1$. Then

$$\frac{1}{n^\mu}>\frac{1}{n}.$$

Thus 3) is divergent for $\mu<1$, if it is for $\mu=1$.

But we saw, I, 141, that

$$\lim J_n=\infty,$$

hence J is divergent.

It is sometimes useful to know that

$$\lim \frac{J_n}{\log n}=1. \tag{6}$$

In fact, by I, 180,

$$\lim\frac{J_n}{\log n}=\lim\frac{J_n-J_{n-1}}{\log n-\log(n-1)}=\lim\frac{\dfrac{1}{n}}{\log\left(\dfrac{n}{n-1}\right)}$$

$$=\lim\frac{1}{\log\left(1-\dfrac{1}{n}\right)^{-n}}=1.$$

Since * $n > \log n > l_2 n \cdots$ we have

$$\lim \frac{J_n}{n} = 0 \quad ; \quad \lim \frac{J_n}{l_r n} = \infty \quad , \quad r > 1. \tag{7}$$

Another useful relation is

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} > \log(n+1). \tag{8}$$

For $$\log(1+m) - \log m = \log\left(1 + \frac{1}{m}\right) < \frac{1}{m}.$$

Let $m = 1, 2 \cdots n$. If we add the resulting inequalities we get 8).

3. *Alternating Series.* This important class of series is defined as follows. Let $a_1 > a_2 > a_3 > \cdots \doteq 0$.

Then $$A = a_1 - a_2 + a_3 - a_4 + \cdots \tag{9}$$

whose signs are alternately positive and negative, is such a series.

The alternating series 9) *is convergent and*

$$|\bar{A}_n| < a_{n+1}. \tag{10}$$

For let $p > 3$. We have

$$A_{n,p} = (-1)^n \{a_{n+1} - a_{n+2} + \cdots (-1)^{p+1} a_{n+p}\}$$
$$= (-1)^n P.$$

If p is even,

$$P = (a_{n+1} - a_{n+2}) + \cdots + (a_{n+p-1} - a_{n+p}).$$

If p is odd,

$$P = (a_{n+1} - a_{n+2}) + \cdots + (a_{n+p-2} - a_{n+p-1}) + a_{n+p}.$$

Thus in both cases,

$$P > a_{n+1} - a_{n+2} > 0. \tag{11}$$

Again, if p is even,

$$P = a_{n+1} - (a_{n+2} - a_{n+3}) - \cdots - (a_{n+p-2} - a_{n+p-1}) - a_{n+p}.$$

* In I, 461, the symbol "lim" in the first relation should be replaced by $\overline{\underline{\lim}}$.

If p is odd,

$$P = a_{n+1} - (a_{n+2} - a_{n+3}) - \cdots - (a_{n+p-1} - a_{n+p}).$$

Thus in both cases,

$$P < a_{n+1} - (a_{n+2} - a_{n+3}) < a_{n+1}. \tag{12}$$

From 11), 12) we have

$$0 < a_{n+1} - a_{n+2} < |\overline{A}_{n,p}| < a_{n+1} - (a_{n+2} - a_{n+3}).$$

Hence passing to the limit $p = \infty$,

$$0 < |\overline{A}_n| < a_{n+1};$$

moreover,

$$a_{n+1} \doteq 0.$$

Example 1. The series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots \tag{13}$$

being alternating, is convergent. The adjoint series is

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots$$

which being the harmonic series is divergent. Thus 13) is an example of a convergent series which is not absolutely convergent.

Example 2. The series

$$A = \frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}-1} - \frac{1}{\sqrt{3}+1} + \cdots$$

is divergent, although its terms are alternately positive and negative, and $a_n \doteq 0$.

For

$$A_{2m} = \sum_{s=2}^{m} \left(\frac{1}{\sqrt{s}-1} - \frac{1}{\sqrt{s}+1} \right) = \sum \frac{2}{s-1}$$

$$= 2\left(1 + \frac{1}{2} + \cdots + \frac{1}{m-1}\right) \doteq \infty.$$

If now A were convergent,

$$\lim A_n = \lim A_{2m}$$

by I, 103, 2.

4. *Telescopic Series.* Such series are

$$A = (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \cdots$$

We note that

$$\begin{aligned} A_n &= (a_1 - a_2) + \cdots + (a_n - a_{n+1}) \\ &= a_1 - a_{n+1}. \end{aligned} \tag{14}$$

Thus the terms of any A_n cancelling out in pairs, A_n reduces to only two terms and so shuts up like a telescope.

The relation 14) gives us the theorem:

A telescopic series is convergent when and only when lim a_n *exists.*

Let

$$A = a_1 + a_2 + \cdots \text{ denote any series.}$$

Then

$$a_n = A_n - A_{n-1} \quad , \quad A_0 = 0.$$

Hence

$$A = (A_1 - A_0) + (A_2 - A_1) + (A_3 - A_2) + \cdots$$

This shows us that

Any series can be written as a telescopic series.

This fact, as we shall see, is of great value in studying the general theory of series.

Example 1.

$$\begin{aligned} A &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots \\ &= \sum_2^\infty \frac{1}{(n-1)n} = \sum \left(\frac{1}{n-1} - \frac{1}{n} \right). \end{aligned}$$

Thus A is a telescopic series and

$$A_n = 1 - \frac{1}{n} \doteq 1.$$

Example 2. Let $a_1, a_2, a_3, \cdots > 0$. Then

$$\begin{aligned} A &= \sum_1^\infty \frac{a_n}{(1 + a_1) \cdots (1 + a_n)} \\ &= \sum \left\{ \frac{1}{(1 + a_1) \cdots (1 + a_{n-1})} - \frac{1}{(1 + a_1) \cdots (1 + a_n)} \right\}, \qquad a_0 = 0, \end{aligned}$$

is telescopic. Thus

$$0 < A_n = 1 - \frac{1}{(1 + a_1) \cdots (1 + a_n)} < 1,$$

and A is convergent and ≤ 1.

Example 3.
$$A=\sum_1^\infty \frac{1}{(x+n-1)(x+n)} \qquad x\neq 0, -1, -2, \cdots$$
$$=\sum\left\{\frac{1}{x+n-1}-\frac{1}{x+n}\right\}$$
is telescopic.
$$A_n=\frac{1}{x}-\frac{1}{x+n}\doteq\frac{1}{x}.$$

82. *Dini's Series. Let $A=a_1+a_2+\cdots$ be a divergent positive term series. Then*
$$D=\frac{a_1}{A_1}+\frac{a_2}{A_2}+\frac{a_3}{A_3}+\cdots$$
is divergent.

For
$$D_{m,p}=\frac{a_{m+1}}{A_{m+1}}+\cdots+\frac{a_{m+p}}{A_{m+p}}$$
$$>\frac{1}{A_{m+p}}(a_{m+1}+\cdots+a_{m+p})$$
$$>\frac{A_{m,p}}{A_m+A_{m,p}}=1-\frac{A_m}{A_{m+p}}.$$

Letting m remain fixed and $p\doteq\infty$, we have $\bar{D}_m\geq 1$, since $A_{m+p}\doteq\infty$. Hence D is divergent.

Let
$$A=1+1+1+\cdots \qquad \text{Then } A_n=n.$$
Hence
$$D=\tfrac{1}{1}+\tfrac{1}{2}+\tfrac{1}{3}+\cdots \text{ is divergent.}$$
Let
$$A=1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots$$
Then
$$D=\frac{1}{1}+\frac{1}{2(1+\frac{1}{2})}+\frac{1}{3(1+\frac{1}{2}+\frac{1}{3})}+\cdots=\sum\frac{1}{nA_n}$$
is divergent, and hence, *a fortiori*,
$$\sum\frac{1}{nA_{n-1}}.$$
But
$$A_{n-1}>\log n.$$
Hence
$$\sum_2^\infty\frac{1}{n\log n}=\frac{1}{2\log 2}+\frac{1}{3\log 3}+\cdots$$
is divergent, as *Abel* first showed.

83. 1. *Abel's Series.*

An important class of series have the form

$$B = a_1 t_1 + a_2 t_2 + a_3 t_3 + \cdots \tag{1}$$

As Abel first showed how the convergence of certain types of these series could be established, they may be appropriately called in his honor. The reasoning depends on the simple identity (*Abel's identity*),

$$B_{n,p} = t_{n+1} A_{n,1} + t_{n+2}(A_{n,2} - A_{n,1}) + \cdots + t_{n+p}(A_{n,p} - A_{n,p-1})$$
$$= A_{n,1}(t_{n+1} - t_{n+2}) + \cdots + A_{n,p-1}(t_{n+p-1} - t_{n+p}) + t_{n+p} A_{n,p}, \tag{2}$$

where as usual $A_{n,m}$ is the sum of the first m terms of the remainder series $\overline{A}_n$. From this identity we have at once the following cases in which the series 1) converges.

2. *Let the series* $A = a_1 + a_2 + \cdots$ *and the series* $\Sigma |t_{n+1} - t_n|$ *converge. Let the* t_n *be limited. Then* $B = a_1 t_1 + a_2 t_2 + \cdots$ *converges.*

For since A is convergent, there exists an m such that

$$|A_{n,p}| < \epsilon ; \qquad n > m, \qquad p = 1, 2, 3 \cdots$$

Hence

$$|B_{n,p}| < \epsilon \{ |t_{n+1} - t_{n+2}| + |t_{n+2} - t_{n+3}| + \cdots + |t_{n+p}| \}.$$

3. *Let the series* $A = a_1 + a_2 + \cdots$ *converge. Let* $t_1, t_2, t_3 \cdots$ *be a limited monotone sequence. Then* B *is convergent.*

This is a corollary of 2.

4. *Let* $A = a_1 + a_2 + \cdots$ *be such that* $|A_n| < G$, $n = 1, 2, \cdots$ *Let* $\Sigma |t_{n+1} - t_n|$ *converge and* $t_n \doteq 0$. *Then* B *is convergent.*

For by hypothesis there exists an m such that

$$|t_{n+1} - t_{n+2}| + |t_{n+2} - t_{n+3}| + \cdots + |t_{n+p}| < \epsilon$$

for any $n > m$.

5. *Let* $|A_n| < G$ *and* $t_1 \geq t_2 \geq t_3 \geq \cdots \doteq 0$. *Then* B *is convergent.*

This is a special case of 4.

6. As an application of 5 we see the alternating series

$$B = t_1 - t_2 + t_3 - \cdots$$

is convergent. For as the A series we may take $A = 1 - 1 + 1 - 1 + \cdots$ as $|A_n| \leq 1$.

84. *Trigonometric Series.*

Series of this type are

$$C = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots \quad (1$$

$$S = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots \quad (2$$

As we see, they are special cases of Abel's series. Special cases of the series 1), 2) are

$$\Gamma = \tfrac{1}{2} + \cos x + \cos 2x + \cos 3x + \cdots \quad (3$$

$$\Sigma = \sin x + \sin 2x + \sin 3x + \cdots \quad (4$$

It is easy to find the sums Γ_n, Σ_n as follows. We have

$$2 \sin mx \sin \tfrac{1}{2} x = \cos \frac{2m-1}{2} x - \cos \frac{2m+1}{2} x.$$

Letting $m = 1, 2, \cdots n$ and adding, we get

$$2 \sin \tfrac{1}{2} x \cdot \Sigma_n = \cos \tfrac{1}{2} x - \cos \frac{2n+1}{2} x. \quad (5$$

Keeping x fixed and letting $n \doteq \infty$, we see Σ_n oscillates between fixed limits when $x \neq 0, \pm 2\pi, \cdots$

Thus Σ is divergent except when $x = 0, \pm \pi, \cdots$

Similarly we find when $x \neq 2m\pi$,

$$\Gamma_n = \frac{\sin (n - \frac{1}{2}) x}{2 \sin \frac{1}{2} x}. \quad (6$$

Hence for such values Γ_n oscillates between fixed limits. For the values $x = 2m\pi$ the equation 3) shows that $\Gamma_n \doteq +\infty$.

From the theorems 4, 5 we have at once now

If $\Sigma |a_{n+1} - a_n|$ converges and $a_n \doteq 0$, and hence in particular if $a_1 \geq a_2 \geq \cdots \doteq 0$, the series 1) converges for every x, and 2) converges for $x \neq 2m\pi$.

If in 3) we replace x by $x + \pi$, it goes over into

$$\Delta = \tfrac{1}{2} - \cos x + \cos 2x - \cos 3x + \cdots \quad (7$$

Thus Δ_n oscillates between fixed limits if $x \neq \pm (2m-1)\pi$, when $n \doteq \infty$. Thus

If $\Sigma |a_{n+1} + a_n|$ *converges and* $a_n \doteq 0$, *and hence in particular if* $a_1 \geq a_2 \geq \cdots \doteq 0$, *the series* $a_0 - a_1 \cos x + a_2 \cos 2x - a_3 \cos 3x + \cdots$ *converges for* $x \neq (2m-1)\pi$.

85. *Power Series.*

An extremely important class of series are those of the type

$$P = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots \tag{1}$$

called power series. Since P reduces to a_0 if we set $x = a$, we see that every power series converges for at least one point. On the other hand, there are power series which converge at but one point, *e.g.*

$$a_0 + 1!(x-a) + 2!(x-a)^2 + 3!(x-a)^3 + \cdots \tag{2}$$

For if $x \neq a$, $\lim n! \, |x-a|^n = \infty$, and thus 2) is divergent.

1. *If the power series* P *converges for* $x = b$, *it converges absolutely within*

$$D_\lambda(a) \quad , \quad \lambda = |a-b|.$$

If P *diverges for* $x = b$, *it diverges without* $D_\lambda(a)$.

Let us suppose first that P converges at b. Let x be a point in D_λ, and set $|x-a| = \xi$. Then the adjoint of P becomes for this point

$$\Pi = \alpha_0 + \alpha_1\xi + \alpha_2\xi^2 + \alpha_3\xi^3 + \cdots$$

$$= \alpha_0 + \alpha_1\lambda \cdot \frac{\xi}{\lambda} + \alpha_2\lambda^2 \cdot \left(\frac{\xi}{\lambda}\right)^2 + \alpha_3\lambda^3 \cdot \left(\frac{\xi}{\lambda}\right)^3 + \cdots$$

But

$$\lim \alpha_n\lambda^n = 0,$$

since series P is convergent for $x = b$.

Hence

$$\alpha_n\lambda^n < M \qquad n > 1, 2, \cdots$$

Thus

$$\Pi_n < M\left(1 + \frac{\xi}{\lambda} + \cdots + \frac{\xi^n}{\lambda^n}\right) < \frac{M}{1 - \frac{\xi}{\lambda}}$$

and Π is convergent.

If P diverges at $x = b$, it must diverge for all b' such that $|b'-a| > \lambda$. For if not, P would converge at b by what we have just proved, and this contradicts the hypothesis.

2. Thus we conclude that the set of points for which P converges form an interval $(a-\rho, a+\rho)$ about the point a, called the *interval of convergence*; ρ is called its norm. We say P is *developed about the point a.* When $a=0$, the series 1) takes on the simpler form

$$a_0 + a_1x + a_2x^2 + \cdots$$

which for many purposes is just as general as 1). We shall therefore employ it to simplify our equations.

We note that the geometric series is a simple case of a power series.

86. *Cauchy's Theorem on the Interval of Convergence.*

The norm ρ of the interval of convergence of the power series,

$$P = a_0 + a_1x + a_2x^2 + \cdots$$

is given by

$$\frac{1}{\rho} = \overline{\lim}\sqrt[n]{\alpha_n} \qquad \alpha_n = |a_n|.$$

We show Π diverges if $\xi > \rho$. For let

$$\frac{1}{\rho} > \beta > \frac{1}{\xi}.$$

Then by I, 338, 1, there exist an infinity of indices $\iota_1, \iota_2 \cdots$ for which

$$\sqrt[\iota_n]{\alpha_{\iota_n}} > \beta.$$

Hence

$$\alpha_{\iota_n} > \beta^{\iota_n},$$

and thus

$$\alpha_{\iota_n}\xi^{\iota_n} > (\xi\beta)^{\iota_n} > 1,$$

since $\xi\beta > 1$. Hence

$$\sum_n \alpha_{\iota_n}\xi^{\iota_n}$$

is divergent and therefore Π.

We show now that Π converges if $\xi < \rho$. For let

$$\xi < \frac{1}{\beta} < \rho.$$

Then there exist only a finite number of indices for which

$$\sqrt[n]{\alpha_n} > \beta.$$

Let m be the greatest of these indices. Then

$$\sqrt{\alpha_n} < \beta \qquad n > m.$$

Hence
$$\alpha_n \leq \beta^n,$$
and
$$\alpha_n \xi^n < (\beta\xi)^n.$$
Thus
$$\alpha_{m+1}\xi^{m+1} + \cdots \alpha_{m+p}\xi^{m+p} < (\beta\xi)^{m+1}\{1 + (\beta\xi) + \cdots + (\beta\xi)^{m+p-1}\}$$
$$< \frac{(\beta\xi)^{m+1}}{1-\beta\xi},$$
and Π is convergent.

Example 1.
$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Here
$$\sqrt[n]{\alpha_n} = \frac{1}{\sqrt[n]{n!}} \doteq 0 \qquad \text{by I, 185, 4.}$$

Hence $\rho = \infty$ and the series converges absolutely for every x.

Example 2.
$$\frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

Here
$$\sqrt[n]{\alpha_n} = \frac{1}{\sqrt[n]{n}} \doteq 1 \qquad \text{by I, 185, 3.}$$

Hence $\rho = 1$, and the series converges absolutely for $|x| < 1$.

Tests of Convergence for Positive Term Series

87. To determine whether a given positive term series
$$A = a_1 + a_2 + \cdots$$
is convergent or not, we may compare it with certain standard series whose convergence or divergence is known. Such comparisons enable us also to establish criteria of convergence of great usefulness.

We begin by noting the following theorem which sometimes proves useful.

1. *Let A, B be two series which differ only by a finite number of terms. Then they converge or diverge simultaneously.*

This follows at once from 80, 2. Hence if a series A whose convergence is under investigation has a certain property only

after the mth term, we may replace A by $\bar{A}_m$, which has this property from the start.

2. The *fundamental theorem* of comparison is the following:

Let $A = a_1 + a_2 + \cdots$, $B = b_1 + b_2 + \cdots$ be two positive term series. Let $r > 0$ denote a constant. If $a_n \leq rb_n$, A converges if B does and $A \leq rB$. If $a_n \geq rb_n$, A diverges if B does.

For on the first hypothesis

$$A_n \leq rB_n.$$

On the second hypothesis

$$A_n \geq rB_n.$$

The theorem follows on passing to the limit.

3. From 2 we have at once:

Let $A = a_1 + a_2 + \cdots$, $B = b_1 + b_2 + \cdots$ be two positive term series. Let r, s be positive constants. If

$$r \leq \frac{a_n}{b_n} \leq s \qquad n = 1, 2, \cdots$$

or if

$$\lim \frac{a_n}{b_n}$$

exists and is $\neq 0$, A and B converge or diverge simultaneously. If B converges and $\frac{a_n}{b_n} \doteq 0$, A also converges. If B diverges and $\frac{a_n}{b_n} \doteq \infty$, A also diverges.

4. *Let $A = a_1 + a_2 + \cdots$, $B = b_1 + b_2 + \cdots$ be positive term series. If B is convergent and*

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n} \qquad n = 1, 2, 3 \cdots$$

A converges. If B is divergent and

$$\frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n},$$

A diverges.

For on the first hypothesis

$$\frac{a_{n+1}}{b_{n+1}} \leq \frac{a_n}{b_n} \leq \cdots \leq \frac{a_1}{b_1}.$$

We may, therefore, apply 3. On the second hypothesis, we have

$$\frac{a_n}{b_n} \geq \frac{a_1}{b_1}$$

and we may again apply 3.

Example 1. $$A = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots$$

is convergent. For

$$a_n = \frac{1}{n \cdot \overline{n+1}} < \frac{1}{n^2}$$

and $\sum \frac{1}{n^2}$ is convergent. The series A was considered in 81, 4, Ex. 1.

Example 2. $A = e^{-x} \cos x + e^{-2x} \cos 2x + \cdots$
is absolutely convergent for $x > 0$.

For

$$|a_n| \leq \frac{1}{e^{nx}}$$

which is thus $\leq$ the nth term in the convergent geometric series

$$\frac{1}{e^x} + \frac{1}{e^{2x}} + \frac{1}{e^{3x}} + \cdots$$

Example 3. $$A = \sum \frac{1}{n} \log \frac{n+1}{n}$$

is convergent.

For

$$\log\left(1 + \frac{1}{n}\right) = \frac{1}{n} - \frac{\theta_n}{n^2} \qquad 0 < \theta_n < 1.$$

Hence

$$0 < a_n = \frac{1}{n^2}\left(1 - \frac{\theta_n}{n}\right) < \frac{1}{n^2}.$$

Thus A is comparable with the convergent series $\sum \frac{1}{n^2}$.

88. We proceed now to deduce various tests for convergence and divergence. One of the simplest is the following, obtained by comparison with the hyperharmonic series.

Let $A = a_1 + a_2 + \cdots$ be a positive term series. It is convergent if

$$\overline{\lim}\, a_n n^{\mu} < \infty \quad , \quad \mu > 1,$$

and divergent if

$$\underline{\lim}\, n a_n > 0.$$

For on the first hypothesis there exists, by I, 338, a constant $G>0$ such that

$$a_n < \frac{G}{n^\mu} \qquad n = 1, 2, \cdots$$

Thus each term of A is less than the corresponding term of the convergent series $G\sum \frac{1}{n^\mu}$.

On the second hypothesis there exists a constant c such that

$$a_n > \frac{c}{n} \qquad n = 1, 2, \cdots$$

and each term of A is greater than the corresponding term of the divergent series $c\sum \frac{1}{n}$.

Example 1. $$A = \sum \frac{1}{\log^m n} \qquad m > 0.$$

Here $$na_n = \frac{n}{\log^m n} \doteq +\infty, \quad \text{by I, 463.}$$

Hence A is divergent.

Example 2. $$A = \sum \frac{1}{n \log n}.$$

Here $$na_n = \frac{1}{\log n} \doteq 0.$$

Thus the theorem does not apply. The series is divergent by 82.

Example 3.

$$L = \Sigma l_n = \Sigma \log\left(1 + \frac{\mu}{n} + \frac{\theta_n}{n^r}\right), \quad r > 1,$$

where μ is a constant and $|\theta_n| < G$.

From I, 413, we have, setting $r = 1 + s$,

$$l_n = \frac{1}{n}\left(\mu + \frac{\theta_n}{n^s}\right) - \frac{\alpha_n}{n^2}\left(\mu + \frac{\theta_n}{n^s}\right)^2 \qquad 0 < \alpha_n < 1.$$

Hence $$nl_n \doteq \mu, \quad \text{if } \mu \neq 0,$$

and L is divergent. If $\mu > 0$, L is an essentially positive term series. Hence $L = +\infty$. If $\mu < 0$, $L = -\infty$.

Let $\mu = 0$. Then

$$l_n = \frac{\theta_n}{n^r}\left(1 - \frac{\alpha_n \theta_n}{n^r}\right) \qquad 0 < \alpha_n < 1,$$

which is comparable with the convergent series

$$\sum \frac{1}{n^r}\ , \quad r > 1.$$

Thus L is convergent in this case.

Example 4. The harmonic series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots$$

is divergent. For

$$\lim na_n = 1.$$

Example 5.

$$A = \sum \frac{1}{n^\alpha \log^\beta n} \qquad \beta \text{ arbitrary.}$$

Here

$$na_n = \frac{n^{1-\alpha}}{\log^\beta n} \doteq \infty \quad , \quad \alpha < 1$$

by I, 463, 1. Hence A is divergent for $\alpha < 1$.

Example 6.

$$A = \sum \frac{1}{\sqrt[n]{n^{n+1}}}.$$

Here

$$na_n = \frac{1}{\sqrt[n]{n}} \doteq 1 \qquad \text{by I, 185, Ex. 3.}$$

Example 7.

$$A = \sum \left(\frac{\log(n+1)}{\log n} - 1\right).$$

Here, if $\mu > 0$,

$$n^{1+\mu} a_n = n^{1+\mu} \frac{\log\left(1 + \frac{1}{n}\right)}{\log n} = \frac{n^\mu}{\log n} \log\left(1 + \frac{1}{n}\right)^n \doteq \infty,$$

since $n^\mu > \log n$ and $\left(1 + \frac{1}{n}\right)^n \doteq e$.

Hence A is divergent.

89. *D'Alembert's Test. The positive term series* $A = a_1 + a_2 + \cdots$ *converges if there exists a constant* $r < 1$ *for which*

$$\frac{a_{n+1}}{a_n} \leq r, \qquad n = 1, 2, \cdots$$

It diverges if

$$\frac{a_{n+1}}{a_n} \geq 1.$$

This follows from 87, 4, taking for B the geometric series $1 + r + r^2 + r^3 + \cdots$

Corollary. Let $\dfrac{a_{n+1}}{a_n} \doteq l$. *If* $l < 1$, A *converges. If* $l > 1$, *it diverges.*

Example 1. The Exponential Series.

Let us find for what values of x the series

$$E = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{1}$$

is convergent. Applying D'Alembert's test to its adjoint, we find

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^n}{n!} \cdot \frac{n-1!}{x^{n-1}}\right| = \frac{|x|}{n}.$$

Thus E converges absolutely for every x.

Let us employ 80, 9 to estimate the remainder $\overline{E}_n$. *Let* $x > 0$. The terms of E are all > 0. Since

$$\frac{x^{n+p}}{(n+p)!} = \frac{x^n}{n!} \cdot \frac{x^p}{n+1 \cdot n+2 \cdot \cdots n+p} \leq \frac{x^n}{n!}\left(\frac{x}{n+1}\right)^p,$$

we have

$$\frac{x^n}{n!} < \overline{E}_n < \frac{x^n}{n!} \sum_0^\infty \left(\frac{x}{n+1}\right)^p. \tag{2}$$

However large x may be, we may take n so large that $x < n + 1$. Then the series on the right of 2) is a convergent geometric series.

Let $x < 0$. Then however large $|x|$ is, $\overline{E}_n$ is alternating for some m. Hence by 81, 3 for $n \geq m$,

$$|\overline{E}_n| < \frac{|x|^n}{n!} \tag{3}$$

Example 2. *The Logarithmic Series.*

Let us find for what values of x the series

$$L = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

is convergent. The adjoint gives

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n}{n+1} \cdot |x| \doteq |x|.$$

Thus L converges absolutely for any $|x| < 1$, and diverges for $|x| > 1$.

When $x = 1$, L becomes

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

which is simply convergent by 81, 4.

When $x = -1$, L becomes

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots$$

which is the divergent harmonic series.

Example 3. $A = \frac{1}{1^\mu} + \frac{1}{2^\mu} + \frac{1}{3^\mu} + \cdots$

$$\frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1}\right)^\mu = 1.$$

As A is convergent when $\mu > 1$ and divergent if $\mu \leq 1$, we see that D'Alembert's test gives us no information when $l = 1$. It is, however, convergent for this case by 81, 2.

Example 4.

$$\sum_1^\infty \frac{n!}{(1+x)\cdots(n+x)} \qquad x > 0.$$

Here

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n+1+x} \doteq 1,$$

and D'Alembert's test does not apply.

Example 5.

$$A = \Sigma n^\mu x^n.$$

Here

$$\left| \frac{a_{n+1}}{a_n} \right| = \left(\frac{n+1}{n}\right)^\mu |x| \doteq |x|.$$

Thus A converges for $|x|<1$ and diverges for $|x|>1$. For $|x|=1$ the test does not apply. For $x=1$ we know by 81, 2 that A is convergent for $\mu<-1$, and is divergent for $\mu\geq-1$.

For $x=-1$, A is divergent for $\mu \geqq 0$, since a_n does not $\doteq 0$. A is an alternating series for $\mu<0$, and is then convergent.

90. *Cauchy's Radical Test. Let $A = a_1 + a_2 + \cdots$ be a positive term series. If there exists a constant $r<1$ such that*

$$\sqrt[n]{a_n} \leq r \qquad n = 1, 2, \cdots$$

A is convergent. If, on the other hand,

$$\sqrt[n]{a_n} \geq 1$$

A is divergent.

For on the first hypothesis,

$$a_n \leq r^n$$

so that each term of A is $\leq$ the corresponding term in $r + r^2 + r^3 + \cdots$ a convergent geometric series. On the second hypothesis, this geometric series is divergent and $a_n \geq r^n$.

Corollary. If $\lim \sqrt[n]{a_n} = l$, *and $l<1$, A is convergent. If $l>1$, A is divergent.*

Example 1. The series

$$\sum_2^\infty \frac{1}{\log^n n} = a_2 + a_3 + \cdots$$

is convergent. For

$$\sqrt[n]{a_n} = \frac{1}{\log n} \doteq 0.$$

Example 2.

$$A = \sum \frac{n^{n^2}}{(n+1)^{n^2}}$$

is convergent. For

$$\sqrt[n]{a_n} = \frac{n^n}{(n+1)^n} = \frac{1}{\left(1+\frac{1}{n}\right)^n} \doteq \frac{1}{e} < 1.$$

Example 3. In the elliptic functions we have to consider series of the type

$$\theta(v) = 1 + 2\sum_1^\infty q^{n^2} \cos 2\pi n v \qquad 0<q<1.$$

This series converges absolutely if

$$q + q^4 + q^9 + \cdots$$

does. But here

$$\sqrt[n]{a_n} = \sqrt[n]{q^{n^2}} = q^n \doteq 0.$$

Thus $\theta(v)$ converges absolutely for every v.

Example 4. Let $0 < a < b < 1$. The series

$$A = a + b^2 + a^3 + b^4 + \cdots$$

is convergent. For if

$$n = 2m$$

$$\sqrt[n]{a_n} = \sqrt[2m]{b^{2m}} = b.$$

If $n = 2m + 1$,

$$\sqrt[n]{a_n} = \sqrt[2m+1]{a^{2m+1}} = a.$$

Thus for all n

$$\sqrt[n]{a_n} \leq b < 1.$$

Let us apply D'Alembert's test. Here

$$\frac{a_{n+1}}{a_n} = b\left(\frac{b}{a}\right)^{2m-1} \doteq \infty \qquad n = 2m + 1,$$

$$= a\left(\frac{a}{b}\right)^{2m} \doteq 0 \qquad n = 2m.$$

Thus the test gives us no information.

91. *Cauchy's Integral Test.*

Let $\phi(x)$ be a positive monotone decreasing function in the interval (a, ∞). The series

$$\Phi = \phi(1) + \phi(2) + \phi(3) + \cdots$$

is convergent or divergent according as

$$\int_a^\infty \phi(x)\, dx$$

is convergent or divergent.

For in the interval $(n, n + 1)$, $\qquad n \geq m \geq a$,

$$\phi(n + 1) \leq \phi(x) \leq \phi(n).$$

Hence

$$\phi(n+1) \leq \int_n^{n+1} \phi dx \leq \phi(n).$$

Letting $n = m,\ m+1, \cdots m+p$, and adding, we have

$$\Phi_{m,\,p+1} \leq \int_m^{m+p} \phi dx \leq \Phi_{m-1,\,p+1}.$$

Passing to the limit $p = \infty$, we get

$$\overline{\Phi}_m \leq \int_m^{\infty} \phi dx \leq \overline{\Phi}_{m-1}, \tag{1}$$

which proves the theorem.

Corollary. *When Φ is convergent*

$$\overline{\Phi}_m \leq \int_m^{\infty} \phi dx.$$

Example 1. We can establish at once the results of 81, 2. For, taking $\phi(x) = \dfrac{1}{x^\mu}$,

$$\int_1^{\infty} \phi dx = \int_1^{\infty} \frac{dx}{x^\mu}$$

is convergent or divergent according as $\mu > 1$, or $\mu \leq 1$, by I, 635, 636.

We also note that if

$$A = \frac{1}{1^{1+\mu}} + \frac{1}{2^{1+\mu}} + \frac{1}{3^{1+\mu}} + \cdots$$

then

$$\overline{A} < \int_n^{\infty} \frac{1}{x^{1+\mu}} = \frac{1}{\mu} \cdot \frac{1}{n^\mu}.$$

Example 2. The *logarithmic series*

$$\sum_n \frac{1}{nl_1nl_2n \cdots l_{s-1}nl_s^\mu n} \qquad s = 1, 2, \cdots$$

for notation See I § 385 p 243.

are convergent if $\mu > 1$; divergent if $\mu \leq 1$.

We take here

$$\phi(x) = \frac{1}{xl_1x \cdots l_{s-1}xl_s^\mu x}$$

and apply I, 637, 638.

92. 1. One way, as already remarked, to determine whether a given positive term series $A = a_1 + a_2 + \cdots$ is convergent or divergent is to compare it with some series whose convergence or divergence is known. We have found up to the present the following *standard series S:*

The geometric series

$$1 + g + g^2 + \cdots \tag{1}$$

The general harmonic series

$$\frac{1}{1^\mu} + \frac{1}{2^\mu} + \frac{1}{3^\mu} + \cdots \tag{2}$$

The logarithmic series

$$\sum \frac{1}{nl_1^\mu n}, \tag{3}$$

$$\sum \frac{1}{nl_1 nl_2^\mu n}, \tag{4}$$

$$\sum \frac{1}{nl_1 nl_2 nl_3^\mu n} \tag{5}$$

. . . .

We notice that none of these series could be used to determine by comparison the convergence or divergence of the series following it.

In fact, let a_n, b_n denote respectively the nth terms in 1), 2). Then for $g < 1$, $\mu > 0$,

$$\frac{b_n}{a_{n+1}} = \frac{1}{n^\mu g^n} = \frac{e^{-n\log g}}{n^\mu} \doteq \infty \qquad \text{by I, 464,}$$

or using the *infinitary notation* of I, 461,

$$b_n > a_n.$$

Thus the terms of 2) converge to 0 infinitely slower than the terms of 1), so that it is useless to compare 2) with 1) for convergence. Let $g \geq 1$. Then

$$\frac{a_{n+1}}{b_n} = n^\mu g^n \doteq \infty,$$

or

$$a_n > b_n.$$

This shows we cannot compare 2) with 1) for divergence.

Again, if a_n, b_n denote the nth terms of 2), 3) respectively, we have, if $\mu > 1$,

$$\frac{b_n}{a_n} = \frac{n^{\mu-1}}{\log^\mu n} \doteq \infty \qquad \text{by I, 463,}$$

or

$$b_n > a_n.$$

If $\mu = 1$,

$$\frac{a_n}{b_n} = \log n \doteq \infty,$$

or

$$a_n > b_n.$$

Thus the convergence or divergence of 3) cannot be found from 2) by comparison. In the same way we may proceed with the others.

2. These considerations lead us to introduce the following notions. Let $A = a_1 + a_2 + \cdots$, $B = b_1 + b_2 + \cdots$ be positive term series. Instead of considering the behavior of a_n/b_n, let us generalize and consider the ratios $A_n : B_n$ for divergent and $\overline{A}_n : \overline{B}_n$ for convergent series. These ratios obviously afford us a measure of the rate at which A_n and B_n approach their limit. If now A, B are divergent and

$$A_n \sim B_n,$$

we say A, B diverge equally fast; if

$$A_n < B_n,$$

A diverges slower than B, and B diverges faster than A. From I, 180, we have:

Let A, B be divergent and

$$\lim \frac{a_n}{b_n} = l.$$

According as l is 0, $\neq 0$, ∞, *A diverges slower, equally fast, or faster than B.*

If A, B are convergent and

$$\overline{A}_n \sim \overline{B}_n,$$

we say A, B converge equally fast; if A converges and

$$\overline{B}_n < \overline{A}_n,$$

B converges faster than A, and A converges slower than B. From I, 184, we have:

Let A, B be convergent and

$$\lim \frac{a_n}{b_n} = l.$$

According as l is 0, $\neq 0$, ∞, A converges faster, equally fast, or slower than B.

Returning now to the set of standard series S, we see that each converges (diverges) slower than any preceding series of the set. Such a set may therefore appropriately be called a *scale* of convergent (divergent) series. Thus if we have a decreasing positive term series, whose convergence or divergence is to be ascertained, we may compare it successively with the scale S, until we arrive at one which converges or diverges equally fast. In practice such series may always be found. It is easy, however, to show that there exist series which converge or diverge slower than any series in the scale S or indeed any other scale.

For let

$$A, \quad B, \quad C, \cdots \tag{Σ}$$

be any scale of positive term convergent or divergent series.

Then, if convergent,

$$\bar{A}_n^{-1} > \bar{B}_n^{-1} > \bar{C}_n^{-1} > \cdots;$$

if divergent,

$$A_n > B_n > C_n > \cdots$$

Thus in both cases we are led to a sequence of functions of the type

$$f_1(n) > f_2(n) > f_3(n) > \cdots$$

Thus to show the existence of a series Ω which converges (diverges) slower than any series in Σ, we have only to prove the theorem:

3. (*Du Bois Reymond.*) *In the interval (a, ∞) let*

$$f_1(x) > f_2(x) > \cdots$$

denote a set of positive increasing functions which $\doteq \infty$ as $x \doteq \infty$. Moreover, let

$$f_1 > f_2 > f_3 > \cdots$$

Then there exist positive increasing functions which $\doteq \infty$ *slower than any* f_n.

For as $f_1 > f_2$ there exists an $a_1 > a$ such that $f_1 > f_2 + 1$ for $x \geq a_1$. Since $f_2 > f_3$, there exists an $a_2 > a_1$ such that $f_2 > f_3 + 2$ for $x \geq a_2$. And in general there exists an $a_n > a_{n-1}$ such that $f_n > f_{n+1} + n$ for $x > a_n$. Let now

$$g(x) = f_n(x) + n - 1 \qquad \text{in } (a_{n-1}, a_n).$$

Then g is an increasing unlimited function in (a, ∞) which finally remains below any $f_m(x) + m - 1$, m arbitrary but fixed.

Thus $$0 \leq \lim_{x=\infty} \frac{g(x)}{f_m(x)} = \lim \frac{g(x)}{f_m + m - 1} \leq \lim \frac{f_{m+1} + m}{f_m + m - 1} = 0.$$

Hence $g < f_m$.

93. From the logarithmic series we can derive a number of tests, for example, the following:

1. (*Bertram's Tests.*) *Let* $A = a_1 + a_2 + \cdots$ *be a positive term series.*

Let

$$Q_s(n) = \frac{\log \dfrac{1}{a_n n l_1 n \cdots l_{s-1} n}}{l_{s+1} n} \qquad s = 1, 2, \cdots \qquad l_0 n = 1.$$

If for some s *and* m,

$$Q_s(n) \geq \mu > 1 \qquad n \geq m, \tag{1}$$

A is convergent. If, however,

$$Q_s(n) \leq 1, \tag{2}$$

A is divergent.

For multiplying 1) by $l_{s+1}n$, we get

$$l_{s+1}n \cdot Q_s(n) \geq \mu l_{s+1} n,$$

or

$$\log \frac{1}{a_n n l_1 n \cdots l_{s-1} n} \geq \mu \log l_s n = \log l_s^\mu n.$$

Hence

$$\frac{1}{a_n n l_1 n \cdots l_{s-1} n} \geq l_s^\mu n,$$

or

$$a_n \leq \frac{1}{n l_1 n \cdots l_{s-1} l_s^\mu n}.$$

Thus A is convergent.

The rest of the theorem follows similarly.

2. *For the positive term series* $A = a_1 + a_2 + \cdots$ *to converge it is necessary that, for* $n = \infty$,

$$\underline{\lim}\, a_n = 0, \quad \underline{\lim}\, na_n = 0, \quad \underline{\lim}\, na_n l_1 n = 0, \quad \underline{\lim}\, na_n l_1 n l_2 n = 0, \cdots$$

We have already noted the first two. Suppose now that

$$\underline{\lim}\, na_n l_1 n \cdots l_s n > 0.$$

Then by I, 338 there exists an m and a $c > 0$, such that

$$na_n l_1 n \cdots l_s n > c \quad , \quad n > m,$$

or

$$a_n > \frac{c}{nl_1 n \cdots l_s n}.$$

Hence A diverges.

Example 1.
$$A = \sum \frac{1}{n^\alpha \log^\beta n}.$$

We saw, 88, Ex. 5, that A is divergent for $\alpha < 1$. For $\alpha = 1$, A is convergent for $\beta > 1$ and divergent if $\beta \gtreqless 1$, according to 91, Ex. 2.

If $\alpha > 1$, let
$$\alpha = \alpha' + \alpha'' \quad , \quad \alpha'' > 1.$$

Then if $\beta \geq 0$,
$$a_n = \frac{1}{n^\alpha \log^\beta n} \leq \frac{1}{n^\alpha} \quad , \quad n > 2,$$

and A is convergent since $\sum \frac{1}{n^\alpha}$ is. If $\beta < 0$, let

$$\beta = -\beta' \quad , \quad \beta' > 0.$$

Then
$$a_n = \frac{\log^{\beta'} n}{n^{\alpha'}} \cdot \frac{1}{n^{\alpha''}}.$$

But
$$\log^{\beta'} n < n^{\alpha'} \qquad \text{by I, 463, 1;}$$

and A is convergent since $\sum \frac{1}{n^{\alpha''}}$ is.

Example 2.
$$A = \sum \frac{1}{n^\mu e^{\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}} = \sum \frac{1}{n^\mu e^{H_n}}.$$

Here
$$Q_1 = \frac{\log \dfrac{1}{a_n n}}{l_2 n} = \frac{-\log n + \mu \log n + H_n}{l_2 n}$$
$$= \frac{\log n}{l_2 n}\left\{\mu - 1 + \frac{H_n}{\log n}\right\} \doteq \mu \infty \qquad \text{by 81, 6).}$$

Hence A is convergent for $\mu > 0$ and divergent for $\mu < 0$. No test for $\mu = 0$.

But for $\mu = 0$,
$$Q_2 = \frac{\log \dfrac{1}{a_n n l_1 n}}{l_3 n} = \frac{H_n - l_1 n - l_2 n}{l_3 n}$$
$$= \frac{l_1 n}{l_3 n}\left\{-1 - \frac{l_2 n}{l_1 n} + \frac{H_n}{l_1 n}\right\}$$
$$\doteq -\infty,$$
since $l_2 n > l_3 n$. Thus A is divergent for $\mu = 0$.

94. A very general criterion is due to *Kummer*, viz.:

Let $A = a_1 + a_2 + \cdots$ be a positive term series. Let $k_1, k_2, \cdots$ be a set of positive numbers chosen at pleasure. A is convergent, if for some constant $k > 0$.
$$K_n = k_n \frac{a_n}{a_{n+1}} - k_{n+1} \geq k \qquad n = 1, 2, \cdots$$
A is divergent if
$$R = \frac{1}{k_1} + \frac{1}{k_2} + \cdots$$
is divergent and
$$K_n \leq 0 \qquad n = 1, 2, \cdots$$

For on the first hypothesis
$$a_2 \leq \frac{1}{k}(k_1 a_1 - k_2 a_2)$$
$$a_3 \leq \frac{1}{k}(k_2 a_2 - k_3 a_3)$$
$$\cdots\cdots\cdots$$
$$a_n \leq \frac{1}{k}(k_{n-1} a_{n-1} - k_n a_n).$$

Hence adding,

$$0 < A_n \leq a_1 + \frac{1}{k}(k_1 a_1 - k_n a_n) < a_1\left(1 + \frac{k_1}{k}\right),$$

and A is convergent by 80, 4.

On the second hypothesis,

$$\frac{a_n}{a_{n+1}} \leq \frac{k_{n+1}}{k_n},$$

or

$$\frac{a_{n+1}}{a_n} \geq \frac{k_{n+1}^{-1}}{k_n^{-1}}.$$

Hence A diverges since R is divergent.

95. 1. From Kummer's test we may deduce D'Alembert's test at once. For take

$$k_1 = k_2 = \cdots = 1.$$

Then $A = a_1 + a_2 + \cdots$ converges if

$$K_n = \frac{a_n}{a_{n+1}} - 1 \geq k > 0,$$

i.e. if

$$\frac{a_{n+1}}{a_n} \leq \rho < 1.$$

Similarly A diverges if $\frac{a_{n+1}}{a_n} \geq 1$.

2. To derive Raabe's test we take

$$k_n = n.$$

Then A converges if

$$K_n = n\frac{a_n}{a_{n+1}} - (n+1) \geq k > 0,$$

i.e. if

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq l > 1.$$

Similarly A diverges if

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) \leq 1.$$

96. 1. *Let $A = a_1 + a_2 + \cdots$ be a positive term series. Let*

$$\lambda_0(n) = n\left(\frac{a_n}{a_{n+1}} - 1\right)$$

$$\lambda_1(n) = l_1 n \left\{ n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1 \right\}$$

$$\lambda_2(n) = l_2 n \left\{ l_1 n \left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1 \right] - 1 \right\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Then A converges if there exists an s such that

$$\lambda_s(n) \geq \delta > 1 \qquad \textit{for some } n > m;$$

it diverges if

$$\lambda_s(n) \leq 1 \qquad \textit{for } n > m.$$

We have already proved the theorem for $\lambda_0(n)$. Let us show how to prove it for $\lambda_1(n)$. The other cases follow similarly.

For the Kummer numbers k_n we take

$$k_n = n \log n.$$

Then A converges if

$$k_n = n \log n \cdot \frac{a_n}{a_{n+1}} - (n+1)\log(n+1) \geq k > 0.$$

As

$$n + 1 = n\left(1 + \frac{1}{n}\right),$$

$$K_n = \lambda_1(n) - \log\left(1 + \frac{1}{n}\right)^n - \log\left(1 + \frac{1}{n}\right)$$

$$= \lambda_1(n) - \log\left(1 + \frac{1}{n}\right)^{n+1}$$

$$= \lambda_1(n) - (1 + \alpha) \qquad \alpha > 0.$$

Thus A converges if $\lambda_1(n) \geq \delta > 1$ for $n > m$.

In this way we see that A diverges if $\lambda_1(n) \leq 1$, $n > m$.

2. *Cahen's Test. For the positive term series to converge it is necessary that*

$$\overline{\lim_{n=\infty}}\, n\left\{ n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1 \right\} = +\infty.$$

For if this upper limit is not $+\infty$,

$$n\left\{n\left(\frac{a_n}{a_{n+1}}-1\right)-1\right\} \leq G$$

for all n. Hence

$$\lambda_1(n) \leq \frac{\log n}{n} \cdot G.$$

But the right side $\doteq 0$. Hence $\lambda_1(n) \leq 1$ for $n >$ some m, and A is divergent by 1.

Example. We note that Raabe's test does apply to the harmonic series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots \tag{1}$$

Here

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = 1.$$

Hence

$$P_n = 0, \qquad \text{and}$$

$$\lim P_n = 0.$$

Hence the series 1) is divergent.

97. *Gauss' Test. Let* $A = \alpha_1 + \alpha_2 + \cdots$ *be a positive term series such that*

$$\frac{\alpha_n}{\alpha_{n+1}} = \frac{n^s + a_1 n^{s-1} + \cdots + a_s}{n^s + b_1 n^{s-1} + \cdots + b_s},$$

where s, $a_1 \cdots b_1 \cdots$ *do not depend on* n. *Then* A *is convergent if* $a_1 - b_1 > 1$, *and divergent if* $a_1 - b_1 \leq 1$.

Using the identity I, 91, 2), we have

$$\lambda_0(n) = n\left(\frac{\alpha_n}{\alpha_{n+1}} - 1\right) = \frac{a_1 - b_1 + \frac{1}{n}\{a_2 - b_2 + \cdots\}}{1 + \frac{1}{n}\{b_1 + \cdots\}}.$$

Thus $\lim \lambda_0(n) = a_1 - b_1$. Hence, if $a_1 - b_1 > 1$, A is convergent; if $a_1 - b_1 < 1$, it is divergent. If $a_1 - b_1 = 1$, Raabe's test does not always apply. To dispose of this case we may apply the test corresponding to $\lambda_1(n)$. Or more simply we may use Cahen's test which depends on $\lambda_1(n)$. We find at once

$$\lim P_n = a_2 - b_2 - b_1 < \infty;$$

and A is divergent.

98. *Let $A = a_1 + a_2 + \cdots$ be a positive term series such that*

$$\frac{a_n}{a_{n+1}} = 1 + \frac{\alpha}{n} + \frac{\beta_n}{n^\mu} \qquad \mu > 1,\ \beta_n < \infty.$$

Then A is convergent if $\alpha > 1$ and divergent if $\alpha \leq 1$.

For

$$\lambda_0(n) = n\left(\frac{a_n}{a_{n+1}} - 1\right) = \alpha + \frac{\beta_n}{n^{\mu-1}} \doteq \alpha,$$

and A converges if $\alpha > 1$ and diverges if $\alpha < 1$. If $\alpha = 1$,

$$\lambda_1(n) = l_1 n\{\lambda_0(n) - 1\} = \frac{l_1 n}{n^{\mu-1}} \cdot \beta_n \doteq 0,$$

and A is divergent.

EXAMPLES

99. *The Binomial Series.* Let us find for what values of x and μ the series

$$B = 1 + \mu x + \frac{\mu \cdot \mu - 1}{1 \cdot 2} x^2 + \frac{\mu \cdot \mu - 1 \cdot \mu - 2}{1 \cdot 2 \cdot 3} x^3 + \cdots$$

converges. If μ is a positive integer, B is a polynomial of degree μ. For $\mu = 0$, $B = 1$. We now exclude these exceptional values of μ. Applying D'Alembert's test to its adjoint we find

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\mu - n + 1}{n}\right| |x| \doteq |x|.$$

Thus B converges absolutely for $|x| < 1$ and diverges for $|x| > 1$.

Let $x = 1$. Then

$$B = 1 + \mu + \frac{\mu \cdot \mu - 1}{1 \cdot 2} + \frac{\mu \cdot \mu - 1 \cdot \mu - 2}{1 \cdot 2 \cdot 3} + \cdots$$

Here D'Alembert's test applied to its adjoint gives

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|\mu - n + 1|}{n} \doteq 1.$$

As this gives us no information unless $\mu < -1$, let us apply Raabe's test. Here

$$n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right) = \frac{1 + \mu}{1 - \frac{1 + \mu}{n}}, \text{ for sufficiently large } n$$

$$\doteq 1 + \mu.$$

Thus B converges absolutely if $\mu > 0$, and its adjoint diverges if $\mu < 0$. Thus B does not converge absolutely for $\mu < 0$.

But in this case we note that the terms of B are alternately positive and negative. Also

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|1 - \frac{1+\mu}{n}\right|,$$

so that the $|a_n|$ form a decreasing sequence from a certain term. We investigate now when $a_n \doteq 0$. Now

$$a_n = (-1)^n \frac{(-\mu)(-\mu+1)\cdots(-\mu+n-1)}{1\cdot 2\cdot\cdots n} = (-1)^n Q_n.$$

In I, 143, let $\alpha = -\mu$, $\beta = 1$. We thus find that $\lim a_n = 0$ only when $\mu > -1$. Thus B converges when $\mu > -1$ and diverges when $\mu \le -1$.

Let $x = -1$. Then

$$B = 1 - \mu + \frac{\mu\cdot\mu-1}{1\cdot 2} - \frac{\mu\cdot\mu-1\cdot\mu-2}{1\cdot 2\cdot 3} + \cdots$$

If $\mu > 0$, the terms of B finally have one sign, and

$$n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right) \doteq 1 + \mu.$$

Hence B converges absolutely.

If $\mu < 0$, let $\mu = -\lambda$. Then B becomes

$$1 + \lambda + \frac{\lambda\cdot\lambda+1}{1\cdot 2} + \frac{\lambda\cdot\lambda+1\cdot\lambda+2}{1\cdot 2\cdot 3} + \cdots$$

Here

$$n\left(\frac{a_n}{a_{n+1}} - 1\right) = \frac{1-\lambda}{1+\dfrac{\lambda-1}{n}} \doteq 1 - \lambda.$$

Hence B diverges in this case. Summing up:

The binomial series converges absolutely for $|x| < 1$ *and diverges for* $|x| > 1$. *When* $x = 1$ *it converges for* $\mu > -1$ *and diverges for* $\mu \le -1$; *it converges absolutely only for* $\mu > 0$. *When* $x = -1$, *it converges absolutely for* $\mu > 0$ *and diverges for* $\mu < 0$.

100. *The Hypergeometric Series*

$$F(\alpha, \beta, \gamma, x) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha \cdot \alpha+1 \cdot \beta \cdot \beta+1}{1 \cdot 2 \cdot \gamma \cdot \gamma+1} x^2$$
$$+ \frac{\alpha \cdot \alpha+1 \cdot \alpha+2 \cdot \beta \cdot \beta+1 \cdot \beta+2}{1 \cdot 2 \cdot 3 \cdot \gamma \cdot \gamma+1 \cdot \gamma+2} x^2 + \cdots$$

Let us find for what values of x this series converges. Passing to the adjoint series, we find

$$\left|\frac{a_{n+2}}{a_{n+1}}\right| = \left|\frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}\right| \cdot |x| \doteq |x|. \tag{1}$$

Thus F converges absolutely for $|x| < 1$ and diverges for $|x| > 1$.

Let $x = 1$. The terms finally have one sign, and

$$\frac{a_{n+1}}{a_{n+2}} = \frac{n^2 + n(1+\gamma) + \gamma}{n^2 + n(\alpha+\beta) + \alpha\beta}.$$

Applying Gauss', test we find F converges when and only when

$$\alpha + \beta - \gamma < 0.$$

Let $x = -1$. The terms finally alternate in sign. Let us find when $a_n \doteq 0$. We have

$$|a_{n+2}| = \frac{\alpha\beta}{\gamma} \cdot \frac{(\alpha+1)\cdots(\alpha+n)(\beta+1)\cdots(\beta+n)}{(1+1)\cdots(1+n)(\gamma+1)\cdots(\gamma+n)}.$$

Now

$$\alpha + m = m\left(1 + \frac{\alpha}{m}\right) \quad , \quad \beta + m = m\left(1 + \frac{\beta}{m}\right),$$

$$1 + m = m\left(1 + \frac{1}{m}\right) \quad , \quad \gamma + m = m\left(1 + \frac{\gamma}{m}\right).$$

Thus

$$|a_{n+2}| = \prod_1^n \frac{\alpha\beta}{\gamma} \cdot \frac{\left(1+\frac{\alpha}{m}\right)\left(1+\frac{\beta}{m}\right)}{\left(1+\frac{1}{m}\right)\left(1+\frac{\gamma}{m}\right)}.$$

But by I, 91, 1),

$$\frac{1}{1+\frac{1}{m}} = 1 - \frac{1}{m} + \frac{\sigma_m}{m^2} \quad ; \quad \frac{1}{1+\frac{\gamma}{m}} = 1 - \frac{\gamma}{m} + \frac{\tau_m}{m^2}$$

where $\sigma_m \doteq 1$, $\tau_m \doteq \gamma^2$ as $m \doteq \infty$.

Hence

$$|a_{n+2}| = \prod_1^n \left(1+\frac{\alpha}{m}\right)\left(1+\frac{\beta}{m}\right)\left(1-\frac{1}{m}+\frac{\sigma_m}{m^2}\right)\left(1-\frac{\gamma}{m}+\frac{\tau_m}{m^2}\right)$$

$$= \prod_1^n \left(1+\frac{\alpha+\beta-\gamma-1}{m}+\frac{\eta_m}{m^2}\right).$$

Hence

$$\log|a_{n+2}| = \sum_1^n \log\left(1+\frac{\alpha+\beta-\gamma-1}{m}+\frac{\eta_m}{m^2}\right) = \sum_1^n l_m = L_n,$$

and thus

$$L = \lim \log|a_{n+2}| = \sum_1^\infty l_m.$$

Now for a_n to $\doteq 0$ it is necessary that $L_n \doteq -\infty$. In 88, Ex. 3, we saw this takes place when and only when $\alpha+\beta-\gamma-1<0$.

Let us find now when $|a_{n+1}|<|a_n|$. Now 1) gives

$$\left|\frac{a_{n+2}}{a_{n+1}}\right| = 1+\frac{\alpha+\beta-\gamma-1}{n}+\frac{\delta_m}{n^2}.$$

Thus when $\alpha+\beta-\gamma-1<0$, $|a_{n+2}|<|a_{n+1}|$. Hence in this case F is an alternating series. We have thus the important theorem:

The hypergeometric series converges absolutely when $|x|<1$ *and diverges when* $|x|>1$. *When* $x=1$, F *converges only when* $\alpha+\beta-\gamma<0$ *and then absolutely. When* $x=-1$, F *converges only when* $\alpha+\beta-\gamma-1<0$, *and absolutely if* $\alpha+\beta-\gamma<0$.

Pringsheim's Theory

101. 1. In the 35th volume of the *Mathematische Annalen* (1890) Pringsheim has developed a simple and uniform theory of convergence which embraces as special cases all earlier criteria, and makes clear their interrelations. We wish to give a brief sketch of this theory here, referring the reader to his papers for more details.

Let M_n denote a positive increasing function of n whose limit is $+\infty$ for $n=\infty$. Such functions are, for example, $\mu>0$,

$$n^\mu \quad , \quad \log^\mu n \quad , \quad l_s^\mu n \quad , \quad l_1 n l_2 n \cdots l_{s-1} n l_s^\mu n$$

A_n^μ, where A is any positive term divergent series.

$\bar{B}_n^{-\mu}$ where B is any positive term convergent series.

$$e^{\mu n}\quad,\quad \left(e^{e^n}\right)^\mu\quad,\quad \left(e^{e^{e^n}}\right)^\mu\quad,\quad \cdots\ e^{\mu M n}$$

It will be convenient to denote in general a convergent positive term series by the symbol

$$C = c_1 + c_2 + \cdots$$

and a divergent positive term series by

$$D = d_1 + d_2 + \cdots$$

2. *The series*

$$C = \sum_1^\infty \frac{M_{n+1} - M_n}{M_n M_{n+1}} = \sum_1^\infty \left(\frac{1}{M_n} - \frac{1}{M_{n+1}}\right) = \sum c_n \tag{1}$$

is convergent, and conversely every positive term convergent series may be brought into this form.

For

$$C_m = \sum_1^m \left(\frac{1}{M_n} - \frac{1}{M_{n+1}}\right)$$

$$= \frac{1}{M_1} - \frac{1}{M_{m+1}} \doteq \frac{1}{M_1}$$

and C is convergent.

Let now conversely $C = c_1 + c_2 + \cdots$ be a given convergent positive term series. Let

$$\bar{C}_{n-1} = \frac{1}{M_n}.$$

Then

$$c_n = \frac{1}{M_n} - \frac{1}{M_{n+1}}.$$

3. *The series*

$$D = \sum_1^\infty (M_{n+1} - M_n) \tag{2}$$

is divergent, and conversely every positive term divergent series may be brought into this form.

For

$$D_n = \sum_1^n (M_{n+1} - M_n)$$

$$= M_{n+1} - M_1 \doteq +\infty.$$

Let now conversely $D = d_1 + d_2 + \cdots$ be a given positive term divergent series. Let

$$M_n = D_{n-1}.$$

Then

$$d_n = M_{n+1} - M_n.$$

102. Having now obtained a general form of all convergent and divergent series, we now obtain another general form of a convergent or divergent series, but which converges slower than 1) or diverges slower than 101, 2). Let us consider first convergence. Let $M_n' < M_n$, then

$$\sum\left(\frac{1}{M_n'} - \frac{1}{M_{n+1}'}\right) \tag{1}$$

is convergent, and if M_n' is properly chosen, not only is each term of 1) greater than the corresponding term of 101, 1), but 1) will converge slower than 101, 1). For example, for M_n' let us take M_n^μ, $0 < \mu < 1$. Then denoting the resulting series by $C' = \Sigma c_n'$, we have

$$\frac{c_n'}{c_n} = \frac{M_{n+1}^\mu - M_n^\mu}{M_n^\mu M_{n+1}^\mu} \frac{M_n M_{n+1}}{M_{n+1} - M_n}$$

$$= \frac{1 - r^\mu}{1 - r} M_n^{1-\mu} \quad , \quad r = \frac{M_n}{M_{n+1}} < 1. \tag{2}$$

Thus C' converges slower than C. But the preceding also shows that C' and

$$\sum \frac{M_{n+1} - M_n}{M_{n+1} M_n^\mu} \tag{3}$$

converge equally fast. In fact 2) states that

$$c' \sim c_n M_n^{1-\mu}.$$

Since M_n is any positive increasing function of n whose limit is ∞, we may replace M_n in 3) by $l_r M_n$ so that

$$\sum \frac{l_r M_{n+1} - l_r M_n}{l_r M_{n+1} l_r^\mu M_n}$$

is convergent and *a fortiori*

$$\sum \frac{l_r M_{n+1} - l_r M_n}{l_r^{1+\mu} M_{n+1}} \quad , \quad r = 1, 2, \cdots \tag{4}$$

is convergent.

Now by I, 413, for sufficiently large n,

$$\log M_{n+1} - \log M_n = -\log\left(1 - \frac{M_{n+1} - M_n}{M_{n+1}}\right) > \frac{M_{n+1} - M_n}{M_{n+1}}.$$

Replacing here M_n by $\log M_n$, we get

$$l_2 M_{n+1} - l_2 M_n > \frac{\log M_{n+1} - \log M_n}{\log M_{n+1}} > \frac{M_{n+1} - M_n}{M_{n+1} \log M_{n+1}};$$

and in general

$$l_2 M_{n+1} - l_2 M_n > \frac{M_{n+1} - M_n}{M_{n+1} l_1 M_{n+1} \cdots l_{r-1} M_{n+1}}. \qquad (5$$

Thus the series

$$\sum \frac{M_{n+1} - M_n}{M_{n+1} l_1 M_{n+1} \cdots l_{r-1} M_{n+1} l_r^{1+\mu} M_{n+1}} \qquad (6$$

converges as is seen by comparing with 4). We are thus led to the theorem:

The series

$$\sum \frac{M_{n+1} - M_n}{M_n M_{n+1}}, \quad \sum \frac{M_{n+1} - M_n}{M_{n+1} M_n^\mu} \qquad (7$$

$$\sum_1^\infty \frac{M_{n+1} - M_n}{M_{n+1} l_1 M_{n+1} \cdots l_{r-1} M_{n+1} l_r^{1+\mu} M_{n+1}} \qquad r = 1, 2, \cdots, \mu > 0$$

form an infinite set of convergent series; each series converging slower than any preceding it.

The last statement follows from I, 463, 1, 2.

Corollary 1 (Abel). Let $D = d_1 + d_2 + \cdots$ denote a positive term divergent series. Then

$$\sum \frac{d_n}{D_n^{1+\mu}} \qquad \mu > 0$$

is convergent.

Follows from 3), setting $M_{n+1} = D_n$.

Corollary 2. If we take $M_n = n$ we get the series 91, Ex. 2.

Corollary 3. Being given a convergent positive term series $C = c_1 + c_2 + \cdots$ we can construct a series which converges slower than C.

For by 101, 2 we may bring C to the form

$$\sum \frac{M_{n+1} - M_n}{M_n M_{n+1}}.$$

Then any of the series 7) converges slower than C.

103. 1. Let us consider now divergent series. Here our problem is simpler and we have at once the theorem:

The series

$$D = \sum_1^\infty \frac{M_{n+1} - M_n}{M_n} = \Sigma d_n \tag{1}$$

diverges slower than

$$\Sigma (M_{n+1} - M_n) = \Sigma d'_n. \tag{2}$$

That 1) is divergent is seen thus: Consider the product

$$P_n = \prod_1^n \left(1 + \frac{M_{m+1} - M_m}{M_m}\right) = \prod_1^n \frac{M_{m+1}}{M_m}$$
$$= \frac{M_{m+1}}{M_1}$$

which obviously $\doteq \infty$.

Now

$$\begin{aligned} P_n &= (1 + d_1)(1 + d_2) \cdots (1 + d_n) \\ &= 1 + (d_1 + \cdots + d_n) + (d_1 d_2 + d_1 d_3 + \cdots) \\ &\quad + (d_1 d_2 d_3 + \cdots) + \cdots + d_1 d_2 \cdots d_n \\ &< 1 + D_n + \frac{1}{2} D_n^2 + \cdots + \frac{1}{n\,!} D_n^n < e^{D_n} < e^D. \end{aligned}$$

Hence $D_n \doteq \infty$ and D is divergent.

As

$$\frac{d_n}{d'_n} = \frac{1}{M_n} \doteq 0$$

we see that 1) converges slower than 2).

2. *Any given positive term series* $D = d_1 + d_2 + \cdots$ *can be put in the form 1).*

For taking $M_1 > 0$ at pleasure, we determine $M_2, M_3 \cdots$ by the relations

$$\frac{M_{n+1}}{M_n} = 1 + d_n \qquad n = 1, 2 \cdots$$

Then $M_{n+1} > M_n$ and

$$d_n = \frac{M_{n+1} - M_n}{M_n}.$$

Moreover $M_n \doteq \infty$. For

$$\frac{M_{n+1}}{M_1} = (1 + d_1) \cdots (1 + d_n)$$
$$> 1 + D_n \qquad \text{by I, 90, 1.}$$

But $D_n \doteq \infty$.

3. *The series*

$$\sum_1^\infty (M_{n+1} - M_n) = \Sigma d_n$$

$$\sum_1^\infty \frac{d_n}{M_n l_1 M_n \cdots l_r M_n} \qquad r = 0, 1, 2, \cdots$$

form an infinite set of divergent series, each series divergent slower than any preceding it. $l_0 M_n = M_n$.

For

$$\log M_{n+1} - \log M_n = \log\left(1 + \frac{M_{n+1} - M_n}{M_n}\right)$$
$$< \frac{M_{n+1} - M_n}{M_n}.$$

This proves the theorem for $r = 0$. Hence as in 102 we find, replacing repeatedly M_n by $\log M_n$,

$$l_{r+1} M_{n+1} - l_{r+1} M_n < \frac{M_{n+1} - M_n}{M_n l_1 M_n \cdots l_r M_n}. \tag{3}$$

Corollary 1. If we take $M_n = n$, we get the series 91, Ex. 2.

Corollary 2 (Abel). Let $D = d_1 + d_2 + \cdots$ *be a divergent positive term series. Then*

$$\Sigma \frac{d_n}{D_{n-1}}$$

is divergent.

We take here $M_n = D_n$.

Corollary 3. Being given a positive term divergent series D, *we can construct a series which diverges slower than* D.

For by 101, 3 we may bring D to the form

$$\Sigma(M_{n+1} - M_n).$$

Then 1) diverges slower than D.

104. In Ex. 3 of I, 454, we have seen that M_{n+1} is not always $\sim M_n$. In case it is we have

1. *The series*

$$\sum \frac{M_{n+1} - M_n}{M_n^{1+\mu}} \qquad \mu > 0 \quad , \quad M_{n+1} \sim M_n$$

is convergent.

Follows from 102, 3).

2. *The series*

$$\sum \frac{M_{n+1} - M_n}{e^{\mu M_n}} \qquad M_{n+1} \sim M_n$$

is convergent if $\mu > 0$; *it is divergent if* $\mu < 0$.

For $$e^{\mu M_n} > \tfrac{1}{2} \mu^2 M_n^2 \sim M_n^2 \qquad \mu > 0.$$

Thus

$$\frac{M_{n+1} - M_n}{e^{\mu M_n}} \lesssim \frac{M_{n+1} - M_n}{M_n^2}.$$

If $\mu < 0$

$$\frac{M_{n+1} - M_n}{e^{\mu M_n}} \gtrsim M_{n+1} - M_n.$$

3. *If* $M_{n+1} \sim M_n$, *we have*

$$l_{r+1} M_{n+1} - l_{r+1} M_n \sim \frac{M_{n+1} - M_n}{M_n l_1 M_n \cdots l_r M_n} \sim \frac{M_{n+1} - M_n}{M_{n+1} l_1 M_{n+1} \cdots l_r M_{n+1}}.$$

For by 102, 5), 103, 3),

$$l_{r+1} M_{n+1} - l_{r+1} M_n < \frac{M_{+1} - M_n}{M_n l_1 M_n \cdots l_r M_n},$$

$$\frac{M_{n+1} - M_n}{M_{n+1} l_1 M_{n+1} \cdots l_r M_n} < l_{r+1} M_{n+1} - l_{r+1} M_n.$$

Now since $M_{n+1} \sim M_n$, we have also obviously

$$l_m M_n \sim l_m M_{n+1} \qquad m = 1, 2, \cdots r.$$

105. Having obtained an unlimited set of series which converge or diverge more and more slowly, we show now how they may be employed to furnish tests of ever increasing strength. To obtain them we go back to the fundamental theorems of comparison of 87. In the first place, if $A = a_1 + a_2 + \cdots$ is a given positive term series, it converges if

$$\frac{a_n}{c_n} \leq G \qquad G > 0. \tag{1}$$

It diverges if

$$\frac{a_n}{d_n} > G. \tag{2}$$

In the second place, A converges if

$$\frac{a_{n+1}}{a_n} - \frac{c_{n+1}}{c_n} \leq 0, \tag{3}$$

and diverges if

$$\frac{a_{n+1}}{a_n} - \frac{d_{n+1}}{d_n} \geq 0. \tag{4}$$

The tests 1), 2) involve only a single term of the given series and the comparison series, while the tests 3), 4) involve two terms. With *Du Bois Reymond* such tests we may call respectively tests of the *first* and *second kinds.* And in general any relation between p terms

$$a_n, a_{n+1}, \cdots a_{n+p-1}$$

of the given series and p terms of a comparison series,

$$c_n, c_{n+1}, \cdots c_{n+p-1}, \quad \text{or } d_n, d_{n+1} \cdots d_{n+p-1}$$

which serves as a criterion of convergence or divergence may be called a *test of the* p^{th} *kind.*

Let us return now to the tests 1), 2), 3), 4), and suppose we are testing A for convergence. If for a certain comparison series C

$$\frac{a_n}{c_n} \quad \text{not always} \leq G \quad , \quad n > m$$

it might be due to the fact that $c_n \doteq 0$ too fast. We would then take another comparison series $C' = \Sigma c'_n$ which converges slower than C. As there always exist series which converge slower than any given positive term series, the test 1) must decide the convergence of A if a proper comparison series is found. To find such series we employ series which converge slower and slower. Similar remarks apply to the other tests. We show now how these considerations lead us most naturally to a set of tests which contain as special cases those already given.

106. 1. *General Criterion of the First Kind. The positive term series* $A = a_1 + a_2 + \cdots$ *converges if*

$$\overline{\lim} \frac{M_n M_{n+1}}{M_{n+1} - M_n} a_n < \infty. \tag{1}$$

It diverges if

$$\underline{\lim}\, \frac{M_n}{M_{n+1} - M_n}\, a_n > 0. \qquad (2$$

This follows at once from 105, 1), 2); and 101, 2; 103, 1.

2. To get tests of greater power we have only to replace the series

$$\sum \frac{M_{n+1} - M_n}{M_{n+1} M_n} \quad , \quad \sum \frac{M_{n+1} - M_n}{M_n}$$

just employed in 1), 2) by the series of 102 and 103, 3 which converge (diverge) slower. We thus get from 1:

The positive term series A converges if

$$\overline{\lim}\, \frac{M_{n+1} M_n^\mu}{M_{n+1} - M_n}\, a_n \qquad \text{or } \overline{\lim}\, \frac{M_n l_1 M_n \cdots l_{r-1} M_{n+1} l_r^{1+\mu} M_n}{M_{n+1} - M_n}\, a_n < \infty.$$

It diverges if

$$\underline{\lim}\, \frac{M_n l_1 M_n \cdots l_r M_n}{M_{n+1} - M_n}\, a_n > 0.$$

Bonnet's Test. The positive term series A converges if

$$\overline{\lim}\; n l_1 n \cdots l_{r-1} n l_r^{1+\mu} n \cdot a_n < \infty \quad , \quad \mu > 0.$$

It diverges if

$$\underline{\lim}\; n l_1 n \cdots l_r n \cdot a_n > 0.$$

Follows from the preceding setting $M_n = n$.

3. *The positive term series A converges or diverges according as*

$$\frac{e^{\mu M_n} a_n}{M_{n+1} - M_n} \begin{matrix} \leq 1 \quad , \quad \mu > 0, \\ \geq 1 \quad , \quad \mu \leq 0. \end{matrix} \qquad M_{n+1} \sim M_n. \qquad (3$$

For in the first case

$$a_n \leq \frac{M_{n+1} - M_n}{e^{\mu M_n}} \quad ; \quad \mu > 0,$$

and in the second case

$$a_n \geq \frac{M_{n+1} - M_n}{e^{\mu M_n}} \qquad \mu < 0.$$

The theorem follows now by 104, 2.

4. *The positive term series A converges if*

$$\underline{\lim}\, \frac{\log \dfrac{M_{n+1} - M_n}{a_n}}{M_n} > 0 \qquad \text{or } \underline{\lim}\, \frac{\log \dfrac{M_{n+1} - M_n}{M_n l_1 M_n \cdots l_r M_n \cdot a_n}}{l_{r+1} M_n} > 0.$$

It diverges if

$$\overline{\lim}\,\frac{\log\dfrac{M_{n+1}-M_n}{a_n}}{M_n}<0 \qquad \text{or } \overline{\lim}\,\frac{\log\dfrac{M_{n+1}-M_n}{M_n l_1 M_n\cdots l_r M_n\cdot a_n}}{l_{r+1}M_n}<0.$$

Here $r=0, 1, 2, \cdots$ *and as before* $l_0 M_n = M_n$.

For taking the logarithm of both sides of 3) we have for convergence

$$q=\frac{\log\dfrac{M_{n+1}-M_n}{a_n}}{M_n}\geq\mu.$$

As μ is an arbitrarily small but fixed positive number, A converges if $\underline{\lim}\, q>0$. Making use of 104, 3 we get the first part of the theorem. The rest follows similarly.

Remark. If we take $M_n=n$ we get Cauchy's radical test 90 and Bertram's tests 93.

For if
$$\frac{\log\dfrac{1}{a_n}}{n}=\log\sqrt[n]{\frac{1}{a_n}}=-\log\sqrt[n]{a_n}\geq\mu>0,$$
it is necessary that
$$\sqrt[n]{a_n}\leq\mu<1.$$

Also if
$$\frac{\log\dfrac{1}{a_n n l_1 n\cdots l_r n}}{l_{r+1}n}=\frac{\log\dfrac{1}{a_n n l_1 n\cdots l_{r-1}n}+\log\dfrac{1}{l_r n}}{l_{r+1}n}$$
$$=-1+\frac{\log\dfrac{1}{a_n n l_1 n\cdots l_{r-1}n}}{l_{r+1}n}\geq\mu>0,$$
it is necessary that
$$\frac{\log\dfrac{1}{a_n n l_1 n\cdots l_{r-1}n}}{l_{r+1}n}\geq\mu+1>1.$$

107. In 94 we have given Kummer's criterion for the convergence of a positive term series. The most remarkable feature about it is the fact that the constants $k_1, k_2\cdots$ which enter it are subject to no conditions whatever except that they shall be positive. On this account this test, which is of the second kind, has stood entirely apart from all other tests, until Pringsheim discovered its counterpart as a test of the first kind, viz.:

Pringsheim's Criterion. Let $p_1, p_2 \cdots$ *be a set of positive numbers chosen at pleasure, and let* $P_n = p_1 + \cdots + p_n$. *The positive term series A converges if*

$$\underline{\lim}\, \frac{\log \dfrac{p_n}{a_n}}{P_n} > 0. \qquad (1$$

For A converges if

$$\underline{\lim}\, \frac{\log \dfrac{M_{n+1} - M_n}{a_n}}{M_n} > 0 \quad , \quad \text{by 106, 4.} \qquad (2$$

But $M_{n+1} - M_n = d_n$ is the general term of the divergent series $D = d_1 + d_2 + \cdots$

Thus 2) may be written

$$\underline{\lim}\, \frac{\log \dfrac{d_n}{a_n}}{D_n} > 0. \qquad (3$$

Moreover A converges if

$$\frac{c_n}{a_n} \geq r > 1,$$

that is, if

$$\underline{\lim}\, \frac{c_n}{a_n} > 0,$$

where as usual $C = c_1 + c_2 + \cdots$ is a convergent series.

Hence A converges if

$$\underline{\lim}\, \frac{\dfrac{c_n}{a_n}}{C_n} > 0. \qquad (4$$

But now the set of numbers $p_1, p_2 \cdots$ gives rise to a series $P = p_1 + p_2 + \cdots$ which must be either convergent or divergent. Thus 3), 4) show that in either case 1) holds.

108. 1. Let us consider now still more briefly *criteria of the second kind.* Here the fundamental relations are 3), 4) of 105, which may be written:

$$c_{n+1} \frac{a_n}{a_{n+1}} - c_n > 0 \text{ for } \textit{convergence;} \qquad (1$$

$$d_{n+1} \frac{a_n}{a_{n+1}} - d_n \leq 0 \text{ for } \textit{divergence.} \qquad (2$$

Or in less general form:

The positive term series A converges if

$$\underline{\lim}\left(c_{n+1}\frac{a_n}{a_{n+1}} - c_n\right) > 0. \tag{3}$$

It diverges if

$$\overline{\lim}\left(d_{n+1}\frac{a_n}{a_{n+1}} - d_n\right) < 0. \tag{4}$$

Here as usual $C = c_1 + c_2 + \cdots$ is a convergent, and $D = d_1 + d_2 + \cdots$ a divergent series.

2. Although we have already given one demonstration of Kummer's theorem we wish to show here its place in Pringsheim's general theory, and also to exhibit it under a more general form. Let us replace c_n, c_{n+1} in 1) by their values given in 101, 2. We get

$$\frac{M_{n+2} - M_{n+1}}{M_{n+2}} \cdot \frac{a_n}{a_{n+1}} - \frac{M_{n+1} - M_n}{M_n} \geq 0,$$

or since

$$M_{n+2} > M_{n+1},$$

$$\frac{M_{n+2} - M_{n+1}}{M_{n+1}} \cdot \frac{a_n}{a_{n+1}} - \frac{M_{n+1} - M_n}{M_n} > 0,$$

or by 103, 2

$$d_{n+1}\frac{a_n}{a_{n+1}} - d_n > 0,$$

where $D = d_1 + d_2 + \cdots$ is *any* divergent positive term series. Since any set of positive numbers $k_1, k_2, \cdots$ gives rise to a series $k_1 + k_2 + \cdots$ which must be either convergent or divergent, we see from 1) that 5) holds when we replace the d's by the k's. We have therefore:

The positive term series A converges if there exists a set of positive numbers $k_1, k_2 \cdots$ such that

$$k_{n+1}\frac{a_n}{a_{n+1}} - k_n > 0. \tag{6}$$

It diverges if

$$d_{n+1}\frac{a_n}{a_{n+1}} - d_n \leq 0$$

where as usual $d_1 + d_2 + \cdots$ denotes a divergent series.

Since the k's are entirely arbitrary positive numbers, the relation 6) also gives

A converges if

$$k_n \frac{a_n}{a_{n+1}} - k_{n+1} > 0 \, ;$$

as is seen by writing

$$k_n = \frac{1}{k'_n}$$

reducing, and then dropping the accent.

3. From Kummer's theorem we may at once deduce a set of tests of increasing power, viz.:

The positive term series A is convergent or divergent according as

$$\frac{M_{n+2} - M_{n+1}}{M_{n+1} l_1 M_{n+1} \cdots l_r M_{n+1}} \cdot \frac{a_n}{a_{n+1}} - \frac{M_{n+1} - M_n}{M_n l_1 M_n \cdots l_r M_n}$$

is >0 *or is* ≤ 0.

For $k_1, k_2 \cdots$ we have used here the terms of the divergent series of 103, 3.

Arithmetic Operations on Series

109. 1. Since an infinite series

$$A = a_1 + a_2 + a_3 \cdots \tag{1}$$

is not a true sum but the limit of a sum

$$A = \lim_{n=\infty} A_n,$$

we now inquire in how far the properties of polynomials hold for the infinite polynomial 1). The *associative property* is expressed in the theorem :

Let $A = a_1 + a_2 + \cdots$ *be convergent. Let* $b_1 = a_1 + \cdots + a_{m_1}$, $b_2 = a_{m_1+1} + \cdots + a_{m_2}, \cdots$ *Then the series* $B = b_1 + b_2 + \cdots$ *is convergent and* $A = B$. *Moreover the number of terms which* b_n *embraces may increase indefinitely with n.*

For

$$B_n = A_{m_n}$$

and

$$\lim_{n=\infty} A_{m_n} = A \qquad \text{by I, 103, 2.}$$

This theorem relates to grouping the terms of A in parentheses. The following relate to removing them.

2. *Let* $B = b_1 + b_2 + \cdots$ *be convergent and let* $b_1 = a_1 + \cdots + a_{m_1}$, $b_2 = a_{m_1+1} + \cdots + a_{m_2}, \cdots$ *If* 1° $A = a_1 + a_2 + \cdots$ *is convergent,* $A = B$. 2° *If the terms* $a_n > 0$, A *is convergent.* 3° *If each* $m_n - m_{n-1} \leq p$ *a constant, and* $a_n \doteq 0$, A *is convergent.*

On the first hypothesis we have only to apply 1, to show $A = B$. On the second hypothesis

$$\epsilon > 0, \qquad m, \qquad \bar{B}_n < \epsilon, \qquad n \geq m.$$

Then

$$B - A_s < \epsilon, \qquad s \geq m_n.$$

On the third hypothesis we may set

$$A_s = B_r + b'_{r+1}$$

where b'_{r+1} denotes a part of the a-terms in b_{r+1}. Since b_{r+1} contains at most p terms of A, $b'_{r+1} \doteq 0$.

Hence

$$\lim A_s = \lim B_r \quad , \quad \text{or} \quad A = B.$$

Example 1. The series

$$B = (1-1) + (1-1) + (1-1) + \cdots$$

is convergent. The series obtained by removing the parentheses

$$A = 1 - 1 + 1 - 1 + \cdots$$

is divergent.

Example 2.

$$A = 1 - \frac{1}{1+x} + \frac{1}{2} - \frac{1}{2+x} + \frac{1}{3} - \frac{1}{3+x} + \cdots; \quad x \neq -1, -2, \ldots$$

$$B = \sum\left(\frac{1}{n} - \frac{1}{n+x}\right) = \sum \frac{x}{n(n+x)}.$$

As B is comparable with $\sum \frac{1}{n^2}$, it is convergent. Hence A is convergent by 3°.

110. 1. Let us consider now the *commutative property*.

Here *Riemann* has established the following remarkable theorem:

The terms of a simply convergent series $A = a_1 + a_2 + \cdots$ *can be arranged to form a series* S, *for which* $\lim S_n$ *is any prescribed number, or* $\pm\infty$.

For let

$$B = b_1 + b_2 + \cdots$$
$$C = c_1 + c_2 + \cdots$$

be the series formed respectively of the positive and negative terms of A, the relative order of the terms in A being preserved.

To fix the ideas let l be a positive number; the demonstration of the other cases is similar. Since $B_n \doteq +\infty$, there exists an m_1 such that

$$B_{m_1} > l. \tag{1}$$

Let m_1 be the least index for which 1) is true. Since $C_n \doteq -\infty$, there exists an m_2 such that

$$B_{m_1} + C_{m_2} < l. \tag{2}$$

Let m_2 be the least index for which 2) is true. Continuing, we take just enough terms, say m_3 terms of B, so that

$$B_{m_1} + C_{m_2} + B_{m_1, m_3} > l.$$

Then just enough terms, say m_4 terms of C, so that

$$B_{m_1} + C_{m_2} + B_{m_1, m_3} + C_{m_2, m_4} < l,$$

etc. In this way we form the series

$$S = B_{m_1} + C_{m_2} + B_{m_1, m_3} + \cdots$$

whose sum is l. For

$$|a_s| < \epsilon \qquad s > \sigma;$$

also

$$r_n = m_1 + m_2 + \cdots + m_n \geqq n.$$

Hence

$$|l - S_n| \le |a_{r_n}| < \epsilon \qquad \text{for } n > \sigma.$$

2. *Let* $A = a_1 + a_2 + \cdots$ *be absolutely convergent. Let the terms of* A *be arranged in a different order, giving the series* B. *Then* B *is absolutely convergent and* $A = B$.

For we may take m so large that

$$\overline{A}_m < \epsilon.$$

We may now take n so large that $A_n - B_n$ contains no term whose index is $\leq m$. Thus the terms of $A_n - B_n$ taken with positive sign are a part of $\overline{\mathrm{A}}_m$ and hence

$$A_n - B_n| < \overline{\mathrm{A}}_m < \epsilon \qquad n > m.$$

Thus B is convergent and $B = A$.

The same reasoning shows that B is convergent, hence B is absolutely convergent.

3. *If* $A = a_1 + a_2 + \cdots$ *enjoys the commutative property, it is absolutely convergent.*

For if only simply convergent we could arrange its terms so as to have any desired sum. But this contradicts the hypothesis.

Addition and Subtraction

111. *Let* $A = a_1 + a_2 + \cdots$, $B = b_1 + b_2 + \cdots$ *be convergent. The series*

$$C = (a_1 \pm b_1) + (a_2 \pm b_2) + \cdots$$

are convergent and $C = A \pm B$.

For obviously $C_n = A_n \pm B_n$. We have now only to pass to the limit.

Example. We saw, 81, 3, Ex. 1, that

$$A = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

is a simply convergent series. Grouping its terms by twos and by fours [109, 1] we get

$$A = \sum_1^\infty \left(\frac{1}{2n-1} - \frac{1}{2n}\right) = \sum_1^\infty \left(\frac{1}{4n-3} - \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{1}{4n}\right)$$

Let us now rearrange A, taking two positive terms to one negative. We get

$$B = 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \cdots$$

We note now that

$$\frac{3}{2}A = A + \frac{1}{2}A$$

$$= \sum\left(\frac{1}{4n-3} - \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{1}{4n}\right) + \frac{1}{2}\sum\left(\frac{1}{2n-1} - \frac{1}{2n}\right)$$

$$= \sum\left(\frac{1}{4n-3} - \frac{1}{4n-2} + \frac{1}{4n-1} - \frac{1}{4n}\right) + \sum\left(\frac{1}{4n-2} - \frac{1}{4n}\right)$$

$$= \sum\left(\frac{1}{4n-1} + \frac{1}{4n-3} - \frac{1}{2n}\right) \qquad (1$$

$= B$ by 109, 2.

Thus $$B = \tfrac{3}{2}A.$$

This example, due to *Dirichlet*, illustrates the non-commutative property of simply convergent series. We have shown the convergence of B by actually determining its sum. As an exercise let us proceed directly as follows :

The series 1) may be written:

$$\sum \frac{8n-3}{2n(4n-1)(4n-3)} = \sum \frac{1}{n^2} \cdot \frac{8 - \frac{3}{n}}{2\left(4 - \frac{1}{n}\right)\left(4 - \frac{3}{n}\right)}.$$

Comparing this with

$$\sum \frac{1}{n^2}$$

we see that it is convergent by 87, 3. Since 1) is convergent, B is also by 109, 2.

112. 1. *Multiplication.* We have already seen, 80, 7, that we may multiply a convergent series by any constant. Let us now consider the multiplication of two series. As customary let

$$\sum_{\iota\kappa} a_\iota b_\kappa \qquad \iota, \kappa = 1, 2, 3, \cdots \qquad (1$$

denote the infinite series whose terms are all possible products $a_\iota \cdot b_\kappa$ without repetition. Let us take two rectangular axes as in analytic geometry ; the points whose coördinates are $x = \iota$, $y = \kappa$ are called *lattice points.* Thus to each term $a_\iota b_\kappa$ of 1), cor-

responds a lattice point ι, κ and conversely. The reader will find it a great help here and later to keep this correspondence in mind.

Let $A = a_1 + a_2 + \cdots$, $B = b_1 + b_2 + \cdots$ *be absolutely convergent. Then* $C = \Sigma a_\iota b_\kappa$ *is absolutely convergent and* $A \cdot B = C$.

Let m be taken large at pleasure; we may take n so large that $\Gamma_n - \mathrm{A}_m \cdot \mathrm{B}_m$ contains no term both of whose indices are $\leq m$. Then

$$\begin{aligned} \Gamma_n - \mathrm{A}_m\mathrm{B}_m &< \alpha_1\overline{\mathrm{B}}_m + \alpha_2\overline{\mathrm{B}}_m + \cdots + \alpha_m\overline{\mathrm{B}}_m \\ &\quad + \beta_1\overline{\mathrm{A}}_m + \beta_2\overline{\mathrm{A}}_m + \cdots + \beta_m\overline{\mathrm{A}}_m \\ &< \mathrm{A}_m\overline{\mathrm{B}}_m + \mathrm{B}_m\overline{\mathrm{A}}_m \\ &< \epsilon \text{ for } m \text{ sufficiently large.} \end{aligned}$$

Hence

$$\lim \Gamma_n = \mathrm{A} \cdot \mathrm{B}$$

and C is absolutely convergent.

To show that $C = A \cdot B$, we note that

$$| C_n - A_m B_m | \leq \Gamma_n - \mathrm{A}_m\mathrm{B}_m < \epsilon \qquad n > n_0.$$

2. We owe the following theorem to *Mertens*.

If A *converges absolutely and* B *converges (not necessarily absolutely), then*

$$C = a_1b_1 + (a_1b_2 + a_2b_1) + (a_1b_3 + a_2b_2 + a_3b_1) + \cdots$$

is convergent and $C = A \cdot B$.

We set

$$C = c_1 + c_2 + c_3 + \cdots$$

where

$$\begin{aligned} c_1 &= a_1b_1 \\ c_2 &= a_1b_2 + a_2b_1 \\ c_3 &= a_1b_3 + a_2b_2 + a_3b_1 \\ &\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ c_n &= a_1b_n + a_2b_{n-1} + a_3b_{n-2} + \cdots + a_nb_1. \end{aligned}$$

Adding these equations gives

$$C_n = a_1B_n + a_2B_{n-1} + a_3B_{n-2} + \cdots + a_nB_1.$$

But

$$B_m = B - \overline{B}_m \qquad m = 1, 2, \cdots$$

Hence

$$C_n = a_1(B - \overline{B}_n) + a_2(B - \overline{B}_{n-1}) + \cdots + a_n(B - \overline{B}_1)$$

$$= B(a_1 + \cdots + a_n) - (a_1\overline{B}_n + \cdots + a_n\overline{B}_1)$$

$$= A_n B - d_n,$$

where

$$d_n = a_1\overline{B}_n + a_2\overline{B}_{n-1} + \cdots + a_n\overline{B}_1.$$

The theorem is proved when we show $d_n \doteq 0$. To this end let us consider the two sets of remainders

$$\begin{matrix} \overline{B}_1 & , & \overline{B}_2 & , & \cdots \overline{B}_{n_1} \\ \overline{B}_{n_1+1} & , & \overline{B}_{n_1+2} & , & \cdots \overline{B}_{n_1+n_2} \end{matrix} \qquad n_1 + n_2 = n.$$

Let * each one in the first set be $| < | M_1$, and each in the second set $| < | M_2$. Then since

$$d_n = (a_1\overline{B}_n + \cdots + a_{n_2}\overline{B}_{n_1+1}) + (a_{n_2+1}\overline{B}_{n_1} + \cdots + a_n\overline{B}_1),$$

$$| d_n | < M_2(\alpha_1 + \cdots + \alpha_{n_2}) + M_1(\alpha_{n_2+1} + \cdots + \alpha_n)$$

$$< M_2 \mathrm{A}_{n_2} + M_1\overline{\mathrm{A}}_{n_2} < M_2\mathrm{A} + M_1\overline{\mathrm{A}}_{n_2}. \tag{1}$$

Now for each $\epsilon > 0$ there exists an n_1 such that

$$M_2 < \frac{\epsilon}{2\,\mathrm{A}},$$

also a ν, such that

$$\overline{\mathrm{A}}_{n_2} < \frac{\epsilon}{2\,M_1} \qquad n_2 > \nu.$$

Thus 1) shows that

$$| d_n | < \epsilon.$$

3. When neither A nor B converges absolutely, the series C may not even converge. The following example due to Cauchy illustrates this.

$$A = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

$$B = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots = A.$$

* The symbols $|<|$, $|\leq|$ mean numerically $<$, numerically $\leq$.

The series A being alternating is convergent by 81, 3. Its adjoint is divergent by 81, 2, since here $\mu = \frac{1}{2}$. Now

$$C = \frac{1}{\sqrt{1}}\frac{1}{\sqrt{1}} - \left(\frac{1}{\sqrt{1}\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{1}}\right)$$
$$+ \left(\frac{1}{\sqrt{1}}\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}}\frac{1}{\sqrt{1}}\right) - \cdots$$
$$= c_2 + c_3 + c_4 + \cdots$$

and
$$|c_n| = \frac{1}{\sqrt{1}}\frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{n-2}} + \cdots \frac{1}{\sqrt{n-1}} \cdot \frac{1}{\sqrt{1}}.$$

By I, 95,
$$\sqrt{m(n-m)} \leq \frac{n}{2}.$$

Hence
$$\frac{1}{\sqrt{m(n-m)}} \geq \frac{2}{n}, \quad |c_n| > \frac{2(n-1)}{n}.$$

Hence C is divergent since c_n does not $\doteq 0$, as it must if C were convergent, by 80, 3.

4. In order to have the theorems on multiplication together, we state here one which we shall prove later.

If all three series A, B, C are convergent, then $C = A \cdot B$.

113. We have seen, 109, 1, that we may group the terms of a convergent series $A = a_1 + a_2 + \cdots$ into a series $B = b_1 + b_2 + \cdots$ each term b_n containing but a finite number of terms of A. It is easy to arrange the terms of A into a finite or even an infinite number of infinite series, B', B'', B''' ... For example, let

$$B' = a_1 + a_{p+1} + a_{2p+1} + \cdots$$
$$B'' = a_2 + a_{p+2} + a_{2p+2} + \cdots$$
$$\cdots\cdots\cdots\cdots$$
$$B^{(p)} = a_p + a_{2p} + a_{3p} + \cdots$$

Then every term of A lies in one of these p series B. To decompose A into an infinite number of series we may proceed thus: In B' put all terms a_n whose index n is a prime number; in B'' put all terms whose index n is the product of two primes; in

$B^{(m)}$ all terms whose index is the product of m primes. We ask now what is the relation between the original series A and the series B', $B'' \cdots$

If $A = a_1 + a_2 + \cdots$ *is absolutely convergent, we may break it up into a finite or infinite number of series* B', B'', $B''', \cdots$ *Each of these series converges absolutely and*

$$A = B' + B'' + B''' + \cdots$$

That each $B^{(m)}$ converges absolutely was shown in 80, 6. Let us suppose first that there is only a finite number of these series, say p of them. Then

$$A_n = B'_{n_1} + B''_{n_2} + \cdots + B^{(p)}_{n_p} \qquad n = n_1 + \cdots + n_p.$$

As $n \doteq \infty$, each n_1, $n_2 \cdots \doteq \infty$. Hence passing to the limit $n = \infty$, the above relation gives

$$A = B' + B'' + \cdots + B^{(p)}.$$

Suppose now there are an infinite number of series $B^{(m)}$.

Set

$$B = B' + B'' + B''' + \cdots$$

We take ν so large that $A - B_n$, $n > \nu$, contains no term a_n of index $\leq m$, and m so large that

$$\bar{\mathrm{A}}_m < \epsilon.$$

Then

$$|A - B_n| \leq \bar{\mathrm{A}}_m < \epsilon. \quad n > \nu.$$

Two-way Series

114. 1. Up to the present the terms of our infinite series have extended to infinity only one way. It is, however, convenient sometimes to consider series which extend both ways. They are of the type

$$\cdots a_{-3} + a_{-2} + a_{-1} + a_0 + a_1 + a_2 + a_3 + \cdots$$

which may be written

$$a_0 + a_1 + a_2 + \cdots + a_{-1} + a_{-2} + \cdots$$

or

$$\sum_{-\infty}^{\infty} a_n. \tag{1}$$

Such series we called *two-way series.* The series is *convergent* if

$$\lim_{r,\, s=\infty} \sum_{n=-r}^{n=s} a_n \tag{2}$$

is finite. If the limit 2) does not exist, A is divergent. The extension of the other terms employed in one-way series to the present case are too obvious to need any comment. Sometimes $n=0$ is excluded in 1); the fact may be indicated by a dash, thus $\sum_{-\infty}^{\infty}{}' a_n$.

2. Let m be an integer; then while n ranges over

$$\cdots -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3 \cdots$$

$\nu = n+m$ will range over the same set with the difference that ν will be m units ahead or behind n according as $m \gtrless 0$. This shows that

$$\sum_{n=-\infty}^{\infty} a_n = \sum_{n=-\infty}^{\infty} a_{n+m}.$$

Similarly,

$$\sum_{n=-\infty}^{\infty} a_n = \sum_{n=-\infty}^{\infty} a_{-n}.$$

3. *Example 1.*

$$\begin{aligned} \Theta &= \sum_{-\infty}^{\infty} e^{nx+an^2} \\ &= 1 + e^{x+a} + e^{2x+4a} + e^{3x+9a} + \cdots \\ &\quad + e^{-x+a} + e^{-2x+4a} + e^{-3x+9a} \cdots \end{aligned}$$

This series is fundamental in the elliptic functions.

Example 2.

$$\begin{aligned} &\frac{1}{x} + \sum_{-\infty}^{\infty}{}' \left(\frac{1}{x+n} - \frac{1}{n}\right) \\ &= \frac{1}{x} + \left(\frac{1}{x+1} - 1\right) + \left(\frac{1}{x+2} - \frac{1}{2}\right) + \cdots \\ &\quad + \left(\frac{1}{x-1} + 1\right) + \left(\frac{1}{x-2} + \frac{1}{2}\right) + \cdots \end{aligned}$$

The sum of this series as we shall see is $\pi \cot \pi x$.

115. *For a two-way series A to converge, it is necessary and sufficient that the series B formed with the terms with negative indices and the series C formed with the terms with non-negative indices be convergent. If A is convergent, $A = B + C$.*

It is necessary. For A being convergent,

$$|A - B_r - C_s| < \epsilon/2 \quad , \quad |A - B_r - C_{s'}| < \epsilon/2$$

if s, $s' >$ some σ and $r >$ some ρ. Hence adding,

$$|C_s - C_{s'}| < \epsilon,$$

which shows C is convergent. Similarly we may show that B is convergent.

It is sufficient. For B, C being convergent,

$$|B - B_r| < \epsilon/2 \quad , \quad |C - C_s| < \epsilon/2$$

for r, $s >$ some p. Hence

$$|B + C - (B_r + C_s)| < \epsilon,$$

or

$$|B + C - \sum_{n=-r}^{n=s} a_n| < \epsilon.$$

Thus

$$\lim \sum_{n=-r}^{n=s} a_n = B + C.$$

Example 1. The series

$$\frac{1}{x} + \sum_{-\infty}^{\infty}{}' \left(\frac{1}{x+n} - \frac{1}{n} \right) \tag{1}$$

is absolutely convergent if $x \neq 0, \pm 1, \pm 2, \cdots$

For

$$|a_n| = \left| \frac{1}{x+n} - \frac{1}{n} \right| = \frac{|x|}{|n^2 + nx|}.$$

Hence

$$\sum_0^\infty a_n \quad \text{and} \quad \sum_{-\infty}^{-1} a_n$$

are comparable with $\sum_1^\infty \frac{1}{n^2}$.

Example 2. The series

$$\Theta(x) = \sum_{-\infty}^{\infty} e^{nx + an^2} \qquad x \text{ arbitrary} \tag{2}$$

is convergent absolutely if $a < 0$. It diverges if $a \geq 0$.

Here

$$n > 0, \qquad \sqrt[n]{a_n} = e^x e^{an} \doteq 0 \qquad \text{if } a < 0$$
$$\doteq \infty \qquad \text{if } a > 0;$$
$$n = -n',\ n' > 0 \qquad \sqrt[n']{a_n} = e^{-x} e^{an'} \doteq 0 \qquad \text{if } a < 0$$
$$\doteq \infty \qquad \text{if } a > 0.$$

The case $a = 0$ is obvious.

Thus the series defines a one-valued function of x when $a < 0$. As an exercise in manipulation let us prove two of its properties.

1° *$\Theta(x)$ is an even function.*

For

$$\Theta(-x) = \sum_{-\infty}^{\infty} e^{-nx+an^2}. \tag{3}$$

If we compare this series with 2) we see that the terms corresponding to $n = m$ and $n = -m$ have simply changed places, as the reader will see if he actually writes out a few terms of 2), 3). *Cf.* 114, 2.

2° $\Theta(x + 2\,ma) = e^{-m(x+ma)}\Theta(x)$. $m = \pm 1, \pm 2, \cdots$

For we can write 2) in the form

$$\Theta(x) = e^{-\frac{x^2}{4a}} \sum_{n=-\infty}^{\infty} e^{\frac{(x+2na)^2}{4a}}. \tag{4}$$

Thus

$$\Theta(x + 2\,ma) = e^{-\frac{(x+2ma)^2}{4a}} \sum_{n=-\infty}^{\infty} e^{\frac{(x+2(m+n)a)^2}{4a}}$$
$$= e^{-m(x+ma)} \cdot e^{-\frac{x^2}{4a}} \sum_{\nu=-\infty}^{\infty} e^{\frac{(x+2\nu a)^2}{4a}}$$

which with 4) gives 3).

CHAPTER IV

MULTIPLE SERIES

116. Let $x = x_1, \cdots x_m$ be a point in m-way space $\mathfrak{R}_m$. If the coördinates of x are all integers or zero, x is called a *lattice point*, and any set of lattice points a *lattice system*. If no coördinate of any point in a lattice system is negative, we call it a *non-negative* lattice system, etc. Let $f(x_1 \cdots x_m)$ be defined over a lattice system $\iota = \iota_1, \cdots \iota_m$. The set $\{f(\iota_1 \cdots \iota_m)\}$ is called an *m-tuple* sequence. It is customary to set

$$f(\iota_1 \cdots \iota_m) = a_{\iota_1 \cdots \iota_m}.$$

Then the sequence is represented by

$$A = \{a_{\iota_1 \cdots \iota_m}\}.$$

The terms

$$\lim A \quad , \quad \overline{\lim}\, A \quad , \quad \underline{\lim}\, A$$

as $\iota_1 \cdots \iota_m$ converges to an ideal point have therefore been defined and some of their elementary properties given in the discussion of I, 314–328; 336–338.

Let $x = x_1 \cdots x_m \quad y = y_1 \cdots y_m$ be two points in $\mathfrak{R}_m$. If $y_1 \geq x_1 \cdots y_m \geq x_m$ we shall write more shortly $y \geq x$. If x ranges over a set of points $x' \geq x'' \geq x''' \cdots$ we shall say that x is monotone decreasing. Similar terms apply as in I, 211.

If now

$$f(y_1 \cdots y_m) \geq f(x_1 \cdots x_m)$$

when $y \geq x$, we say f is a *monotone increasing function*. If

$$f(y_1 \cdots y_m) \leq f(x_1 \cdots x_m), \qquad y \geq x,$$

we say f is a *monotone decreasing function*.

Similar terms apply as in I, 211.

117. A very important class of multiple sequences is connected with multiple series as we now show. Let $a_{\iota_1 \dots \iota_m}$ be defined over a non-negative lattice system. The symbol

$$\Sigma a_{\iota_1 \dots \iota_m} \qquad \iota_1 = 0, 1, \cdots \nu_1 \quad , \quad \cdots \iota_m = 0, 1, \cdots \nu_m \tag{1}$$

or

$$\sum_0^{\nu_1 \dots \nu_m} a_{\iota_1 \dots \iota_m} \quad , \quad \text{or } A_{\nu_1 \dots \nu_m}$$

denotes the sum of all the a's whose lattice points lie in the rectangular cell

$$0 \leq x_1 \leq \nu_1 \qquad \cdots 0 \leq x_m \leq \nu_m.$$

Let us denote this cell by $R_{\nu_1 \dots \nu_m}$ or by R_ν. The sum 1) may be effected in a variety of ways. To fix the ideas let $m = 3$. Then

$$A_{\nu_1 \nu_2 \nu_3} = \sum_0^{\nu_1} \sum_0^{\nu_2 \nu_3} a_{\iota_1 \iota_2 \iota_3} = \sum_0^{\nu_2 \nu_3} \sum_0^{\nu_1} a_{\iota_1 \iota_2 \iota_3} = \sum_0^{\nu_1} \sum_0^{\nu_2} \sum_0^{\nu_3} a_{\iota_1 \iota_2 \iota_3}$$

etc. In the first sum, we sum up the terms in each plane and then add these results. In the second sum, we sum the terms on parallel lines and then add the results. In the last sum, we sum the terms on the parallel lines lying in a given plane and add the results; we then sum over the different planes.

Returning now to the general case, the symbol

$$A = \Sigma a_{\iota_1 \dots \iota_m} \qquad \iota_1, \cdots \iota_m = 0, 1, \cdots \infty,$$

or

$$A = \sum_0^\infty a_{\iota_1 \dots \iota_m}$$

is called an m-tuple infinite series. For $m = 2$ we can write it out more fully thus

$$\begin{array}{l} a_{00} + a_{01} + a_{02} + \cdots \\ + a_{10} + a_{11} + a_{12} + \cdots \\ + a_{20} + a_{21} + a_{22} + \cdots \\ + \cdots \cdots \cdots \cdots \cdots \cdots \end{array}$$

In general, we may suppose the terms of any m-tuple series displayed in a similar array, the term $a_{\iota_1 \dots \iota_m}$ occupying the lattice point $\iota = (\iota_1 \cdots \iota_m)$. This affords a geometric image of great service. The terms in the cell R_ν may be denoted by A_ν.

If
$$\lim_{\nu_1 \cdots \nu_m = \infty} A_{\nu_1 \cdots \nu_m} = \lim_{\nu = \infty} A_\nu \tag{2}$$

is finite, A is *convergent* and the limit 2) is called the sum of the series A. When no confusion will arise, we may denote the series and its sum by the same letter. If the limit 2) is infinite or does not exist, we say A is *divergent.*

Thus every m-tuple series gives rise to an m-tuple sequence $\{A_{\nu_1 \cdots \nu_m}\}$. Obviously if all the terms of A are ≥ 0 and A is divergent, the limit 2) is $+\infty$. In this case we say A is *infinite.*

Let us replace certain terms of A by zeros, the resulting series may be called the *deleted series.* If we delete A by replacing all the terms of the cell $R_{\nu_1 \cdots \nu_m}$ by zero, the resulting series is called the *remainder* and is denoted by $\bar{A}_{\nu_1 \cdots \nu_m}$ or by $\bar{A}_\nu$. Similarly if the cell R_ν contains the cell R_μ, the terms lying in R_ν and not in R_μ may be denoted by $A_{\mu, \nu}$.

The series obtained from A by replacing each term of A by its numerical value is called the *adjoint series.* In a similar manner most of the terms employed for simple series may be carried over to m-tuple series. In the series $\Sigma a_{\iota_1 \cdots \iota_m}$ the indices ι all began with 0. There is no necessity for this; they may each begin with any integer at pleasure.

118. *The Geometric Series.* We have seen that

$$\frac{1}{1-a} = 1 + a + a^2 + \cdots \qquad |a| < 1,$$

$$\frac{1}{1-b} = 1 + b + b^2 + \cdots \qquad |b| < 1.$$

Hence

$$\frac{1}{(1-a)(1-b)} = \sum_0^\infty a^m b^n$$

for all points a, b within the unit square.

In general we see that

$$G = \Sigma x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$$

is absolutely convergent for any point x within the unit cube

$$0 \leq |x_\iota| < 1 \qquad \iota = 1, 2, \cdots n,$$

and

$$G = \frac{1}{(1-x_1)(1-x_2)\cdots(1-x_n)}.$$

119. 1. It is important to show how any term of $A = \Sigma a_{\iota_1 \cdots \iota_m}$ can be expressed by means of the $A_{\nu_1 \cdots \nu_m}$.

Let $$D_{\nu_1 \nu_2 \cdots \nu_{m-1}} = A_{\nu_1 \nu_2 \cdots \nu_m} - A_{\nu_1 \nu_2 \cdots \nu_{m-1}}. \quad (1$$

Then $$D_{\nu_1 \nu_2 \cdots \nu_{m-1}-1} = A_{\nu_1 \nu_2 \cdots \nu_{m-1}-1 \nu_m} - A_{\nu_1 \nu_2 \cdots \nu_{m-1}-1,\, \nu_{m-1}}. \quad (2$$

Let $$D_{\nu_1 \nu_2 \cdots \nu_{m-2}} = D_{\nu_1 \nu_2 \cdots \nu_{m-1}} - D_{\nu_1 \nu_2 \cdots \nu_{m-1}-1}. \quad (3$$

Similarly
$$D_{\nu_1 \nu_2 \cdots \nu_{m-3}} = D_{\nu_1 \nu_2 \cdots \nu_{m-2}} - D_{\nu_1 \nu_2 \cdots \nu_{m-2}-1}, \quad (4$$

$$D_{\nu_1 \nu_2 \cdots \nu_{m-4}} = D_{\nu_1 \nu_2 \cdots \nu_{m-3}} - D_{\nu_1 \nu_2 \cdots \nu_{m-3}-1}, \quad (5$$

. . . .

Finally $$D_{\nu_1} = D_{\nu_1 \nu_2} - D_{\nu_1 \nu_2 - 1}, \quad (6$$

and $$a_{\nu_1 \nu_2 \cdots \nu_m} = D_{\nu_1} - D_{\nu_1 - 1}. \quad (7$$

If now we replace the D's by their values in terms of the A's, the relation 7) shows that $a_{\nu_1 \cdots \nu_m}$ may be expressed linearly in terms of a number of $A_{\mu_1 \cdots \mu_m}$ where each $\mu_r = \nu_r$ or $\nu_r - 1$.

For $m = 2$ we find

$$a_{\nu_1 \nu_2} = A_{\nu_1 \nu_2} + A_{\nu_1 - 1,\, \nu_2 - 1} - A_{\nu_1,\, \nu_2 - 1} - A_{\nu_1 - 1,\, \nu_2}. \quad (8$$

2. From 1 it follows that we may take *any* sequence $\{A_{\iota_1 \cdots \iota_m}\}$ to form a multiple series
$$A = \Sigma a_{\iota_1 \cdots \iota_m}.$$

This fact has theoretic importance in studying the peculiarities that multiple series present.

120. We have now the following theorems analogous to 80.

1. *For A to be convergent it is necessary and sufficient that*

$$\epsilon > 0, \qquad p, \qquad |A_{\mu, \nu}| < \epsilon \qquad R_p \leq R_\mu < R_\nu.$$

2. *If A is convergent, so is $\overline{A}_\mu$ and*

$$\overline{A}_\mu = A - A_\mu = \lim_{\nu = \infty} A_{\mu, \nu}.$$

Conversely if $\overline{A}_\mu$ is convergent, so is A.

3. *For A to converge it is necessary and sufficient that*

$$\lim_{\nu=\infty} \bar{A}_\nu = 0.$$

4. *A series whose adjoint converges is convergent.*

5. *Let A be absolutely convergent. Any deleted series B of A is absolutely convergent and* $|B| < \mathrm{A}$.

6. *If $A = \Sigma a_{\iota_1 \cdots \iota_m}$ is convergent, so is $B = \Sigma k a_{\iota_1 \cdots \iota_m}$ and*

$$B = kA, \qquad k \text{ a constant.}$$

121. 1. *For A to converge it is necessary that*

$$D_{\nu_1\nu_2\cdots\nu_{m-1}}\ ,\quad D_{\nu_1\nu_2\cdots\nu_{m-2}}\ ,\quad \cdots D_{\nu_1}\ ,\quad a_{\nu_1\nu_2\cdots\nu_2} \doteq 0, \text{ as } \nu \doteq \infty.$$

For by 120, 1
$$|A_{\lambda_1\cdots\lambda_m} - A_{\mu_1\cdots\mu_m}| < \epsilon$$

if $\lambda_1 \cdots \lambda_m,\quad \mu_1 \cdots \mu_m > p$.

Thus by 119, 1)
$$|D_{\nu_1\nu_2\cdots\nu_{m-1}}| < \epsilon \qquad \nu > p.$$

Hence passing to the limit $p = \infty$,

$$\lim_{\nu=\infty} D_{\nu_1\cdots\nu_{m-1}} \leq \epsilon.$$

As ϵ is small at pleasure, this shows that $D_{\nu_1\cdots\nu_{m-1}} \doteq 0$. In this way we may continue.

2. Although
$$\lim_{\nu_\iota\cdots\nu_m=\infty} a_{\nu_1\cdots\nu_m} = 0$$

when A converges, we must guard against the error of supposing that $a_\nu \doteq 0$ when $\nu = (\nu_1 \cdots \nu_m)$ converges to an ideal point, all of whose coördinates are not ∞ as they are in the limits employed in 1.

This is made clear by the following example due to *Pringsheim*. Let
$$A_{r,s} = \frac{(-1)^{r+s}}{2(a+1)}\left\{\frac{1}{a^r} + \frac{1}{a^s}\right\}\quad ,\quad a > 1.$$

Then by 119, 8)
$$|a_{rs}| = \frac{1}{a^r} + \frac{1}{a^s}.$$

As

$$\lim_{r, s=\infty} A_{r, s} = 0$$

A is convergent. But

$$\lim_{r=\infty} |a_{r, s}| = \frac{1}{a^s} \quad , \quad \lim_{s=\infty} |a_{rs}| = \frac{1}{a^r}.$$

That is when the point (r, s) converges to the ideal point (∞, s), or to the ideal point (r, ∞), a_{rs} does not $\doteq 0$.

3. However, we do have the theorem:

Let

$$A = \Sigma a_{\iota_1 \cdots \iota_m} \qquad a_\iota \geq 0$$

converge. Then for each $\epsilon > 0$ *there exists a* λ *such that* $a_{\iota_1 \cdots \iota_m} < \epsilon$ *for any* ι *outside the rectangular cell* R_λ.

This follows at once from 120, 1, since

$$a_\iota \leq A_{\mu, \nu}.$$

122. 1. *Let* $f(x_1 \cdots x_m)$ *be monotone. Then*

$$\lim_{x=a} f(x_1 \cdots x_m) = l \qquad x_1 < a_1, \cdots x_m < a_m, \text{ } a \text{ may be ideal.} \qquad (1$$

exists, finite or infinite. If f *is limited,* l *is finite. If* f *is unlimited,* $l = +\infty$ *when* f *is monotone increasing, and* $l = -\infty$ *when* f *is monotone decreasing.*

For, let f be limited. Let $A = \alpha_1 < \alpha_2 < \cdots \doteq a$.

Then

$$\lim_{n=\infty} f(\alpha_n) = l$$

is finite by I, 109.

Let now $B = \beta_1, \beta_2, \cdots \doteq a$ be any other sequence.

Let

$$\underline{\lim}_B f(\beta_n) = \underline{l} \qquad \overline{\lim}_B f(\beta_n) = \bar{l}.$$

Then there exists by I, 338 a partial sequence of B, say $C = \gamma_1, \gamma_2 \cdots$ such that

$$\lim f(\gamma_n) = \underline{l},$$

also a partial sequence $D = \delta_1, \delta_2 \cdots$ such that

$$\lim f(\delta_n) = \bar{l}.$$

But for each α_n there exists a $\gamma_{\iota_n} \geq \alpha_n$;
hence
$$f(\gamma_{\iota_n}) \geq f(\alpha_n)$$
and therefore
$$\underline{l} \geq l. \qquad (2$$

Similarly, for each d_n there exists an $\alpha_{\kappa_n} \geq \delta_n$;
hence
$$f(\delta_n) \leq f(\alpha_{\kappa_n})$$
and therefore
$$\bar{l} \leq l. \qquad (3$$

Thus 2), 3) give
$$\lim_B f(x) = l.$$

Hence by I, 316, 2 the relation 1) holds.

The rest of the theorem follows along the same lines.

2. As a corollary we have

The positive term series $A = \Sigma a_{\iota_1 \dots \iota_m}$ *is convergent if* $A_{\nu_1 \dots \nu_m}$ *is limited.*

123. 1. *Let* $A = \Sigma a_{\iota_1 \dots \iota_s} = \Sigma a_\iota$, $B = \Sigma b_{\iota_1 \dots \iota_s} = \Sigma b_\iota$ *be two non-negative term series. If they differ only by a finite number of terms, they converge or diverge simultaneously.*

This follows at once from 120, 2.

2. *Let* A, B *be two non-negative term series. Let* $r > 0$ *denote a constant. If* $a_\iota \leq rb_\iota$, A *converges if* B *is convergent and* $A \leq rB$. *If* $a_\iota \geq rb_\iota$, A *diverges if* B *is divergent.*

For on the first hypothesis
$$A_\lambda \leq rB_\lambda,$$
and on the second
$$A_\lambda \geq rB_\lambda.$$

3. *Let* A, B *be two positive term series. Let* r, s *be positive constants. If*
$$r \leq \frac{a_\iota}{b_\iota} \leq s$$
or if
$$\lim_{\iota=\infty} \frac{a_\iota}{b_\iota}$$
exists and is $\neq 0$, A *and* B *converge or diverge simultaneously. If* B *converges and* $\frac{a_n}{b_n} \doteq 0$, A *is convergent. If* B *diverges and* $\frac{a_n}{b_n} \doteq \infty$, A *is divergent.*

4. *The infinite non-negative term series*

$$\Sigma a_{\iota_1 \cdots \iota_s} \quad \text{and} \quad \Sigma \log (1 + a_{\iota_1 \cdots \iota_s})$$

converge or diverge simultaneously.

This follows from 2.

5. *Let the power series*

$$P = \Sigma c_{m_1 m_2 \cdots m_s} x_1^{m_1} x_2^{m_2} \cdots x_s^{m_s}$$

converge at the point $a = (a_1, \cdots a_s)$, *then it converges absolutely for all points* x *within the rectangular cell* R *whose center is the origin, and one of whose vertices is* a; *that is for* $|x_\iota| < |a_\iota|$, $\iota = 1, 2, \cdots s$.

For since P converges at a,

$$\lim_{m=\infty} c_{m_1 m_2 \cdots} a_1^{m_1} \cdots a_s^{m_s} = 0.$$

Thus there exists an M such that each term

$$| c_{m_1 \cdots} a_1^{m_1} \cdots a_s^{m_s} | \leq M.$$

Hence

$$| c_{m_1 \cdots} x_1^{m_1} \cdots x_s^{m_s} | = | c_{m_1 \cdots} a_1^{m_1} \cdots a_s^{m_s} | \cdot \left| \frac{x_1}{a_1} \right|^{m_1} \cdots \left| \frac{x_s}{a_s} \right|^{m_s}$$

$$\leq M \left| \frac{x_1}{a_1} \right|^{m_1} \cdots \left| \frac{x_s}{a_s} \right|^{m_s}.$$

Thus each term of P is numerically $\leq$ than M times the corresponding term in the convergent geometric series

$$\Sigma \left| \frac{x_1}{a_1} \right|^{m_1} \cdots \left| \frac{x_s}{a_s} \right|^{m_s}.$$

We apply now 2.

We shall call R a *rectangular cell of convergence.*

124. 1. Associated with any m-tuple series $A = \Sigma a_{\iota_1 \cdots \iota_n}$ are an infinite number of simple series called *associate simple series,* as we now show.

Let

$$R_{\lambda_1} \ , \quad R_{\lambda_2} \ , \quad R_{\lambda_3} \ , \quad \cdots$$

be an infinite sequence of rectangular cells each lying in the following. Let

$$a_1 \ , \quad a_2 \ , \quad \cdots a_{s_1}$$

be the terms of A arranged in any order lying in R_{λ_1}. Let

$$a_{s_1+1} \ , \quad a_{s_1+2} \ , \quad \cdots a_{s_2}$$

be the terms of A arranged in order lying in $R_{\lambda_2} - R_{\lambda_1}$, and so on indefinitely.

Then
$$\mathfrak{A} = a_1 + a_2 + \cdots + a_{s_1} + a_{s_1+1} + \cdots$$
is an associate simple series of A.

2. Conversely associated with any simple series $\mathfrak{A} = \Sigma a_n$ are an infinity of *associate m-tuple series.* In fact we have only to arrange the terms of $\mathfrak{A}$ over the non-negative lattice points, and call now the term a_n which lies at the lattice point $\iota_1 \cdots \iota_m$ the term $a_{\iota_1 \ldots \iota_m}$.

3. *Let $\mathfrak{A}$ be an associate series of $A = \Sigma a_{\iota_1 \ldots \iota_m}$. If $\mathfrak{A}$ is convergent, so is A and*
$$A = \mathfrak{A}.$$

For
$$A_{\nu_1 \ldots \nu_m} = \mathfrak{A}_n.$$

Let now $\nu \doteq \infty$, then $n \doteq \infty$. But $\mathfrak{A}_n \doteq \mathfrak{A}$, hence $A_{\nu_1 \ldots \nu_m} \doteq \mathfrak{A}$.

4. *If the associate series $\mathfrak{A}$ is absolutely convergent, so is A.*

Follows from 3.

5 *If $A = \Sigma a_{\nu_1 \ldots \nu_m}$ is a non-negative term convergent series, all its associate series $\mathfrak{A}$ converge.*

For, any $\mathfrak{A}_{m,p}$ lies among the terms of some $A_{\mu,\nu}$. But for λ sufficiently large
$$A_{\mu,\nu} < \epsilon \qquad \lambda < \mu < \nu.$$
Hence
$$\mathfrak{A}_{m,p} < \epsilon \qquad m > m_0.$$

6. *Absolutely convergent series are commutative.*

For let B be the series resulting from rearranging the given series A.

Then any associate $\mathfrak{B}$ of B is simply a rearrangement of an associate series $\mathfrak{A}$ of A. But $\mathfrak{A} = \mathfrak{B}$, hence $A = B$.

7. *A simply convergent m-tuple series A can be rearranged, producing a divergent series.*

For let $\mathfrak{A}$ be an associate of A. $\mathfrak{A}$ is not absolutely convergent, since A is not. We can therefore rearrange $\mathfrak{A}$, producing a series $\mathfrak{B}$ which is divergent. Thus for some $\mathfrak{B}$
$$\lim \mathfrak{B}_n$$
does not exist. Let $\mathfrak{B}'$ be the series formed of the positive, and $\mathfrak{B}''$ the series formed of the negative, terms of $\mathfrak{B}$ taken in order.

Then either $\mathfrak{B}'_n \doteq +\infty$ or $\mathfrak{B}''_n \doteq -\infty$, or both. To fix the ideas suppose the former. Then we can arrange the terms of $\mathfrak{B}$ to form a series $\mathfrak{C}$ such that $\mathfrak{C}_n \doteq +\infty$. Let now $\mathfrak{C}$ be an associate series of C. Then

$$C_\nu = C_{\nu_1\nu_2\cdots\nu_m} = \mathfrak{C}_n$$

and thus

$$\lim C_\nu = \lim \mathfrak{C}_n = +\infty.$$

Hence C is divergent.

8. *If the multiple series A is commutative, it is absolutely convergent.*

For if simply convergent, we can rearrange A so as to make the resulting series divergent, which contradicts the hypothesis.

9. In 121, 2 we exhibited a convergent series to show that $a_{\iota_1\cdots\iota_m}$ does not need to converge to 0 if $\iota_1 \cdots \iota_m$ converges to an ideal point some of whose coördinates are finite. As a counterpart we have the following:

Let A be absolutely convergent. Then for each $\epsilon > 0$ there exists a λ, such that any finite set of terms B lying without R_λ satisfy the relation

$$|B| < \epsilon; \tag{1}$$

and conversely.

For let $\mathfrak{A}$ be an associate simple series of Adj A. Since $\mathfrak{A}$ is convergent there exists an n, such that

$$\overline{\mathfrak{A}}_n < \epsilon.$$

But if λ is taken sufficiently large, each term of B lies in $\mathfrak{A}_n$, which proves 1).

Suppose now A were simply convergent. Then, as shown in 7, there exists an associate series $\mathfrak{D}$ which is infinite.

Hence, however large n is taken, there exists a p such that

$$|\mathfrak{D}_{n,p}| > \epsilon.$$

Hence, however large λ is taken, there exist terms $B = \mathfrak{D}_{n,p}$ which do not satisfy 1).

10. We have seen that associated with any m-tuple series

$$A = \Sigma a_{\iota_1\cdots\iota_m}$$

extended over a lattice system $\mathfrak{M}$ in $\mathfrak{R}_m$ is a simple series in $\mathfrak{R}_1$. We can generalize as follows. Let $\mathfrak{M} = \{i\}$ be associated with a lattice system $\mathfrak{M} = \{j\}$ in $\mathfrak{R}_n$ such that to each ι corresponds a j and conversely.

If $\iota \sim j$ we set
$$a_{\iota_1 \cdots \iota_m} = a_{j_1 \cdots j_n}.$$

Then A gives rise to an infinity of n-tuple series as
$$B = \Sigma a_{j_1 \cdots j_n}.$$

We say B is a *conjugate n-tuple series.*

We have now the following:

Let A be absolutely convergent. Then the series B is absolutely convergent and $A = B$.

For let A', B' be associate simple series of A, B. Then A', B' are absolutely convergent and hence $A' = B'$. But $A = A'$, $B = B'$. Hence $A = B$, and B is absolutely convergent.

11. Let $A = \Sigma a_{\iota_1 \cdots \iota_m}$ be absolutely convergent. Let $B = \Sigma a_{\kappa_1 \cdots \kappa_p}$ be any p-tuple series formed of a part or all the terms of A. Then B is absolutely convergent and
$$|B| \lesseqgtr \text{Adj } A.$$

For let A', B' be associate simple series of A and B. Then B' converges absolutely and $|B'| \leq \text{Adj } A$.

125. 1. Let
$$A = \Sigma a_{\iota_1 \cdots \iota_m}. \tag{1}$$

Set
$$f(x_1 \cdots x_m) = a_{\iota_1 \cdots \iota_m}$$
in the cell
$$\iota_1 - 1 < x_1 \leq \iota_1, \qquad \cdots \qquad \iota_m - 1 < x_m \leq \iota_m.$$
Then
$$A_{\nu_1 \cdots \nu_m} = \int_{R_{\nu_1, \cdots \nu_m}} f dx_1 \cdots dx_m. \tag{2}$$

Let R denote that part of $\mathfrak{R}_m$ whose points have non-negative coördinates. Let
$$J = \int_R f dx_1 \cdots dx_m. \tag{3}$$

If J is convergent, $A = J$. We cannot in general state the converse, for A is obtained from A_ν by a special passage to the limit, viz.

by employing a sequence of rectangular cells. If, however, $a_\nu \geq 0$ we may, and we have

For the non-negative term series 1) *to converge it is necessary and sufficient that the integral* 3) *converges.*

2. *Let* $f(x_1 \cdots x_m) \geq 0$ *be a monotone decreasing function of* x *in* R, *the aggregate of points all of whose coördinates are non-negative. Let*

$$a_{\iota_1 \cdots \iota_m} = f(\iota_1 \cdots \iota_m).$$

The series

$$A = \Sigma a_{\iota_1 \cdots \iota_m}$$

is convergent or divergent with

$$J = \int_R f dx_1 \cdots dx_m.$$

For let $R_1, R_2, \cdots$ be a sequence of rectangular cubes each R_n contained in R_{n+1}.
Let

$$R_{n,s} = R_s - R_n \qquad s > n.$$

Then λ, μ being taken at pleasure but $>$ some ν, there exist an l, m such that

$$A_{\lambda\mu} < \int_R f_{lm}.$$

But the integral on the right can be made small at pleasure if J is convergent on taking $l > m >$ some n. Hence A is convergent if J is. Similarly the other half of the theorem follows.

Iterated Summation of Multiple Series

126. Consider the finite sum

$$\Sigma a_{\iota_1 \cdots \iota_m} \qquad \iota_1 = 0, 1, \cdots n_1 \quad \cdots \quad \iota_m = 0, 1, \cdots n_m. \tag{1}$$

One way to effect the summation is to keep all the indices but one fixed, say all but ι_1, obtaining the sum

$$\sum_{\iota_1=0}^{m_1} a_{\iota_1 \cdots \iota_m}.$$

Then taking the sum of these sums when only ι_2 is allowed to vary obtaining the sum

$$\sum_{\iota_2=0}^{m_2} \sum_{\iota_1=0}^{m_1} a_{\iota_1 \cdots \iota_m}$$

and so on arriving finally at

$$\sum_{\iota_m=0}^{m_n} \cdots \sum_{\iota_1=0}^{m_1} a_{\iota_1 \cdots \iota_m} \tag{2}$$

whose value is that of 1). We call this process *iterated summation.* We could have taken the indices $\iota_1 \cdots \iota_m$ in any order instead of the one just employed; in each case we would have arrived at the same result, due to the commutative property of finite sums.

Let us see how this applies to the infinite series,

$$A = \Sigma a_{\iota_1 \cdots \iota_m}, \qquad \iota_1 \cdots \iota_m = 0,\, 1, \cdots \infty. \tag{3}$$

The corresponding process of iterated summation would lead us to a series

$$\mathfrak{A} = \sum_{\iota_m=0}^{\infty} \sum_{\iota_{m-1}=0}^{\infty} \cdots \sum_{\iota_1=0}^{\infty} a_{\iota_1 \cdots \iota_m}, \tag{4}$$

which is an *m-tuple iterated series.* Now by definition

$$\mathfrak{A} = \lim_{\nu_m=\infty} \sum_{\iota_m=0}^{\nu_m} \lim_{\nu_{m-1}=\infty} \sum_{\iota_{m-1}=0}^{\nu_{m-1}} \cdots \lim_{\nu_1=\infty} \sum_{\iota_1=0}^{\nu_1} a_{\iota_1 \cdots \iota_m} \tag{5}$$

$$= \lim_{\nu_m=\infty} \lim_{\nu_{m-1}=\infty} \cdots \lim_{\nu_1=\infty} A_{\nu_1 \cdots \nu_m}, \tag{6}$$

while

$$A = \lim_{\nu_1 \cdots \nu_m} A_{\nu_1 \cdots \nu_m}. \tag{7}$$

Thus A is defined by a general limit while $\mathfrak{A}$ is defined by an iterated limit. These two limits may be quite different. Again in 6) we have passed to the limit in a certain order. Changing this order in 6) would give us another iterated series of the type 4) with a sum which may be quite different. However in a large class of series the summation may be effected by iteration and this is one of the most important ways to evaluate 3).

The relation between iterated summation and iterated integration will at once occur to the reader.

127. 1. Before going farther let us note some peculiarities of iterated summation. For simplicity let us restrict ourselves to double series. Obviously similar anomalies will occur in m-tuple series.

Let
$$A = a_{00} + a_{01} + a_{02} + \cdots + a_{10} + a_{11} + a_{12} + \cdots + \cdots$$

be a double series. The m^{th} row forms a series

$$R^{(m)} = a_{m,0} + a_{m1} + \cdots = \sum_{n=0}^{\infty} a_{mn};$$

and the n^{th} column, the series

$$C^{(n)} = a_{0n} + a_{1n} + \cdots = \sum_{m=0}^{\infty} a_{mn}.$$

Then
$$R = \sum_{m=0}^{\infty} R^{(m)} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn},$$

$$C = \sum_{n=0}^{\infty} C^{(n)} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}$$

are the series formed by *summing by rows* and *columns*, respectively.

2. *A double series may converge although every row and every column is divergent.*

This is illustrated by the series considered in 121, 2. For A is convergent while $\sum_{r=0}^{\infty} a_{rs}$, $\sum_{s=0}^{\infty} a_{rs}$ are divergent, since their terms are not evanescent.

3. *A double series A may be divergent although the series R obtained by summing A by rows or the series C obtained by summing by columns is convergent.*

For let
$$A_{rs} = 0 \qquad \text{if } r \text{ or } s = 0$$

$$= \frac{r}{r+s} \qquad \text{if } r, s > 0.$$

Obviously by I, 318, lim A_{rs} does not exist and $A \doteq \Sigma a_{rs}$ is divergent.

On the other hand,
$$R = \lim_{r=\infty} \lim_{s=\infty} A_{rs} = 0,$$

$$C = \lim_{s=\infty} \lim_{r=\infty} A_{rs} = 1.$$

Thus both R and C are convergent.

4. In the last example R and C converged but their sums were different. We now show:

A double series may diverge although both R and C converge and have the same sum.

For let
$$A_{r,s} = 0 \qquad \text{if } r \text{ or } s = 0$$
$$= \frac{rs}{r^2 + s^2} \qquad \text{if } r, s > 0.$$

Then by I, 319, $\lim A_{rs}$ does not exist and A is divergent. On the other hand,
$$R = \lim_{r=\infty} \lim_{s=\infty} A_{rs} = 0,$$
$$C = \lim_{s=\infty} \lim_{r=\infty} A_{rs} = 0.$$

Then R and S both converge and have the same sum.

128. We consider now some of the cases in which iterated summation is permissible.

Let $A = \sum_0^\infty a_{\iota_1 \cdots \iota_m}$ be convergent. Let $\iota_1', \iota_2', \cdots \iota_m'$ be any permutation of the indices $\iota_1, \iota_2, \cdots \iota_m$. If all the $m - 1$-tuple series
$$\sum_{\iota_2'=0}^\infty \sum_{\iota_3'=0}^\infty \cdots \sum_{\iota_m'=0}^\infty a_{\iota_1 \cdots \iota_m}.$$
are convergent,
$$A = \sum_{\iota_1'=0}^\infty \cdots \sum_{\iota_m'=0}^\infty a_{\iota_1 \cdots \iota_m}.$$

This follows at once from I, 324. For simplicity the theorem is there stated only for two variables; but obviously the demonstration applies to any number of variables.

129. 1. *Let $f(x_1 \cdots x_m)$ be a limited monotone function. Let the point $a = (a_1 \cdots a_m)$ be finite or infinite. When f is limited, all the s-tuple iterated limits*
$$\lim_{x_{\iota_1}=a_{\iota_1}} \cdots \lim_{x_{\iota_s}=a_{\iota_s}} f \tag{1}$$
exist. When $s = m$, these limits equal
$$\lim_{x=a} f(x_1 \cdots x_m). \tag{2}$$

In these limits we suppose $x < a$.

For if f is limited, $$\lim_{x_{\iota_s}=a_{\iota_s}} f \quad , \quad x_{\iota_s} < a_{\iota_s} \tag{3}$$

exists by 122, 1. Moreover 3) is a monotone function of the remaining $m-1$ variables.

Hence similarly $$\lim_{x_{\iota_{s-1}}=a_{\iota_{s-1}}} \lim_{x_{\iota_s}=a_{\iota_s}} f$$

exists and is a monotone function of the remaining $m-2$ variables, etc. The rest of the theorem follows as in I, 324.

2. As a corollary we have

Let A be a non-negative term m-tuple series. If A or any one of its m-tuple iterated series is convergent, A and all the $m!$ iterated m-tuple series are convergent and have the same sum. If one of these series is divergent, they all are.

3. *Let a be a non-negative term m-tuple series. Let $s < m$. All the s-tuple iterated series of A are convergent if A is, and if one of these iterated series is divergent, so is A.*

130. 1. *Let $A = \Sigma a_{\iota_1 \cdots \iota_m}$ be absolutely convergent. Then all its s-tuple iterated series $s = 1, 2 \cdots m$, converge absolutely and its m-tuple iterated series all $= A$.*

For as usual let $\alpha_{\iota_1 \cdots \iota_m} = | a_{\iota_1 \cdots \iota_m} |$. Since $\mathrm{A} = \mathrm{Adj}\ A$ is convergent, all the s-tuple iterated series of A are convergent. Thus $s_1 = \sum\limits_{\iota_1=0}^{\infty} a_{\iota_1 \cdots \iota_m}$ is convergent since $\sum\limits_{\iota_1=0}^{\infty} \alpha_{\iota_1 \cdots \iota_m} = \sigma_1$. Moreover $| s_1 | < \sigma_1$. Similarly $\sum\limits_{\iota_2=0}^{\infty} \sum\limits_{\iota_1=0}^{\infty} a_{\iota_1 \cdots \iota_m} = \sum\limits_{\iota_2} s_1$ is convergent since $\sum\limits_{\iota_2=0}^{\infty} \sum\limits_{\iota_1=0}^{\infty} \alpha_{\iota_1 \cdots \iota_m} = \sum\limits_{\iota_2} \sigma_1$ is convergent; etc. Thus every s-tuple iterated series of A is absolutely convergent. The rest follows now by 128.

2. *Let $A = \Sigma a_{\iota_1 \cdots \iota_m}$. If one of the m-tuple iterated series* B *formed from the adjoint* A *of A is convergent, A is absolutely convergent.*

Follows from 129, 2.

3. The following example may serve to guard the reader against a possible error.

Consider the series

$$A = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots$$
$$+ 1 + 2a + \frac{(2a)^2}{2!} + \frac{(2a)^3}{3!} + \cdots$$
$$+ 1 + 3a + \frac{(3a)^2}{2!} + \frac{(3a)^3}{3!} + \cdots$$
$$+ \cdots$$

Here
$$R^{(n)} = 1 + \frac{na}{1!} + \frac{(na)^2}{2!} + \cdots = e^{na}$$

and
$$R = e^a + e^{2a} + e^{3a} +$$

This is a geometric series and converges *absolutely* for $a < 0$. Thus one of the double iterated series of A is absolutely convergent. We cannot, however, infer from this that A is convergent, for the theorem of 2 requires that one of the iterated series formed from the *adjoint* of A should converge. Now both those series are divergent. The series A is divergent, for $|a_{rs}| \doteq \infty$, as $r, s \doteq \infty$.

131. 1. Up to the present the series

$$\Sigma a_{\iota_1 \cdots \iota_m} \tag{1}$$

have been extended only over non-negative lattice points. This restriction was imposed only for convenience; we show now how it may be removed. Consider the signs of the coördinates of a point $x = (x_1, \cdots x_m)$. Since each coördinate can have two signs, there are 2^m combinations of signs. The set of points x whose coördinates belong to a given one of these combinations form a *quadrant* for $m = 2$, an *octant* for $m = 3$, and a 2^m-*tant* or *polyant* in $\Re_m$. The polyant consisting of the points all of whose coördinates are ≥ 0 may be called the *first* or *principal polyant.*

Let us suppose now that the indices i in 1) run over one or more polyants. Let R_λ be a rectangular cell, the coördinates of each of its vertices being each numerically $\leq \lambda$. Let A_λ denote the terms of A lying in R_λ. Then l is the limit of A_λ for $\lambda = \infty$, if for each $\epsilon > 0$ there exists a λ_0 such that

$$|A_\lambda - A_{\lambda_0}| < \epsilon \qquad \lambda \geq \lambda_0. \tag{2}$$

If
$$\lim_{\lambda=\infty} A_\lambda \tag{3}$$
exists, we say A is *convergent*, otherwise A is *divergent*. In a similar manner the other terms employed in multiple series may be extended to the present case. The rectangular cell R_{λ_0} which figures in the above definition may without loss of generality be replaced by the cube

$$|x_1| \leq \lambda_0 \quad \cdots \quad |x_m| \leq \lambda_0.$$

Moreover the condition necessary and sufficient for the existence of the limit 3) is that

$$|A_\lambda - A_\mu| < \epsilon \qquad \lambda, \mu \geq \lambda_0.$$

132. The properties of series lying in the principal polyant may be readily extended to series lying in several polyants. For the convenience of the reader we bring the following together, omitting the proof when it follows along the same lines as before.

1. *For A to converge it is necessary and sufficient that*

$$\lim_{\lambda=\infty} \overline{A}_\lambda = 0.$$

2. *A series whose adjoint converges is convergent.*

3. *Any deleted series B of an absolutely convergent series A is absolutely convergent and*

$$|B| < \mathrm{Adj}\, A.$$

4. *If $A = \Sigma a_{\iota_1 \cdots \iota_n}$ is convergent, so is $B = \Sigma k a_{\iota_1 \cdots \iota_n}$ and $A = kB$.*

5. *The non-negative term series A is convergent if A_λ is limited,* $\lambda \doteq \infty$.

6. *If the associate simple series $\mathfrak{A}$ of an m-tuple series A converges, A is convergent. Moreover if $\mathfrak{A}$ is absolutely convergent, so is A. Finally if A converges absolutely, so does $\mathfrak{A}$.*

7. *Absolutely convergent series are commutative and conversely.*

8. *Let $f(x_1 \cdots x_m) \geq 0$ be a monotone decreasing function of the distance of x from the origin.*

Let
$$a_{\iota_1 \cdots \iota_m} = f(\iota_1 \cdots \iota_m).$$

Then

$$A = \Sigma a_{\iota_1 \cdots \iota_m}$$

converges or diverges with

$$\int_{\mathfrak{R}} f dx_1 \cdots dx_m,$$

the integration extended over all space containing terms of A.

133. 1. *Let B, C, D $\cdots$ denote the series formed of the terms of A lying in the different polyants. For A to converge it is sufficient although not necessary that B, C, $\cdots$ converge. When they do,*

$$A = B + C + D + \cdots \tag{1}$$

For if B_λ, C_λ $\cdots$ denote the terms of B, C $\cdots$ which lie in a rectangular cell R_λ,

$$A_\lambda = B_\lambda + C_\lambda + \cdots$$

Passing to the limit we get 1).

That A may converge when B, C, $\cdots$ do not is shown by the following example. Let all the terms of $A = \Sigma a_{\iota_1 \cdots \iota_m}$ vanish except those lying next to the coördinate axes. Let these have the value $+1$ if $\iota_1, \iota_2 \cdots \iota_m > 0$ and let two a's lying on opposite sides of the coördinate planes have the same numerical value but opposite signs. Obviously, $A_\lambda = 0$, hence A is convergent. On the other hand, every B, C $\cdots$ is divergent.

2. Thus when B, C $\cdots$ converge, the study of the given series A may be referred to series whose terms lie in a single polyant. But obviously the theory of such series is identical with that of the series lying in the first polyant.

3. The preceding property enables us at once to extend the theorems of 129, 130 to series lying in more than one polyant. The iterated series will now be made up, in general of two-way simple series.

CHAPTER V

SERIES OF FUNCTIONS

134. 1. Let $\iota=(\iota_1, \iota_2 \cdots \iota_p)$ run over an infinite lattice system $\mathfrak{L}$. Let the one-valued functions

$$f_{\iota_1 \cdots \iota_p}(x_1 \cdots x_m)=f_\iota(x)=f_\iota$$

be defined over a domain $\mathfrak{A}$, finite or infinite. If the p-tuple series

$$F=F(x)=F(x_1 \cdots x_m)=\Sigma f_{\iota_1 \cdots \iota_p}(x_1 \cdots x_m) \qquad (1$$

extended over the lattice system $\mathfrak{L}$ is convergent, it defines a one-valued function $F(x_1 \cdots x_m)$ over $\mathfrak{A}$. We propose to study the properties of this function with reference to continuity, differentiation and integration.

2. Here, as in so many parts of the theory of functions depending on changing the order of an iterated limit, *uniform convergence* is fundamental.

We shall therefore take this opportunity to develop some of its properties in an entirely general manner so that they will apply not only to infinite series, but to infinite products, multiple integrals, etc.

3. In accordance with the definition of I, 325 we say the series 1) is *uniformly convergent in* $\mathfrak{A}$ when F_μ converges uniformly to its limit F. Or in other words when for each $\epsilon>0$ there exists a λ such that

$$|F-F_\mu|<\epsilon \qquad \mu \geq \lambda,$$

for any x in $\mathfrak{A}$. Here, as in 117, F_μ denotes the terms of 1) lying in the rectangular cell R_μ, etc.

As an immediate consequence of this definition we have :

Let 1) *converge in* $\mathfrak{A}$. *For it to converge uniformly in* $\mathfrak{A}$ *it is necessary and sufficient that* $|\overline{F}_\lambda|$ *is uniformly evanescent in* $\mathfrak{A}$, *or in other words that for each* $\epsilon>0$, *there exists a* λ *such that* $|\overline{F}_\mu|>\epsilon$ *for any* x *in* $\mathfrak{A}$, *and* $\mu \geq \lambda$.

135. 1. Let

$$\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$$

in $\mathfrak{A}$. Here $\mathfrak{A}$, τ may be finite or infinite. If there exists an $\eta > 0$ such that $f \doteq \phi$ uniformly in $V_\eta(a)$, a finite or infinite, we shall say f converges uniformly at a; if there exists no $\eta < 0$, we say f *does not* converge uniformly at a.

2. Let now a range over $\mathfrak{A}$. Let $\mathfrak{B}$ denote the points of $\mathfrak{A}$ at which no η exists or those points, they may lie in $\mathfrak{A}$ or not, in whose vicinity the minimum of η is 0. Let D denote a cubical division of space of norm d. Let $\mathfrak{B}_D$ denote as usual the cells of D containing points of $\mathfrak{B}$. Let $\mathfrak{C}_D$ denote the points of $\mathfrak{A}$ not in $\mathfrak{B}^D$. Then $f \doteq \phi$ uniformly in $\mathfrak{C}_D$ however small d is taken, but then fixed. The converse is obviously true.

3. *If f converges uniformly in $\mathfrak{A}$, and if moreover it converges at a finite number of other points $\mathfrak{B}$, it converges uniformly in $\mathfrak{A} + \mathfrak{B}$.*

For if $f \doteq \phi$ uniformly in $\mathfrak{A}$,

$$|f - \phi| < \epsilon \qquad x \text{ in } \mathfrak{A}, \qquad t \text{ in } V_{\delta_0}{}^*(\tau).$$

Then also at each point b_s of $\mathfrak{B}$,

$$|f - \phi| < \epsilon \qquad x = b_s \qquad t \text{ in } V_{\delta_s}{}^*(\tau).$$

If now $\delta < \delta_0, \delta_1, \delta_2 \cdots$ these relations hold for any x in $\mathfrak{A} + \mathfrak{B}$ and any t in $V_\delta{}^*(\tau)$.

4. *Let $f(x_1 \cdots x_m, t_1 \cdots t_n) \doteq \phi(x_1 \cdots x_m)$ uniformly in $\mathfrak{A}$. Let f be limited in $\mathfrak{A}$ for each t in $V_\delta{}^*(\tau)$. Then ϕ is limited in $\mathfrak{A}$.*

For $$\phi = f(x, t) + \epsilon' \qquad |\epsilon'| < \epsilon \tag{1}$$

for any x in $\mathfrak{A}$ and t in $V_\delta{}^*(\tau)$. Let us therefore fix t. The relation 1) shows that ϕ is limited in $\mathfrak{A}$.

5. *If $\Sigma\, |f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)|$ converges uniformly in $\mathfrak{A}$, so does $\Sigma f_{\iota_1 \cdots \iota_s}$.* For any remainder of a series is numerically $\leq$ than the corresponding remainder of the adjoint series.

6. Let the s-tuple series

$$F = \Sigma f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)$$

converge *uniformly* in $\mathfrak{A}$. Then for each $\epsilon > 0$ there exists a λ such that

$$|F_{\mu\nu}| < \epsilon \tag{1}$$

for any $R_\nu \geq R_\mu \geq R_\lambda$. When $s = 1$, these rectangular cells reduce to intervals, and thus we have in particular

$$|f_n(x_1 \cdots x_m)| < \epsilon \qquad \text{for any } n \geq n'.$$

When $s > 1$ we cannot infer from 1) that

$$|f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)| < \epsilon \quad , \quad \text{in } \mathfrak{A}, \tag{2}$$

for any ι lying outside the above mentioned cell R_λ.

A similar difference between simple and multiple series was mentioned in 121, 2.

However if $f_\iota \geq 0$ in $\mathfrak{A}$, the relation does hold. Cf. 121, 3.

136. 1. *Let $f(x_1 \cdots x_m, t_1 \cdots t_n)$ be defined for each x in $\mathfrak{A}$, and t in $\mathfrak{T}$. Let*

$$\lim_{t=\tau} f = \phi(x_1 \cdots x_m) \qquad \textit{in } \mathfrak{A},$$

τ finite or infinite. The convergence is uniform if for any x in $\mathfrak{A}$

$$|f - \phi| < \psi(t_1 \cdots t_n) \qquad t \textit{ in } V_\delta^*(\tau), \ \delta \textit{ fixed}$$

while

$$\lim_{t=\tau} \psi = 0.$$

For taking $\epsilon > 0$ at pleasure there exists an $\eta > 0$ such that

$$|\psi| < \epsilon \quad , \quad t \text{ in } V_\eta^*(\tau).$$

But then if $\delta < \eta$,

$$|f - \phi| < \epsilon$$

for any t in $V_\delta^*(\tau)$ and any x in $\mathfrak{A}$.

Example.

$$\lim_{y=\frac{\pi}{2}} \frac{\sin x \sin y}{1 + x \tan^2 y} = 0 = \phi, \qquad \text{in } \mathfrak{A} = (0, \infty).$$

Is the convergence uniform?

Let

$$y = \frac{\pi}{2} - u;$$

then $u \doteq 0$, as $y \doteq \frac{\pi}{2}$.

Then $$\left|f-\phi\right| = \left|\frac{\sin x \cos u}{1+x\cot^2 u}\right| = \left|\frac{\sin x \cos u \sin^2 u}{\sin^2 u + x\cos^2 u}\right|$$

$$\leq \left|\frac{\sin x \sin^2 u}{x\cos^2 u}\right| \leq \tan^2 u \doteq 0.$$

Hence the convergence is uniform in $\mathfrak{A}$.

2. As a corollary we have

Weierstrass' Test. For each point in $\mathfrak{A}$, let $|f_{\iota_1 \cdots \iota_p}| \leq M_{\iota_1 \cdots \iota_p}$ The series $\Sigma f_{\iota_1 \cdots \iota_p}(x_1 \cdots x_m)$ is uniformly convergent in $\mathfrak{A}$ if $\Sigma M_{\iota_1 \cdots \iota_p}$ is convergent.

Example 1.
$$F = \sum \frac{e^{nx}-1}{2^n e^{nx}} \qquad \mathfrak{A} = (0, \infty)$$

Here
$$|f_n| < \frac{1}{2^n}$$

and F is uniformly convergent in $\mathfrak{A}$ since
$$\sum \frac{1}{2^n}$$
is convergent.

Example 2.
$$F(x) = \Sigma a_n \sin \lambda_n x$$

is uniformly convergent for $(-\infty, \infty)$ if
$$\Sigma \, |a_n|$$
is convergent.

137. 1. *The power series $P = \Sigma a_{m_1 \cdots m_p} x_1^{m_1} \cdots x_p^{m_p}$ converges uniformly in any rectangle R lying within its rectangle of convergence.*

For let $b = (b_1, \cdots b_p)$ be that vertex of R lying in the principal polyant. Then P is absolutely convergent at b, *i.e.*

$$\Sigma a_{m_1 \cdots m_p} b_1^{m_1} \cdots b_p^{m_p} \tag{1}$$

is convergent. Let now x be any point of R. Then each term in

$$\Sigma a_{m_1 \cdots m_p} \xi_1^{m_1} \cdots \xi_p^{m_p}$$

is $\leq$ than the corresponding term in 1).

2. *If the power series* $P = a_0 + a_1x + a_2x^2 + \cdots$ *converges at an end point of its interval of convergence, it converges uniformly at this point.*

Suppose P converges at the end point $x = R > 0$. Then

$$| a_{m+1}R^{m+1} + \cdots + a_nR^n | < \epsilon$$

however large n is taken. But for $0 < x \leq R$

$$| a_{m+1}x^{m+1} + \cdots + a_nx^n |$$

$$= \left| a_{m+1}R^{m+1}\left(\frac{x}{R}\right)^{m+1} + \cdots + a_nR^n\left(\frac{x}{R}\right)^n \right|$$

$$< \epsilon \qquad \text{by Abel's identity, 83, 1.}$$

Thus the convergence is uniform at $x = R$. In a similar manner we may treat $x = -R$.

3. Let $f_n(x_1 \cdots x_m)$, $n = 1, 2 \cdots$ be defined over a set $\mathfrak{A}$. If each $|f_n| <$ some constant c_n in $\mathfrak{A}$, f_n is limited in $\mathfrak{A}$. If moreover the c_n are all $<$ some constant C, we say the $f_n(x)$ are *uniformly limited* in $\mathfrak{A}$. In general if each function in a set of functions $\{f\}$ defined over at point set $\mathfrak{A}$ satisfy the relation

$$|f| < \text{a fixed constant } C, \ x \text{ in } \mathfrak{A},$$

we say the f's are uniformly limited in $\mathfrak{A}$.

The series $F = \Sigma g_n h_n$ *is uniformly convergent in* $\mathfrak{A}$, *if* $G = g_1 + g_2 + \cdots$ *is uniformly convergent in* $\mathfrak{A}$, *while* $\Sigma | h_{n+1} - h_n |$ *and* $| h_n |$ *are uniformly limited in* $\mathfrak{A}$.

This follows at once from Abel's identity as in 83, 2.

4. *The series* $F = \Sigma g_n h_n$ *is uniformly convergent in* $\mathfrak{A}$, *if in* $\mathfrak{A}$, $\Sigma | h_{n+1} - h_n |$ *is uniformly convergent*, h_n *is uniformly evanescent, and the* G_n *uniformly limited.*

Follows from Abel's identity, 83, 1.

5. *The series* $F = \Sigma g_n h_n$ *is uniformly convergent in* $\mathfrak{A}$ *if* $G = g_1 + g_2 + \cdots$ *is uniformly convergent in* $\mathfrak{A}$ *while* $h_1, h_2 \cdots$ *are uniformly limited in* $\mathfrak{A}$ *and* $\{h_n\}$ *is a monotone sequence for each point of* $\mathfrak{A}$.

For by 83, 1, $$| F_{n,p} | < H \cdot | G_{n,p} |.$$

6. *The series $F = \Sigma g_n h_n$ is uniformly convergent in $\mathfrak{A}$ if $G_1 = g_1$, $G_2 = g_1 + g_2, \cdots$ are uniformly limited in $\mathfrak{A}$ and if $h_1, h_2, \cdots$ not only form a monotone decreasing sequence for x in $\mathfrak{A}$ but also are uniformly evanescent.*

For by 83, 1,
$$|F_{n,p}| < |h_{n+1}| G.$$

Example. Let $A = a_1 + a_2 + \cdots$ be convergent. Let $b_1, b_2 \cdots \neq 0$ be a limited monotone sequence. Then

$$F(x) = \sum \frac{a_n}{1 - b_n x}$$

converges uniformly in any interval $\mathfrak{A}$ which does not contain a point of $\left\{\frac{1}{b_n}\right\}$.

For obviously the numbers

$$h_n = \frac{1}{1 - b_n x}$$

form a monotone sequence at each point of $\mathfrak{A}$. We now apply 5.

7. As an application of these theorems we have, using the results of 84,

The series
$$a_0 + a_1 \cos x + a_2 \cos 2x + \cdots$$

converges uniformly in any complete interval not containing one of the points $\pm 2m\pi$ provided $\Sigma | a_{n+1} - a_n |$ is convergent and $a_n \doteq 0$, and hence in particular if $a_1 \geq a_2 \geq \cdots \doteq 0$.

8. *The series*
$$a_0 - a_1 \cos x + a_2 \cos 2x - \cdots$$

converges uniformly in any complete interval not containing one of the points $\pm (2m-1)\pi$ provided $\Sigma | a_{n+1} + a_n |$ is convergent and $a_n \doteq 0$, and hence in particular if $a_1 \geq a_2 \geq \cdots \doteq 0$.

9. *The series*
$$a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \cdots$$

converges uniformly in any complete interval not containing one of the points $\pm 2m\pi$ provided $\Sigma | a_{n+1} - a_n |$ is convergent and $a_n \doteq 0$, and hence in particular if $a_1 \geq a_2 \geq \cdots \doteq 0$.

10. *The series*
$$a_1 \sin x - a_2 \sin 2x + a^3 \sin 3x - \cdots$$
converges uniformly in any complete interval not containing one of the points $\pm(2m-1)\pi$ *provided* $\Sigma \mid a_{n+1} + a_n \mid$ *is convergent and* $a_n \doteq 0$, *and hence in particular if* $a_1 \geq a_2 \geq \cdots \doteq 0$.

138. 1. *Let*
$$F = \Sigma f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)$$
be uniformly convergent in $\mathfrak{A}$. *Let* A, B *be two constants and*
$$Af_\iota(x) \leq g_\iota(x) \leq Bf_\iota(x) \qquad \text{in } \mathfrak{A}.$$
Then
$$G = \Sigma g_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)$$
is uniformly convergent in $\mathfrak{A}$.

For then
$$AF_{\lambda, \mu} \leq G_{\lambda, \mu} \leq BF_{\lambda, \mu}.$$
But F being uniformly convergent,
$$|F_{\lambda, \mu}| < \epsilon.$$

2. *Let*
$$F = \Sigma f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m) \qquad f_\iota \geq 0$$
converge uniformly in $\mathfrak{A}$. *Then*
$$L = \Sigma \log (1 + f_\iota)$$
is uniformly convergent in $\mathfrak{A}$. *Moreover if* F *is limited in* $\mathfrak{A}$, *so is* L.

For $f_\iota \geq 0$ in $\mathfrak{A}$, hence
$$|f_\iota| < \epsilon$$
for any ι outside some rectangular cell R_λ.

Thus for such i
$$Af_\iota \leq \log (1 + f_\iota) \leq Bf_\iota \qquad \text{in } \mathfrak{A}.$$

139. 1. *Preserving the notation of* 136, *let* $g_1, g_2, \cdots g_m$ *be chosen such that if we set*
$$x_1 = g_1(t_1 \cdots t_n) \quad , \quad \cdots \quad x_m = g_m(t_1 \cdots t_n),$$
then $x = (x_1 \cdots x_m)$ *lies in* $\mathfrak{A}$ *as* $t = (t_1 \cdots t_n) \doteq \tau$. *If* $f \doteq \phi$ *uniformly in* $\mathfrak{A}$,
$$\lim_{t=\tau} \Delta = \lim_{t=\tau} \{f(g_1 \cdots g_m, t_1 \cdots t_n) - \phi(t_1 \cdots t_n)\} = 0.$$

For if $f \doteq \phi$ uniformly in $\mathfrak{A}$,

$$\epsilon > 0, \qquad \delta > 0 \qquad |f - \phi| < \epsilon$$

for any x in $\mathfrak{A}$ and any t in $V_\delta^*(\tau)$, δ independent of x.

But then $\qquad |\Delta| < \epsilon \qquad t$ in $V_\delta^*(\tau)$.

2. As a corollary we have :

Let $a_1, a_2, \cdots \doteq a$. *Let* $F = \Sigma f_s$ *be uniformly convergent at* a. *Then*

$$\bar{F}_n(a_n) \doteq 0.$$

140. *Example 1.*

$$\lim_{u=0} f = \lim_{u=0} \frac{\sin u \sin 2u}{\sin^2 u + x \cos^2 u} = \phi(x) = \begin{cases} 2 \text{ for } x = 0, \\ 0 \text{ for } x \neq 0. \end{cases}$$

The convergence is not uniform at $x = 0$. For

$$f = \frac{2 \cos u}{1 + x \cot^2 u}.$$

Hence if we set $x = u^2$

$$\lim_{u=0} f = 1, \qquad \text{since } u^2 \cot^2 u \doteq 1.$$

Thus on this assumption

$$\lim |f - g| = |1 - 2| = 1.$$

Example 2. $\quad F = 1 - x + x(1 - x) + x^2(1 - x) + x^3(1 - x) + \cdots$

Here

$$F = \sum_0^\infty (1 - x) \cdot x^n.$$

Hence F is uniformly convergent in any $(-r, r)$, $0 < r < 1$, by 136, 2.

We can see this directly. For

$$F_n = (1 - x)(1 + x + \cdots + x^{n-1}) = 1 - x^n.$$

Hence F is convergent for $-1 < x \leq 1$, and then $F(x) = 1$, except at $x = 1$ where $F = 0$.

Thus $|\bar{F}_n(x)| = |x|^n$, except at $x = 1$.

But we can choose m so large that $r^m < \epsilon$.

Then $|\bar{F}_m(x)| < \epsilon$ for any x in $(-r, r)$.

We show now that F does not converge uniformly at $x=1$. For let

$$a_n = 1 - \frac{1}{n}.$$

Then

$$|\overline{F}_n(a_n)| = \left(1 - \frac{1}{n}\right)^n \doteq \frac{1}{e}$$

and F does not converge uniformly at $x=1$, by 139, 2.

Example 3.

$$F(x) = \sum_1^\infty \frac{x^2}{(1+nx^2)(1+(n+1)x^2)}.$$

Here

$$f_n = \frac{1}{1+nx^2} - \frac{1}{1+(n+1)x^2}$$

and F is telescopic. Hence

$$F_n = \frac{1}{1+x^2} - \frac{1}{1+(n+1)x^2}$$

$$\doteq \frac{1}{1+x^2} \quad , \quad x \neq 0$$

$$\doteq 0 \quad , \quad x = 0.$$

Thus

$$|\overline{F}_n| = \frac{1}{1+(n+1)x^2} \quad , \quad x \neq 0.$$

Let us take

$$a_n = \frac{1}{\sqrt{n+1}}.$$

Then

$$\overline{F}_n(a_n) = \tfrac{1}{2}$$

and F is not uniformly convergent at $x=0$. It is, however, in $(-\infty, \infty)$ except at this point. For let us take x at pleasure such, however, that $|x| \geq \delta$. Then

$$|\overline{F}_n(x)| \leq \frac{1}{1+(n+1)\delta^2}.$$

We now apply 136, 1.

Example 4.

$$F(x) = \Sigma x \frac{n(n+1)x^2 - 1}{(1+n^2x^2)(1+(n+1)^2x^2)}.$$

Here

$$f_n = x\left\{\frac{n}{1+n^2x^2} - \frac{n+1}{1+(n+1)^2x^2}\right\}$$

and F is telescopic. Hence

$$F_n = \frac{x}{1+x^2} - \frac{(n+1)x}{1+(n+1)^2x^2}$$

$$\doteq \frac{x}{1+x^2} \quad \text{in } \mathfrak{A} = (-R, R).$$

The convergence is not uniform at $x = 0$.

For set $a_n = \dfrac{1}{n+1}$. Then

$$|\bar{F}_n(a_n)| = \tfrac{1}{2}, \text{ does not } = 0.$$

It is, however, uniformly convergent in $\mathfrak{A}$ except at 0. For if $|x| > \delta$,

$$|\bar{F}_n(x)| = \left|\frac{(n+1)x}{1+(n+1)^2x^2}\right| \leq \frac{(n+1)R}{1+(n+1)^2\delta^2}.$$

$$< \epsilon \qquad \text{for } n > \text{some } m.$$

141. Let us suppose that the series F converges absolutely and uniformly in $\mathfrak{A}$. Let us rearrange F, obtaining the series G. Since F is absolutely convergent, so is G and $F = G$. We cannot, however, state that G is uniformly convergent in $\mathfrak{A}$, as *Bôcher* has shown.

Example.

$$F = \frac{1-x}{x}\{1 - 1 + x - x + x^2 - x^2 + x^3 - x^3 + \cdots\}.$$

Here

$$F_{2n} = 0.$$

$$F_{2n+1} = x^{n-1}(1-x).$$

Hence F is uniformly convergent in $\mathfrak{A} = (0, 1)$.

Let

$$G = \frac{1-x}{x}\{1 - 1 + x + x^2 - x + x^3 + x^4 - x^2 + \cdots\}.$$

Then

$$G_{2n+2} = \frac{1-x}{x}\{(1-1) + (x + x^2 - x) + (x^3 + x^4 - x^2) + \cdots$$

$$+ (x^{2n} + x^{2n-1} - x^n)\} = \frac{1-x}{x}\{x^{n+1} + \cdots + x^{2n}\} = x^n(1 - x^n).$$

Let

$$a_n = 1 - \frac{1}{n}.$$

Then

$$G_{2n+2}(a_n) = \left(1 - \frac{1}{n}\right)^n \left\{1 - \left(1 - \frac{1}{n}\right)^n\right\}$$

$$\doteq \frac{1}{e}\left(1 - \frac{1}{e}\right) \qquad \text{as } n \doteq \infty.$$

Hence G does not converge uniformly at $x = 1$.

142. 1. *Let $f \doteq \phi$ uniformly in a finite set of aggregates $\mathfrak{A}_1, \mathfrak{A}_2, \cdots \mathfrak{A}_p$. Then f converges uniformly in their union $(\mathfrak{A}_1, \cdots \mathfrak{A}_p)$.*

For by definition

$$\epsilon > 0,\ \delta_s > 0,\ |f - \phi| < \epsilon \qquad x \text{ in } \mathfrak{A}_s, \qquad t \text{ in } V_{\delta_s}{}^*(\tau). \qquad (1$$

Since there are only p aggregates, the minimum δ of $\delta_1, \cdots \delta_p$ is > 0. Then 1) holds if we replace δ_s by δ.

2. The preceding theorem may not be true when the number of aggregates $\mathfrak{A}_1, \mathfrak{A}_2 \cdots$ is infinite. For consider as an example

$$F = \Sigma(1 - x)x^n,$$

which converges uniformly in $\mathfrak{A} = (0, 1)$ except at $x = 1$. Let

$$\mathfrak{A}_s = \left(\frac{s-1}{s}, \frac{s}{s+1}\right) \qquad s = 1, 2, \cdots \infty.$$

Then F is uniformly convergent in each $\mathfrak{A}_s$, but is not in their union, which is $\mathfrak{A}$.

3. *Let $f \doteq \phi$, $g \doteq \psi$ uniformly in $\mathfrak{A}$.*

Then $$f \pm g \doteq \phi \pm \psi \qquad \textit{uniformly}.$$

If ϕ, ψ remain limited in $\mathfrak{A}$,

$$fg \doteq \phi\psi \qquad \textit{uniformly}. \qquad (1$$

If moreover $|\psi| >$ some positive number in $\mathfrak{A}$,

$$\frac{f}{g} \doteq \frac{\phi}{\psi} \qquad \textit{uniformly}. \qquad (2$$

The demonstration follows along the lines of I, 49, 50, 51.

4. To show that 1), 2) may be false if ϕ, ψ are not limited.

Let
$$f = g = \frac{1}{x} + t, \qquad \mathfrak{A} = (0^*, 1), \qquad \tau = 0.$$

Then $\phi = \psi = \frac{1}{x}$ and the convergence is uniform.

But
$$\Delta = fg - \phi\psi = \frac{2t}{x} + t^2.$$

Let $x = t$. Then $\Delta \doteq 2$ as $t \doteq 0$, and fg does not $\doteq \phi\psi$ uniformly.

Again, let
$$f = \frac{1}{x} + t, \qquad g = x + t,$$
the rest being as before.

Then
$$\phi = \frac{1}{x}, \qquad \psi = x.$$

But setting $x = t$,
$$|\Delta| = \left|\frac{f}{g} - \frac{\phi}{\psi}\right| = \frac{t^2 - 1}{2t^2} \doteq -\infty \text{ as } t \doteq 0$$
and $\frac{f}{g}$ does not converge uniformly to $\frac{\phi}{\psi}$.

143. 1. As an extension of I, 317, 2 we have:

Let
$$\lim_{y=\eta} f(x_1 \cdots x_m, y_1 \cdots y_p) = \phi(x_1 \cdots x_m)$$
uniformly in $\mathfrak{A}$. *Let*
$$\lim_{t=\tau} y_1(t_1 \cdots t_n) = \eta_1 \cdots \lim_{t=\tau} y_p(t_1 \cdots t_n) = \eta_p.$$

Let $y \neq \eta$ *in* $V^*(\tau)$. *Then*
$$\lim_{t=\tau} f(x_1 \cdots x_m, y_1 \cdots y_p) = \phi(x_1 \cdots x_m), \textit{ uniformly.}$$

The demonstration is entirely analogous to that of I, 292.

2. *Let*
$$\lim_{t=\tau} u_\iota(x_1 \cdots x_m, t_1 \cdots t_n) = v_\iota(x_1 \cdots x_m) \quad , \quad \iota = 1, 2, \cdots p,$$
uniformly in $\mathfrak{A}$. *Let the points*
$$v = (v_1, v_2, \cdots v_p)$$

form a limited set $\mathfrak{B}$. *Let* $F(u_1 \cdots u_p)$ *be continuous in a complete set containing* $\mathfrak{B}$. *Then*

$$\lim_{t=\tau} F(u_1 \cdots u_p) = F(v_1 \cdots v_p)$$

uniformly in $\mathfrak{A}$.

For F, being continuous in the complete set containing $\mathfrak{B}$, is uniformly continuous. Hence for a given $\epsilon > 0$ there exists a fixed $\sigma > 0$, such that

$$| F(u) - F(v) | < \epsilon \qquad u \text{ in } V_\sigma(v) \quad , \quad v \text{ in } \mathfrak{B}.$$

But as $u_\iota \doteq v_\iota$ uniformly there exists a fixed $\delta > 0$ such that

$$| u_\iota - v_\iota | < \epsilon' \quad , \quad x \text{ in } \mathfrak{A} \quad , \quad t \text{ in } V_\delta^*(\tau).$$

Thus if ϵ' is sufficiently small, $u = (u_1, \cdots u_p)$ lies in $V_\sigma(v)$ when x is in $\mathfrak{A}$ and t in $V_\delta^*(\tau)$.

144. 1. *Let* $$\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$$

uniformly in $\mathfrak{A}$. *Then* $$\lim_{t=\tau} e^f = e^\phi$$

uniformly in $\mathfrak{A}$, *if* ϕ *is limited.*

This is a corollary of 143, 2.

2. *Let* $$\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$$

uniformly in $\mathfrak{A}$. *Let* ϕ *be greater than some positive constant in* $\mathfrak{A}$. *Then*

$$\lim_{t=\tau} \log f = \log \phi,$$

uniformly in $\mathfrak{A}$, *if* ϕ *remains limited in* $\mathfrak{A}$.

Also a corollary of 143, 2.

3. *Let* $f \doteq \phi$ *and* $g \doteq \psi$ *uniformly, as* $t \doteq \tau$.

Let ϕ, ψ *be limited in* $\mathfrak{A}$, *and* $\phi >$ *some positive number. Then*

$$f^g \doteq \phi^\psi \qquad \textit{uniformly in } \mathfrak{A}. \tag{1}$$

For $$f^g = e^{g \log f}. \tag{2}$$

But by 2), $\log f \doteq \log \phi$ uniformly in $\mathfrak{A}$; and by 142, 3 $g \log f \doteq \psi \log \phi$, uniformly in $\mathfrak{A}$. Hence 2) gives 1) by 1.

145. 1. The definition of uniform convergence may be given a slightly different form which is sometimes useful. The function

$$f(x_1 \cdots x_m, t_1 \cdots t_n)$$

is a function of two sets of variables x and t, one ranging in an $\mathfrak{R}_m$ the other in an $\mathfrak{R}_n$.

Let us set now $w = (x_1 \cdots x_m, t_1 \cdots t_n)$ and consider w as a point in $m + p$ way space.

As x ranges over $\mathfrak{A}$ and t over $V_\delta^*(\tau)$, let w range over $\mathfrak{B}_\delta$. Then

$$\lim_{t=\tau} f = \phi$$

uniformly in $\mathfrak{A}$ when and only when

$$\epsilon > 0, \quad \delta > 0 \qquad |f - \phi| < \epsilon \qquad w \text{ in } \mathfrak{B}_\delta, \quad \delta \text{ fixed.}$$

By means of this second definition we obtain at once the following theorem:

2. *Instead of the variables $x_1 \cdots x_m, t_1 \cdots t_n$ let us introduce the variables $y_1 \cdots y_m, u_1 \cdots u_n$ so that as w ranges over $\mathfrak{B}_\delta$,*

$$z = (y_1 \cdots y_m, u_1 \cdots u_n)$$

ranges over $\mathfrak{C}_\delta$, the correspondence between $\mathfrak{B}_\delta$, $\mathfrak{C}_\delta$ being uniform. Then $f \doteq \phi$ uniformly in $\mathfrak{A}$ when and only when

$$\epsilon > 0, \qquad \delta > 0 \qquad |f - \phi| < \epsilon \quad , \quad z \textit{ in } \mathfrak{C}_\delta, \quad \delta \textit{ fixed.}$$

3. *Example.* Let $f(x, n) = \dfrac{n^\lambda x^\alpha}{e^{n^\mu x^\beta}}$.

where

$$\alpha, \beta, \mu > 0 \quad ; \quad \lambda \geq 0. \tag{1}$$

Then

$$\phi(x) = \lim_{n=\infty} f(x, n) = 0 \quad , \quad \text{in } \mathfrak{A} = (0, \infty).$$

Let us investigate whether the convergence is uniform at the point x in $\mathfrak{A}$.

First let $x > 0$. If $0 < a \leq x \leq b$, we have

$$|f - \phi| \leq \frac{n^\lambda b^\alpha}{e^{a^\beta n^\mu}}.$$

As the term on the right $\doteq 0$ as $n \doteq \infty$, we see $f \doteq \phi$ uniformly in (a, b).

When, however, $a = 0$, or $b = \infty$, this reasoning does not hold. In this case we set

$$t = e^{n^\mu x^\beta}, \tag{2}$$

which gives

$$x = \frac{\log^{1/\beta} \cdot t}{n^{\mu/\beta}}.$$

As the point (x, n) ranges over $\mathfrak{X}$ defined by

$$x \geq 0 \quad , \quad n \geq 1,$$

the point (t, n) ranges over a field $\mathfrak{T}$ defined by

$$t \geq 1 \quad , \quad n \geq 1,$$

and the correspondence between $\mathfrak{X}$ and $\mathfrak{T}$ is uniform. Here

$$|f - \phi| = \frac{1}{n^{\frac{a\mu}{\beta} - \lambda}} \cdot \frac{\log^{a/\beta} \cdot t}{t}.$$

The relation 2) shows that when $x > 0$, $t \doteq \infty$ as $n \doteq \infty$; also when $x = 0$, $t = 1$ for any n. Thus the convergence at $x = 0$ is uniform when

$$\frac{\alpha}{\beta} > \frac{\lambda}{\mu}. \tag{3}$$

The convergence is not uniform at $x = 0$ when 3) is not satisfied. For take

$$x = \frac{1}{n^{\lambda/a}} \quad , \quad n = 1, 2, \cdots$$

For these values of x

$$|f - \phi| = e^{n^{\frac{\beta\lambda}{a}} - n^\mu},$$

which does not $\doteq 0$ as $n \doteq \infty$.

146. 1. (*Moore, Osgood.*) *Let*

$$\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$$

uniformly in $\mathfrak{A}$. *Let a be a limiting point of* $\mathfrak{A}$ *and*

$$\lim_{x=a} f(x_1 \cdots x_m, t_1 \cdots t_n) = \psi(t_1 \cdots t_n)$$

for each t in $V_\delta^*(\tau)$. Then

$$\Phi = \lim_{x=a} \phi(x_1 \cdots x_m) \quad , \quad \Psi = \lim_{t=\tau} \psi(t_1 \cdots t_n)$$

exist and are equal. Here a, τ are finite or infinite.

We first show Φ exists. To this end we show that

$$\epsilon>0 \quad , \quad \delta>0 \quad , \quad |\phi(x')-\phi(x'')|<\epsilon \qquad x', x'' \text{ in } V_\delta^*(a). \quad (1$$

Now since $f(x, t)$ converges uniformly, there exists an $\eta>0$ such that for any x', x'' in $\mathfrak{A}$

$$\phi(x')=f(x', t)+\epsilon' \qquad t \text{ in } V_\eta^*(\tau) \qquad (2$$

$$\phi(x'')=f(x'', t)+\epsilon''. \qquad |\epsilon'|, |\epsilon''|<\frac{\epsilon}{4}. \qquad (3$$

On the other hand, since $f \doteq \psi$ there exists a $\delta>0$ such that

$$f(x', t)=\psi(t)+\epsilon''' \qquad (4$$

$$f(x'', t)=\psi(t)+\epsilon^{\text{IV}} \qquad |\epsilon'''|, |\epsilon^{\text{IV}}|<\frac{\epsilon}{4} \qquad (5$$

for any x', x'' in $V_\delta^*(a)$; t fixed.

From 2), 3), 4), 5) we have at once 1). Having established the existence of Φ, we show now that $\Phi=\Psi$. For since f converges uniformly to ϕ, we have

$$|f(x, t)-\phi(x)|<\frac{\epsilon}{3} \qquad x \text{ in } \mathfrak{A} \quad , \quad t \text{ in } V_\eta^*(\tau). \qquad (6$$

Since $f \doteq \psi$, we have

$$|f(x, t)-\psi(t)|<\frac{\epsilon}{3} \qquad x \text{ in } V_{\delta'}^*(a) \quad , \quad t \text{ fixed in } V_\eta^*(\tau). \qquad (7$$

Since $\phi \doteq \Phi$,

$$|\phi(x)-\Phi|<\frac{\epsilon}{3} \qquad x \text{ in } V_{\delta''}^*(a). \qquad (8$$

Thus 7), 8) hold simultaneously for $\delta<\delta'$, δ''.

Hence
$$|\psi(t)-\Phi|<\epsilon \qquad t \text{ in } V_\eta^*(\tau),$$

or
$$\lim_{t=\tau} \psi(t)=\Phi.$$

2. Thus under the conditions of 1)

$$\lim_{x=a} \lim_{t=\tau} f=\lim_{t=\tau} \lim_{x=a} f;$$

in other words, we may interchange the order of passing to the limit.

3. The theorem in 1 obviously holds when we replace the unrestricted limits, by limits which are subjected to some condition; *e.g.* the variables are to approach their limits along some curve.

4. As a corollary we have:

Let $F = \Sigma f_s(x_1 \cdots x_m)$ be uniformly convergent in $\mathfrak{A}$, of which $x = a$ is a limiting point. Let $\lim_{x=a} f_s = l_s$, and set $L = \Sigma l_s$. Then

$$\lim_{x=a} F = L; \qquad a \text{ finite or infinite,}$$

or in other words

$$\lim \Sigma f_s = \Sigma \lim f_s.$$

Example 1.

$$F(x) = \sum \frac{e^{nx} - 1}{2^n e^{nx}}$$

converges uniformly in $\mathfrak{A} = (0, \infty)$ as we saw 136, 2, Ex. 1. Here

$$\lim_{x=\infty} f_n = \frac{1}{2^n} = l_n,$$

and

$$L = \Sigma l_n = \sum \frac{1}{2^n} = 1.$$

Hence

$$\lim_{x=\infty} F(x) = 1.$$

Also

$$R \lim_{x=0} f_n = 0;$$

hence

$$R \lim_{x=0} F(x) = 0.$$

Example 2.

$$F(x) = 1 + \sum_1^\infty \frac{1}{n!}\left(\frac{\sin x}{x}\right)^n$$

converges uniformly in any interval finite or infinite, excluding $x = 0$, where F is not defined. For

$$|f_n| \leq \frac{1}{n!},$$

and

$$1 + \sum \frac{1}{n!} = e.$$

Hence

$$\lim_{x=0} F(x) = e.$$

Example 3.

$$F(x) = \sum_1^\infty \frac{x^2}{(1+nx^2)(1+(n+1)x^2)} = \Sigma f_n(x)$$

$$= \frac{1}{1+x^2} \qquad \text{for } x \neq 0$$

$$= 0 \qquad \text{for } x = 0.$$

Here

$$\lim_{x=0} F(x) = 1,$$

while

$$\Sigma \lim_{x=0} f_n(x) = \Sigma 0 = 0.$$

Thus here

$$\lim_{x=0} \Sigma f_n(x) \neq \Sigma \lim_{x=0} f_n(x),$$

But F does not converge uniformly at $x = 0$. On the other hand, it does converge uniformly at $x = \pm\infty$.

Now

$$\lim_{x=\pm\infty} F(x) = 0 \quad , \quad \lim_{x=\pm\infty} f_n(x) = 0,$$

and

$$\lim_{x=\pm\infty} \Sigma f_n(x) = \Sigma \lim_{x=\pm\infty} f_n(x),$$

as the theorem requires.

Example 4. $F(x) = \sum_1^\infty \left\{ \frac{nx^2}{e^{nx^2}} - \frac{(n+1)x^2}{e^{(n+1)x^2}} \right\} = \frac{x^2}{e^{x^2}},$

which converges about $x = 0$ but not uniformly.

However,

$$\lim_{x=0} \Sigma f_n(x) = \Sigma \lim_{x=0} f_n(x) = 0.$$

Thus the uniform convergence is not a necessary condition.

147. 1. *Let* $\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$ *uniformly at* $x = a$. *Let* $f(x, t)$ *be continuous at* $x = a$ *for each* t *in* $V_\delta^*(\tau)$. *Then* ϕ *is continuous at* a.

This is a corollary of the Moore-Osgood theorem.

For by 146, 1

$$\lim_{h=0} \lim_{t=\tau} f(a+h, t) = \lim_{t=\tau} \lim_{h=0} f(a+h, t).$$

Hence

$$\lim_{h=0} \phi(a+h) = \lim_{t=\tau} f(a, t) = \phi(a). \qquad \text{Q.E.D.}$$

A direct proof may be given as follows:

$$f(x, t) = \phi(x) + \epsilon' \qquad |\epsilon'| < \epsilon,\ x \text{ in } V_\delta(a)$$

$$\phi(x') - \phi(x'') = f(x', t) - f(x'', t) + \epsilon''.$$

But $\qquad |f(x'', t) - f(x', t)| < \epsilon \quad , \quad \text{if } |x' - x''| < \epsilon.$

2. *Let $F = \Sigma f_{s_1 \cdots s_p}(x_1 \cdots x_m)$ be uniformly convergent at $x = a$. Let each $f_{s_1 \cdots s_p}$ be continuous at a. Then $F(x_1 \cdots x_m)$ is continuous at $x = a$.*

Follows at once from 1).

3. In Ex. 3 of 140 we saw that

$$F = \sum \frac{x^2}{(1 + nx^2)(1 + (n+1)x^2)}$$

is discontinuous at $x = 0$ and does not converge uniformly there.

In Ex. 4 of 140 we saw that

$$F = \sum x \frac{n(n+1)x^2 - 1}{(1 + n^2x^2)(1 + (n+1)^2x^2)} \tag{1}$$

does not converge uniformly at $x = 0$ and yet is continuous there. We have thus the result: *The condition of uniform convergence in 1, is sufficient but not necessary.*

Finally, let us note that

$$F(x) = \sum \left\{ \frac{nx^a}{e^{nx^\beta}} - \frac{(n+1)x^a}{e^{(n+1)x^\beta}} \right\} \qquad 0 < \alpha < \beta$$

$$= \frac{x^a}{e^{x^\beta}}, \qquad x \geq 0$$

is a series which is not uniformly convergent at $x = 0$, although $F(x)$ is continuous at this point.

4. *Let each term of $F = \Sigma f_{s_1 \cdots s_p}(x_1 \cdots x_m)$ be continuous at $x = a$ while F itself is discontinuous at a. Then F is not uniformly convergent.*

For if it were, F would be continuous at a, by 2.

Remark. This theorem sometimes enables us to see at once that a given series is not uniformly convergent. Thus 140, Exs. 2, 3.

5. *The power series* $P = \Sigma a_{s_1 \cdots s_m} x_1^{s_1} \cdots x_m^{s_m}$ *is continuous at any inner point of its rectangular cell of convergence.*

For we saw P converges uniformly at this point.

6. *The power series* $P = a_0 + a_1 x + a_2 x^2 + \cdots$ *is a continuous function of* x *in its interval of convergence.*

For we saw P converges uniformly in this interval. In particular we note that if P converges at an end point $x = e$ of its interval of convergence, P is continuous at e.

This fact enables us to prove the theorem on multiplication of two series which we stated 112, 4, viz.:

148. *Let*
$$A = a_0 + a_1 + a_2 + \cdots \quad , \quad B = b_0 + b_1 + b_2 + \cdots$$
$$C = a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots$$
converge. Then $AB = C$.

For consider the auxiliary series
$$F(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$G(x) = b_0 + b_1 x + b_2 x^2 + \cdots$$
$$H(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0) x + \cdots$$

Since A, B, C converge, F, G, H converge for $x = 1$, and hence absolutely for $|x| < 1$. But for all $|x| < 1$,
$$H(x) = F(x)\, G(x).$$

Thus
$$L\lim_{x=1} H(x) = L\lim_{x=1} F(x) \cdot L\lim_{x=1} G(x),$$
or
$$C = A \cdot B.$$

149. 1. We have seen that we cannot say that $f \doteq \phi$ uniformly although f and ϕ are continuous. There is, however, an important case noted by Dini.

Let $f(x_1 \cdots x_m, t_1 \cdots t_n)$ be a function of two sets of variables such that x ranges over $\mathfrak{A}$, and t over a set having τ as limiting point, τ finite or ideal. Let
$$\lim_{t=\tau} f(x, t) = \phi(x) \quad \text{in } \mathfrak{A}.$$
Then we can set
$$f(x, t) = \phi(x) + \psi(x, t).$$

Suppose now $|\psi(x, t')| \leq |\psi(x, t)|$ for any t' in the rectangular cell one of whose vertices is t and whose center is τ. We say then that the convergence of f to ϕ is *steady or monotone at* x. If for each x in $\mathfrak{A}$, there exists a rectangular cell such that the above inequality holds, we say the convergence is *monotone or steady in* $\mathfrak{A}$.

The modification in this definition for the case that τ is an ideal point is obvious. See I, 314, 315.

2. We may now state *Dini's* theorem.

Let $f(x_1 \cdots x_m, t_1 \cdots t_n) \doteq \phi(x_1 \cdots x_m)$ *steadily in the limited complete field* $\mathfrak{A}$ *as* $t \doteq \tau$; τ *finite or ideal. Let* f *and* ϕ *be continuous functions of* x *in* $\mathfrak{A}$. *Then* f *converges uniformly to* ϕ *in* $\mathfrak{A}$.

For let x be a given point in $\mathfrak{A}$, and

$$f(x, t) = \phi(x) + \psi(x, t).$$

We may take t' so near τ that $|\psi(x, t')| < \frac{\epsilon}{3}$.

Let x' be a point in $V_\eta(x)$. Then

$$f(x', t') = \phi(x') + \psi(x', t').$$

As f is continuous in x,

$$|f(x', t') - f(x, t')| < \frac{\epsilon}{3}.$$

Similarly,

$$|\phi(x') - \phi(x)| < \frac{\epsilon}{3}.$$

Thus

$$|\psi(x', t')| < \epsilon \qquad x' \text{ in } V_\eta(x).$$

Hence

$$|\psi(x', t)| < \epsilon \qquad \text{for any } x' \text{ in } V_\eta(x)$$

and for any t in the rectangular cell determined by t'.

As corollaries we have :

3. *Let* $G = \Sigma |f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)|$ *converge in the limited complete domain* $\mathfrak{A}$. *Let* G *and each* f_ι *be continuous in* $\mathfrak{A}$. *Then* G *and a fortiori* $F = \Sigma f_{\iota_1 \cdots \iota_s}$ *converge uniformly in* $\mathfrak{A}$, *furthermore* $f_{\iota_1 \cdots \iota_s} \doteq 0$ *uniformly in* $\mathfrak{A}$.

4. *Let* $G = \Sigma |f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)|$ *converge in the limited complete domain* $\mathfrak{A}$, *having* a *as limiting point. Let* G *and each* f_ι *be con-*

tinuous at a. Then G and a fortiori $F = \Sigma f_{\iota_1 \cdots \iota_s}$ *converge uniformly at a.*

5. *Let* $G = \Sigma \, | f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m) |$ *converge in the limited complete domain* $\mathfrak{A}$, *having a as limiting point. Let* $\lim_{x=a} G$ *and each* $\lim_{x=a} f_\iota$ *exist. Moreover, let*

$$\lim G = \Sigma \lim f_\iota.$$

Then G is uniformly convergent at a.

For if in 4 the function had values assigned them at $x = a$ different from their limits, we could redefine them so that they are continuous at a.

150. 1. *Let* $\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$ *uniformly in the limited field* $\mathfrak{A}$. *Let* ϕ *be limited in* $\mathfrak{A}$. *Then*

$$\lim_{t=\tau} \overline{\underline{\int_{\mathfrak{A}}}} f = \overline{\underline{\int_{\mathfrak{A}}}} \phi = \overline{\underline{\int_{\mathfrak{A}}}} \lim_{t=\tau} f.$$

For let $$f = \phi + \psi.$$

Since $f \doteq \phi$ uniformly $$|\psi| < \epsilon$$

for any t in some $V^*(\tau)$ and for any x in $\mathfrak{A}$.

Thus

$$\left| \overline{\underline{\int_{\mathfrak{A}}}} f - \overline{\underline{\int_{\mathfrak{A}}}} \phi \right| < \epsilon \overline{\mathfrak{A}}.$$

Remark. Instead of supposing ϕ to be limited we may suppose that $f(x, t)$ is limited in $\mathfrak{A}$ for each t near τ.

2. As corollary we have

Let $\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$ *uniformly in the limited field* $\mathfrak{A}$. *Let f be limited and integrable in* $\mathfrak{A}$ *for each t in* $V_\delta^*(\tau)$. *Then* ϕ *is integrable in* $\mathfrak{A}$ *and*

$$\lim_{t=\tau} \int_{\mathfrak{A}} f = \int_{\mathfrak{A}} \phi = \int_{\mathfrak{A}} \lim_{t=\tau} f.$$

3. From 1, 2, we have at once:

Let $F = \Sigma f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)$ *be uniformly convergent in the limited field* $\mathfrak{A}$. *Let each* $f_{\iota_1 \cdots \iota_s}$ *be limited and integrable in* $\mathfrak{A}$. *Then F is integrable and*

$$\int_{\mathfrak{A}} F = \Sigma \int_{\mathfrak{A}} f_{\iota_1 \cdots \iota_s}.$$

If the $f_{\iota_1 \cdots \iota_s}$ *are not integrable, we have*

$$\underline{\int_{\mathfrak{A}}} F = \Sigma \underline{\int_{\mathfrak{A}}} f_{\iota_1 \cdots \iota_s}.$$

Example. $$F = \sum_0^\infty \frac{x^2}{(1+nx^2)(1+(n+1)x^2)}$$

does not converge uniformly at $x = 0$. Cf. 140, Ex. 3.

Here $$F_n = 1 - \frac{1}{1+nx^2}$$

and $$F(x) = \begin{cases} 1 & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Hence $$\int_0^1 F dx = 1,$$

$$\int_0^1 F_n dx = 1 - \int_0^1 \frac{dx}{1+nx^2}$$

$$= 1 - \frac{\operatorname{arctg} \sqrt{n}}{\sqrt{n}} \doteq 1.$$

Thus we can integrate F termwise although F does not converge uniformly in $(0, 1)$.

151. That uniform convergence of the series

$$F(x) = f_1(x) + f_2(x) + \cdots \tag{1}$$

with integrable terms, in the interval $\mathfrak{A} = (a < b)$ is a sufficient condition for the validity of the relation

$$\int_a^b F dx = \int_a^b f_1 dx + \int_a^b f_2 dx + \cdots$$

is well illustrated graphically, as Osgood has shown.*

Since 1) converges uniformly in $\mathfrak{A}$ by hypothesis, we have

$$F_n(x) = F(x) - \overline{F}_n(x) \tag{2}$$

and

$$|\overline{F}_n(x)| < \epsilon \qquad n \geq m \tag{3}$$

for any x in $\mathfrak{A}$.

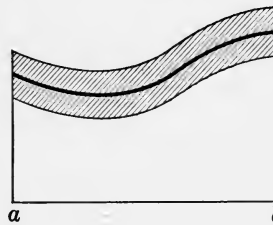
* Bulletin Amer. Math. Soc. (2), vol. 3, p. 59.

In the figure, the graph of $F(x)$ is drawn heavy. On either side of it are drawn the curves $F-\epsilon$, $F+\epsilon$ giving the shaded band which we call the *ϵ-band.*

From 2), 3) we see that the graph of each F_n, $n \geq m$ lies in the ϵ-band. The figure thus shows at once that

$$\int_a^b F dx \quad \text{and} \quad \int_a^b F_n dx$$

can differ at most by the area of the ϵ-band, *i.e.* by at most



$$\int_a^b 2\,\epsilon\, dx = 2\,\epsilon\,(b-a).$$

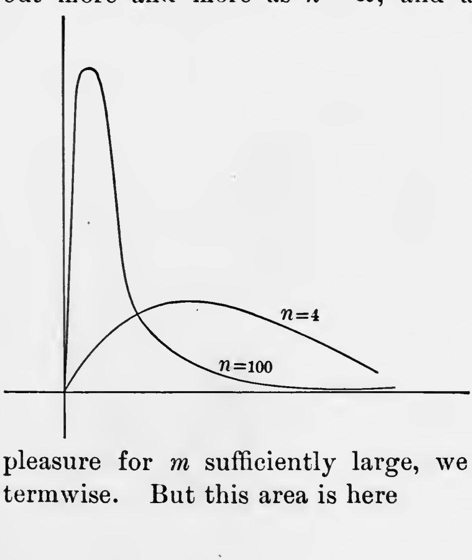
152. 1. Let us consider a case where the convergence is not uniform, as

$$F(x) = \sum_1^\infty \left\{\frac{nx}{e^{nx^2}} - \frac{(n-1)x}{e^{(n-1)x^2}}\right\} = 0.$$

Here

$$F_n(x) = \frac{nx}{e^{nx^2}}.$$

If we plot the curves $y = F_n(x)$, we observe that they flatten out more and more as $n \doteq \infty$, and approach the x-axis except near the origin, where they have peaks which increase indefinitely in height. The curves $F_n(x)$, $n \geq m$, and m sufficiently large, lie within an ϵ-band about their limit $F(x)$ in any interval which does not include the origin.



If the area of the region under the peaks could be made small at pleasure for m sufficiently large, we could obviously integrate termwise. But this area is here

$$\int_0^a F_n dx = \frac{1}{2}\int_0^a \frac{d}{dx}\left[-\frac{1}{e^{nx^2}}\right]dx = \frac{1}{2}\left[-\frac{1}{e^{nx^2}}\right]_0^a = \frac{1}{2}\left(1-\frac{1}{e^{na^2}}\right)$$

$$\doteq \frac{1}{2} \qquad \text{as } n \doteq \infty.$$

Thus we cannot integrate the F series termwise.

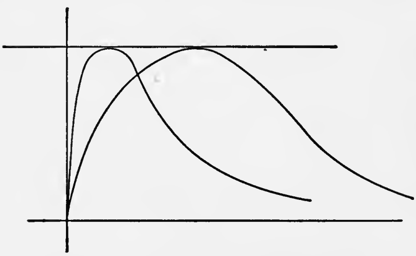
2. As another example in which the convergence is not uniform let us consider

$$F(x) = \sum_0^\infty \left\{ \frac{(n+1)x}{e^{(n+1)x}} - \frac{nx}{e^{nx}} \right\} = 0.$$

Here

$$F_n = \frac{nx}{e^{nx}}.$$

The convergence of F is uniform in $\mathfrak{A} = (0, 1)$ except at $x = 0$. The peaks of the curves $F_n(x)$ all have the height e^{-1}.



Obviously the area of the region under the peaks can be made small at pleasure if m is taken sufficiently large. Thus in this case we can obviously integrate termwise, although the convergence is not uniform in $\mathfrak{A}$.

We may verify this analytically. For

$$\int_0^x F_n dx = \int_0^x \frac{nx}{e^{nx}} dx = \frac{1}{n} - \frac{1+nx}{ne^{nx}} \doteq 0 \qquad \text{as } n \doteq \infty.$$

3. Finally let us consider

$$F(x) = \sum_0^\infty \left\{ \frac{(n+1)^2 x}{1+(n+1)^3 x^2} - \frac{n^2 x}{1+n^3 x^2} \right\} = 0.$$

Here

$$F_n(x) = \frac{n^2 x}{1+n^3 x^2}.$$

The convergence is not uniform at $x = 0$.

The peaks of $F_n(x)$ are at the points $x = n^{-\frac{3}{2}}$, at which points

$$F_n = \tfrac{1}{2}\sqrt{n}.$$

Their height thus increases indefinitely with n. But at the same time they become so slender that the area under them $\doteq 0$. In fact

$$\int_0^a F_n(x)dx = \int_0^a \frac{1}{2n} d \log(1+n^3x^2)$$

$$= \frac{1}{2n}\Big[\log(1+n^3x^2)\Big]_0^a = \frac{1}{2}\ \frac{\log(1+n^3a^2)}{n} \doteq 0.$$

We can therefore integrate termwise in $(0 < a)$.

153. 1. *Let* $\lim\limits_{t=\tau} G(x, t_1 \cdots t_n) = g(x)$ *in* $\mathfrak{A} = (a, a+\delta)$, τ *finite or infinite. Let each* $G'_x(x, t)$ *be continuous in* $\mathfrak{A}$; *also let* $G'_x(x, t)$ *converge to a limit uniformly in* $\mathfrak{A}$ *as* $t \doteq \tau$. *Then*

$$\lim_{t=\tau} G'_x(x, t) = g'(x) \qquad \text{in } \mathfrak{A}, \tag{1}$$

and $g'(x)$ *is continuous.*

For by 150, 2,

$$\lim_{t=\tau} \int_a^x G'_x dx = \int_a^x \lim_{t=\tau} G'_x dx.$$

By I, 538,

$$\int_a^x G'_x dx = G(x, t) - G(a, t).$$

Also by hypothesis, $\lim\limits_{t=\tau}\{G(x, t) - G(a, t)\} = g(x) - g(a)$.

Hence

$$g(x) - g(a) = \int_a^x \lim_{t=\tau} G'_x(x, t)dx. \tag{2}$$

But by 147, 1, the integrand is continuous in $\mathfrak{A}$.

Hence by I, 537, the derivative of the right side of 2) is this integrand. Differentiating 2), we get 1).

2. *Let* $F(x) = \Sigma f_{\iota_1 \cdots \iota_s}(x)$ *converge in* $\mathfrak{A} = (a, a+\delta)$. *Let each* $f'_\iota(x)$ *be continuous, also let*

$$\Sigma f'_{\iota_1 \cdots \iota_s}(x)$$

be uniformly convergent in $\mathfrak{A}$. *Then*

$$F'(x) = \Sigma f'_\iota(x), \qquad \text{in } \mathfrak{A}.$$

This is a corollary of 1.

3. The more general case that the terms $f_{\iota_1 \cdots \iota_s}$ are functions of several variables $x_1, \cdots x_m$ follows readily from 2.

154. *Example.*

$$F(x) = \sum_0^\infty \left\{ \frac{n^\lambda x^\alpha}{e^{n^\mu x^\beta}} - \frac{(n+1)^\lambda x^\alpha}{e^{(n+1)^\mu x^\beta}} \right\} = \Sigma f_n ; \qquad \alpha,\ \beta,\ \lambda \geq 0,\ \mu > 0.$$

Here

$$F_n = -\frac{n^\lambda x^\alpha}{e^{n^\mu x^\beta}},$$

a function whose uniform convergence was studied, 145, 3. We saw

$$F(x) = 0 \qquad \text{for any } x \geq 0.$$

Hence

$$F'(x) = 0 \qquad x \geq 0.$$

Let

$$G(x) = \Sigma f_n'(x). \tag{1}$$

Then

$$G_n(x) = F_n'(x) = -\frac{\alpha n^\lambda x^{\alpha-1}}{e^{n^\mu x^\beta}} + \frac{\beta n^{\lambda+\mu} x^{\alpha+\beta-1}}{e^{n^\mu x^\beta}}.$$

If $x > 0$,

$$G_n(x) \doteq 0,$$

hence

$$F'(x) = \Sigma f_n'(x), \tag{2}$$

and we may differentiate the series termwise.

If $x = 0$, and $\alpha = 1$, $\lambda > 0$; $G_n(0) = -n^\lambda \doteq -\infty$ as $n \doteq \infty$.

In this case 2) does not hold, and we cannot differentiate the series termwise.

For $x = 0$, and $\alpha > 1$, $G_n(0) = 0$, and now 2) holds; we may therefore differentiate the series termwise. But if we look at the uniform convergence of the series 1), we see this takes place only when

$$\frac{\alpha - 1}{\beta} > \frac{\lambda}{\mu}.$$

155. 1. *(Porter.) Let*

$$F(x) = \Sigma f_{\iota_1 \cdots \iota_s}(x)$$

converge in $\mathfrak{A} = (a, b)$. *For every* x *in* $\mathfrak{A}$ *let* $|f_\iota'(x)| \leq g_\iota$, *a constant. Let* $G = \Sigma g_\iota$ *converge. Then* $F(x)$ *has a derivative in* $\mathfrak{A}$ *and*

$$F'(x) = \Sigma f'_{\iota_1 \cdots \iota_s}(x); \tag{1}$$

or we may differentiate the given series termwise.

For simplicity let us take $s = 1$. Let the series on the right of 1) be denoted by $\phi(x)$. For each x in $\mathfrak{A}$ we show that

$$\epsilon > 0, \qquad \delta > 0, \qquad D = \left|\frac{\Delta F}{\Delta x} - \phi(x)\right| < \epsilon, \qquad |\Delta x| < \delta.$$

For
$$\frac{\Delta F}{\Delta x} = \Sigma \frac{f_n(x+\Delta x) - f_n(x)}{\Delta x} = \Sigma f'_n(\xi_n)$$

where ξ_n lies in $V_\delta(x)$.

Thus
$$D = \sum_1^\infty \{ f'_n(\xi_n) - f'_n(x) \} = D_m + \overline{D}_m.$$

But G being convergent, $\overline{G}_m < \epsilon/3$ if m is taken sufficiently large. Hence

$$|\overline{D}_m| \leq \sum_{m+1}^\infty |f'_n(\xi_n)| + \sum_{m+1}^\infty |f'_n(x)| \leq 2\,\overline{G}_m < \tfrac{2}{3}\epsilon.$$

On the other hand, since $\dfrac{\Delta f_n}{\Delta x} \doteq f'_n(x)$ and since there are only m terms in D_m, we may take δ so small that

$$|D_m| < \epsilon/3.$$

Thus
$$|D| < \epsilon \qquad \text{for } |\Delta x| < \delta.$$

2. *Example 1.* Let

$$F(x) = \sum_0^\infty \frac{(-1)^n}{n!} \cdot \frac{1}{1+a^n x} = \Sigma f_n(x) \qquad a > 1. \tag{1}$$

This series converges uniformly in $\mathfrak{A} = (0 < b)$, since

$$|f_n(x)| \leq \frac{1}{n!}.$$

Also
$$f'_n(x) = \frac{(-1)^{n+1}}{n!} \frac{a^n}{(1+a^n x)^2}.$$

Hence
$$|f'_n(x)| \leq \frac{a^n}{n!} = g_n \qquad \text{in } \mathfrak{A}.$$

As Σg_n converges, we may differentiate 1) termwise. In general we have

$$F^{(\lambda)}(x) = \Sigma f_n^{(\lambda)}(x) = (-1)^\lambda \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{\lambda!\, a^{n\lambda}}{(1+a^n x)^{\lambda+1}} \tag{2}$$

valid in $\mathfrak{A}$.

3. *Example 2. The ϑ functions.*

These are defined by

$$\vartheta_1(x) = 2\sum_0^\infty (-1)^n q^{(n+\frac{1}{2})^2} \sin(2n+1)\pi x$$
$$= 2q^{\frac{1}{4}} \sin \pi x - 2q^{\frac{9}{4}} \sin 3\pi x + \cdots$$

$$\vartheta_2(x) = 2\sum_0^\infty q^{(n+\frac{1}{2})^2} \cos(2n+1)\pi x$$
$$= 2q^{\frac{1}{4}} \cos \pi x + 2q^{\frac{9}{4}} \cos 3\pi x + \cdots$$

$$\vartheta_3(x) = 1 + 2\sum_1^\infty q^{n^2} \cos 2n\pi x$$
$$= 1 + 2q \cos 2\pi x + 2q^4 \cos 4\pi x + \cdots$$

$$\vartheta_0(x) = 1 + 2\sum_1^\infty (-1)^n q^{n^2} \cos 2n\pi x$$
$$= 1 - 2q \cos 2\pi x + 2q^4 \cos 4\pi x - \cdots$$

Let us take

$$|q| < 1.$$

Then these series converge uniformly at every point x. For let us consider as an example ϑ_1. The series

$$T = |q| + |q|^4 + |q|^9 + \cdots$$

is convergent since the ratio of two successive terms is

$$\frac{q^{(n+1)^2}}{q^{n^2}} = q^{2n+1};$$

and this $\doteq 0$. Now each term in ϑ_1 is numerically

$$\leq |q|^{(n+\frac{1}{2})^2} < |q|^{n^2},$$

and hence $<$ the corresponding term in T.

Thus $\vartheta_1(x)$ is a continuous function of x for every x by 147, 2. The same is true of the other ϑ's. These functions were discovered by Abel, and were used by him to express the elliptic functions.

Let us consider now their derivatives.

Making use of 155, 1 it is easy to show that we may differentiate these series termwise. Then

$$\vartheta_1'(x) = 2\pi\sum_0^\infty (-1)^n(2n+1)\, q^{(n+\frac{1}{2})^2} \cos(2n+1)\pi x$$
$$= 2\pi(q^{\frac{1}{4}} \cos \pi x - 3q^{\frac{9}{4}} \cos 3\pi x + \cdots).$$

$$\vartheta_2'(x) = -2\pi\sum_0^\infty (2n+1)q^{(n+\frac{1}{2})^2}\sin(2n+1)\pi x$$
$$= -2\pi(q^{\frac{1}{4}}\sin\pi x + 3q^{\frac{9}{4}}\sin 3\pi x + \cdots).$$

$$\vartheta_3'(x) = -4\pi\sum_1^\infty nq^{n^2}\sin 2n\pi x$$
$$= -4\pi(q\sin 2\pi x + 2q^4\sin 4\pi x + \cdots).$$

$$\vartheta_0'(x) = -4\pi\sum_1^\infty (-1)^n nq^{n^2}\sin 2n\pi x$$
$$= +4\pi(q\sin 2\pi x - 2q^4\sin 4\pi x + \cdots).$$

To show the uniform convergence of these series, let us consider the first and compare it with

$$S = 1 + 3|q| + 5|q|^4 + 7|q|^9 + \cdots$$

The ratio of two successive terms of this series is

$$\frac{2n+3}{2n+1}\,\frac{|q|^{(n+1)^2}}{|q|^{n^2}} = \frac{2n+3}{2n+1}\,|q|^{2n+1},$$

which $\doteq 0$. Thus S is convergent. The rest follows now as before.

156. 1. *Let*

$$\lim_{t=\tau}\frac{G(a+h, t_1\cdots t_n) - G(a, t_1\cdots t_n)}{h} = \frac{g(a+h)-g(a)}{h}$$

uniformly for $0 < |h| \leq \eta$, τ *finite or infinite.*

Let

$$G_x'(a, t) \text{ exist}$$

for each t *near* τ. *Then* $g'(a)$ *exists and*

$$\lim_{t=\tau} G_x'(a, t) = g'(a).$$

This is a corollary of 146, 1. Here

$$\frac{G(a+h, t) - G(a, t)}{h}$$

takes the place of $f(x, t)$.

2. From 1 we have as corollary:

(*Dini*), *Let*

$$F(x) = \Sigma f_{\iota_1\ldots\iota_n}(x)$$

converge for each x *in* $\mathfrak{A}$ *which has* $x = a$ *as a proper limiting point. Let* $f_\iota'(a)$ *exist for each* $\iota = (\iota_1, \cdots \iota_n)$. *Let*

$$\sum \frac{f_\iota(a+h) - f_\iota(a)}{h}$$

converge uniformly with respect to h. *Then*

$$F'(a) = \Sigma f'_{\iota_1 \cdots \iota_n}(a).$$

CHAPTER VI

POWER SERIES

157. On account of their importance in analysis we shall devote a separate chapter to power series.

We have seen that without loss of generality we may employ the series

$$a_0 + a_1x + a_2x^2 + \cdots \tag{1}$$

instead of the formally more general one

$$a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots$$

We have seen that if 1) converges for $x = c$ it converges absolutely and uniformly in $(-\gamma, \gamma)$ where $0 < \gamma < |c|$. Finally, we saw that if c is an end point of its interval of convergence, it is unilaterally continuous at this point. The series 1) is, of course, a continuous function of x at any point *within* its interval of convergence.

158. 1. *Let* $P(x) = a_0 + a_1x + a_2x^2 + \cdots$ *converge in the interval* $\mathfrak{A} = (-\alpha, \alpha)$ *which may not be complete. The series*

$$P_n = 1 \cdot 2 \cdot \cdots na_n + 2 \cdot 3 \cdot \cdots (n+1)a_{n+1}x + \cdots$$

obtained by differentiating each term of P n times is absolutely and uniformly convergent in $\mathfrak{B} = (-\beta, \beta)$, $\beta < \alpha$, *and*

$$\frac{d^nP}{dx^n} = P_n(x), \qquad \text{in } \mathfrak{B}.$$

For since P converges absolutely for $x = \beta$,

$$\alpha_n\beta^n \leq M, \qquad n = 1, 2, \cdots$$

Let now x lie within $\mathfrak{B}$. Then the adjoint series of $P_1(x)$ is

$$\alpha_1 + 2\,\alpha_2\xi + 3\,\alpha_3\xi^2 + \cdots$$

Now its m^{th} term

$$m\alpha_m\xi^{m-1} = \frac{m\alpha_m\beta^m}{\beta}\left(\frac{\xi}{\beta}\right)^{m-1} \leq \frac{mM}{\beta}\left(\frac{\xi}{\beta}\right)^{m-1}.$$

But the series whose general term is the last term of the preceding inequality is convergent.

2. *Let*
$$P = a_0 + a_1x + a_2x^2 + \cdots$$
converge in the interval $\mathfrak{A}$. *Then*

$$Q = \int_a^x P dx = \int_a^x a_0 dx + \int_a^x a_1 x dx + \cdots$$

where α, x *lie in* $\mathfrak{A}$. *Moreover* Q *considered as a function of* x *converges uniformly in* $\mathfrak{A}$.

For by 137, P is uniformly convergent in (α, x). We may therefore integrate termwise by 150, 3. To show that Q is uniformly convergent in $\mathfrak{A}$ we observe that P being uniformly convergent in $\mathfrak{A}$ we may set

$$P = P_m + \overline{P}_m$$

where

$$|\overline{P}_m| < \sigma, \qquad m > m_0, \ \sigma \text{ small at pleasure.}$$

Then

$$Q = Q_m + \overline{Q}_m$$

where

$$|\overline{Q}_m| = \left| \int_a^x \overline{P}_m dx \right| \leq \sigma \widehat{\mathfrak{A}} < \epsilon$$

on taking σ sufficiently small.

159. 1. Let us show how the theorems in 2 may be used to obtain the developments of some of the elementary functions in power series.

The Logarithmic Series. We have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

for any x in $\mathfrak{A} = (-1^*, 1^*)$. Thus

$$\int_0^x \frac{dx}{1-x} = -\log(1-x) = \int_0^x dx + \int_0^x x dx + \cdots$$

Hence

$$\log(1-x) = -\left\{ x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \right\} \quad ; \quad x \text{ in } \mathfrak{A}.$$

This gives also

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad ; \quad x \text{ in } \mathfrak{A}. \qquad (1$$

The series 1) is also valid for $x = 1$. For the series is convergent for $x = 1$, and $\log(1 + x)$ is continuous at $x = 1$. We now apply 147, 6.

For $x = 1$, we get

$$\log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

2. *The Development of arcsin x.* We have by the Binomial Series

$$\frac{1}{\sqrt{1 - x^2}} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots$$

for x in $\mathfrak{A} = (-1^*, 1^*)$. Thus

$$\int_0^x \frac{dx}{\sqrt{1 - x^2}} = \arcsin x = x + \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \cdots \tag{2}$$

It is also valid for $x = 1$. For the series on the right is convergent for $x = 1$. We can thus reason as in 1.

For $x = 1$ we get

$$\frac{\pi}{2} = 1 + \frac{1}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7} + \cdots$$

3. *The Arctan Series.* We have

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

for x in $\mathfrak{A} = (-1^*, 1^*)$. Thus

$$\int_0^x \frac{dx}{1 + x^2} = \arctan x = \int_0^x dx - \int_0^x x^2 dx + \cdots$$
$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \tag{3}$$

valid in $\mathfrak{A}$. The series 3) is valid for $x = 1$ for the same reason as in 2.

For $x = 1$ we get $\quad \dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$

4. *The Development of e^x.* We have seen that

$$E(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

converges for any x. Differentiating, we get

$$E'(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$$

Hence
$$E'(x) = E(x) \tag{a}$$
for any x. Let us consider now the function
$$f(x) = \frac{E(x)}{e^x}.$$
We have
$$f'(x) = \frac{e^x E' - E e^x}{e^{2x}} = \frac{E' - E}{e^x} = 0$$
by (a). Thus by I, 400, $f(x)$ is a constant. For $x = 0, f(x) = 1$. Hence
$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$
valid for any x.

5. *Development of* $\cos x$, $\sin x$.

The series
$$C = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
converges for every x. Hence, differentiating,
$$C' = -\frac{x}{1} + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots$$
$$C'' = -1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots$$
Hence adding,
$$C + C'' = 0. \tag{b}$$

Let us consider now the function
$$f(x) = C \sin x + C' \cos x.$$
Then
$$\begin{aligned} f'(x) &= C \cos x + C' \sin x - C' \sin x + C'' \cos x \\ &= (C + C'') \cos x \\ &= 0 \qquad \text{by (b).} \end{aligned}$$

Thus $f(x)$ is a constant. But $C = 1$, $C' = 0$, for $x = 0$, hence $f(x) = 0$,

or
$$C \sin x + C' \cos x = 0. \tag{c}$$

In a similar manner we may show that

or
$$g(x) = C \cos x - C' \sin x = 1. \tag{d}$$

If we multiply (c) by $\sin x$ and (d) by $\cos x$ and add, we get $C = \cos x$. Similarly we get $C' = -\sin x$. Thus finally

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

valid for any x.

160. 1. *Let* $P = a_m x^m + a_{m+1}x^{m+1} + \cdots$, $a_m \neq 0$, *converge in some interval* $\mathfrak{A}$ *about the origin. Then there exists an interval* $\mathfrak{B} < \mathfrak{A}$ *in which* P *does not vanish except at* $x = 0$.

For
$$P = x^m(a_m + a_{m+1}x + \cdots)$$
$$= x^m Q.$$

Obviously Q converges in $\mathfrak{A}$. It is thus continuous at $x = 0$. Since $Q \neq 0$ at $x = 0$ it does not vanish in some interval $\mathfrak{B}$ about $x = 0$ by I, 351.

In analogy to polynomials, we say P has a *zero or root of order* m at the origin.

2. *Let* $P = a_0 + a_1 x + a_2 x^2 + \cdots$ *vanish at the points* $b_1, b_2, \cdots \doteq 0$. *Then all the coefficients* $a_n = 0$. *The points* b_n *are supposed to be different from each other and from* 0.

For by hypothesis $P(b_n) = 0$. But P being continuous at $x = 0$,

$$P(0) = \lim P(b_n).$$

Hence
$$P(0) = 0,$$

and thus
$$a_0 = 0.$$

Hence
$$P = xP_1.$$

Thus P_1 vanishes also at the points b_n. We can therefore reason on P_1 as on P and thus $a_1 = 0$. In this way we may continue.

3. *If*
$$P = a_0 + a_1 x + \cdots$$
$$Q = b_0 + b_1 x + \cdots$$

be equal for the points of an infinite sequence B whose limit is $x = 0$, then $a_n = b_n$, $n = 0, 1, 2 \cdots$

For $P - Q$ vanishes at the points B.

Hence

$$a_n - b_n = 0 \quad , \quad n = 0, 1, 2 \cdots$$

4. Obviously if the two series P, Q are equal for all x in a little interval about the origin, the coefficients of like powers are equal; that is

$$a_n = b_n \quad , \quad n = 0, 1, 2 \cdots$$

161. 1. Let

$$y = a_0 + a_1 x + a_2 x^2 + \cdots \tag{1}$$

converge in an interval $\mathfrak{A}$. As x ranges over $\mathfrak{A}$, let y range over an interval $\mathfrak{B}$. Let

$$z = b_0 + b_1 y + b_2 y^2 + \cdots \tag{2}$$

converge in $\mathfrak{B}$. Then z may be considered as a function of x defined in $\mathfrak{A}$. We seek to develop z in a power series in x.

To this end let us raise 1) to the 2°, 3°, 4° ⋯ powers; we get series

$$\begin{aligned} y^2 &= a_{20} + a_{21} x + a_{22} x^2 + \cdots \\ y^3 &= a_{30} + a_{31} x + a_{32} x^2 + \cdots \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{3}$$

which converge absolutely within $\mathfrak{A}$.

We note that a_{mn} is a polynomial.

$$a_{m,n} = F_{m,n}(a_0, a_1 \cdots a_n)$$

in $a_0, \cdots a_n$ with coefficients which are positive integers.

If we put 3) in 2), we get a double series

$$\begin{aligned} D = (b_0 + b_1 a_0) &+ b_1 a_1 x + b_1 a_2 x^2 + \cdots \\ &+ b_2 a_{20} + b_2 a_{21} x + b_2 a_{22} x^2 + \cdots \\ &+ b_3 a_{30} + b_3 a_{31} x + b_3 a_{32} x^2 + \cdots \\ &+ \cdots\cdots\cdots\cdots \end{aligned} \tag{4}$$

If we sum by rows, we get a series whose sum is evidently z, since each row of D is a term of z. Summing by columns we get a series we denote by

$$C = c_0 + c_1 x + c_2 x^2 + \cdots \tag{5}$$

where
$$c_0 = b_0 + b_1 a_0 + b_2 a_{20} + b_3 a_{30} + \cdots$$
$$c_1 = b_1 a_1 + b_2 a_{21} + b_3 a_{31} + \cdots \qquad (6$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

We may now state the following theorem, which is a solution of our problem.

Let the adjoint y-series,
$$\eta = \alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \cdots \qquad (7$$
converge for $\xi = \xi_0$ to the value $\eta = \eta_0$. Let the adjoint z series
$$\zeta = \beta_0 + \beta_1 \eta + \beta_2 \eta^2 + \cdots \qquad (8$$
converge for $\eta = \eta_0$. Then the z series 2) can be developed into a power series in x, viz. the series 5), which is valid for $|x| \leq \xi_0$.

For in the first place, the series 8) converges for $\eta \leq \eta_0$. We show now that the positive term series
$$\begin{aligned}\mathfrak{D} = (\beta_0 + \beta_1 \alpha_0) &+ \beta_1 \alpha_1 \xi + \beta_1 \alpha_2 \xi^2 + \cdots \\ &+ \beta_2 \alpha_{20} + \beta_2 \alpha_{21} \xi + \beta_2 \alpha_{22} \xi^2 + \cdots \\ &+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\end{aligned}$$
converges for $\xi \leq \xi_0$. We observe that $\mathfrak{D}$ differs from Adj D, at most by its first term. To show the convergence of $\mathfrak{D}$ we have, raising 7) to successive powers,
$$\eta^2 = A_{20} + A_{21} \xi + A_{22} \xi^2 + \cdots$$
$$\eta^3 = A_{30} + A_{31} \xi + A_{32} \xi^2 + \cdots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

We note that A_{mn} is the same function $F_{m,n}$ of $\alpha_0, \alpha_1, \cdots \alpha_n$ as a_{mn} is of $a_0, \cdots a_n$, *i.e.*
$$A_{m,n} = F_{m,n}(\alpha_0, \cdots \alpha_n).$$

As the coefficients of $F_{m,n}$ are positive integers,
$$\alpha_{m,n} = | a_{m,n} | \leq A_{m,n}. \qquad (9$$

Putting these values of η, η^2, $\eta^3 \cdots$ in 8), we get

$$\begin{aligned}\Delta = (\beta_0 + \beta_1\alpha_0) &+ \beta_1\alpha_1\xi + \beta_1\alpha_2\xi^2 + \cdots \\ &+ \beta_2 A_{20} + \beta_2 A_{21}\xi + \beta_2 A_{22}\xi^2 + \cdots \\ &+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

Summing by rows we get a convergent series whose sum is ζ or 8). But this series converges for $\xi \leq \xi_0$ since then $\eta \leq \eta_0$, and 8) converges by hypothesis for $\eta = \eta_0$. Now by 9) each term of $\mathfrak{D}$ is $\leq$ than the corresponding term in Δ. Hence $\mathfrak{D}$ converges for $\xi \leq \xi_0$.

2. As a corollary of 1 we have:

Let

$$y = a_0 + a_1x + a_2x^2 + \cdots$$

converge in $\mathfrak{A}$, *and*

$$z = b_0 + b_1y + b_2y^2 + \cdots$$

converge for all $-\infty < y < +\infty$. *Then* z *can be developed in a power series in* x,

$$z = c_0 + c_1x + c_2x^2 + \cdots = C$$

for all x *within* $\mathfrak{A}$.

3. *Let the series*

$$y = a_mx^m + a_{m+1}x^{m+1} + \cdots, \qquad m \geq 1$$

converge for some $x > 0$. *If the series*

$$z = b_0 + b_1y + b_2y^2 + \cdots$$

converges for some $y > 0$, *it can be developed in a power series*

$$z = c_0 + c_1x + c_2x^2 + \cdots$$

convergent for some $x > 0$.

For we may take $\xi = |x| > 0$ so small that

$$\eta = \alpha_m\xi^m + \alpha_{m+1}\xi^{m+1} + \cdots$$

has a value which falls within the interval of convergence of

$$\zeta = \beta_0 + \beta_1\eta + \beta_2\eta^2 + \cdots$$

4. Another corollary of 1 is the following:

Let

$$y = a_0 + a_1x + a_2x^2 + \cdots$$

converge in $\mathfrak{A} = (-A, A)$. *Then y can be developed in a power series about any point c of* $\mathfrak{A}$,

$$y = c_0 + c_1(x - c) + c_2(x - c)^2 + \cdots$$

which is valid in an interval $\mathfrak{B}$ *whose center is c and lying within* $\mathfrak{A}$.

162. 1. As an *application* of the theorem 161, 1 let us take

$$z = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$$

$$y = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

As the reader already knows,

$$z = e^y \quad , \quad y = \sin x,$$

hence z considered as a function of x is

$$z = e^{\sin x}.$$

We have

$$\begin{array}{rl} z = 1 + x + 0 \cdot x^2 - \frac{1}{6} x^3 + 0 \cdot x^4 + \frac{1}{120} x^5 + 0 \cdot x^6 + \cdots \\ + \frac{1}{2} x^2 + \ 0 \ - \frac{1}{6} x^4 + \ 0 \ + \frac{1}{45} x^6 + \cdots \\ + \frac{1}{6} x^3 + \ 0 \ - \frac{1}{12} x^5 + \ 0 \ + \cdots \\ + \frac{1}{24} x^4 + \ 0 \ - \frac{1}{36} x^6 + \cdots \\ + \frac{1}{120} x^5 + \ 0 \ + \cdots \\ + \frac{1}{720} x^6 + \cdots \end{array}$$

Summing by columns, we get

$$z = e^{\sin x} = 1 + x + \frac{1}{2} x^2 - \frac{1}{8} x^4 - \frac{1}{15} x^5 - \frac{1}{240} x^6 \cdots$$

2. As a *second application* let us consider the power series

$$z = P(y) = a_0 + a_1 y + a_2 y^2 + \cdots \qquad (1$$

convergent in the interval $\mathfrak{A} = (-R, R)$. Let x be a point in $\mathfrak{A}$. Let us take $\eta > 0$ so small that $y = x + h$ lies within $\mathfrak{A}$ for all $|h| \leq \eta$.

Then

$$\begin{aligned} z = a_0 &+ a_1 (x + h) \\ &+ a_2 (x^2 + 2\,xh + h^2) \\ &+ a_3 (x^3 + 3\,x^2 h + 3\,xh^2 + h^3) \\ &+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

This may be regarded as a double series. By 161, 1 it may be summed by columns. Hence

$$\begin{aligned} P(x+h) &= a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots \\ &\quad + h(a_1 + 2\,a_2x + 3\,a_3x^2 + \cdots) \\ &\quad + \frac{h^2}{2!}(2\,a_2 + 2\cdot 3\,a_3x + 3\cdot 4\,a_4x^2 + \cdots) \qquad (2 \\ &\quad + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &= P(x) + hP'(x) + \frac{h^2}{2!}P''(x) + \frac{h^3}{3!}P'''(x) + \cdots \qquad (3 \end{aligned}$$

on using 158, 1.

This, as the reader will recognize, is Taylor's development of the series 1) about the point x. We thus have the theorem:

A power series 1) *may be developed in Taylor's series* 3) *about any point x within its interval of convergence. It is valid for all h such that $x + h$ lies within the interval of convergence of* 1).

163. 1. The addition, subtraction, and multiplication of power series may be effected at once by the principles of 111, 112. We have if

$$P = a_0 + a_1x + a_2x^2 + \cdots$$

$$Q = b_0 + b_1x + b_2x^2 + \cdots$$

converge in a common interval $\mathfrak{A}$:

$$P + Q = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$$

$$P - Q = (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + \cdots$$

$$P \cdot Q = a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + \cdots$$

These are valid within $\mathfrak{A}$, and the first two in $\mathfrak{A}$.

2. Let us now consider the division of P by R. Since

$$\frac{P}{R} = P \cdot \frac{1}{R}$$

the problem of dividing P by R is reduced to that of finding the reciprocal of a power series.

Let
$$P = a_0 + a_1x + a_2x^2 + \cdots \quad , \quad a_0 \neq 0$$
converge absolutely in $\mathfrak{R} = (-R, R)$. Let
$$Q = a_1x + a_2x^2 + \cdots$$
be numerically $< |a_0|$ in $\mathfrak{B} = (-r, r)$ $\qquad r < R$.

Then $1/P$ *can be developed in a power series*

$$\frac{1}{P} = c_0 + c_1 x + c_2 x^2 + \cdots$$

valid in $\mathfrak{B}$. *The first coefficient* $c_0 = \dfrac{1}{a_0}$.

For

$$\frac{1}{P} = \frac{1}{a_0 + Q} = \frac{1}{a_0} \cdot \frac{1}{1 + \dfrac{Q}{a_0}}$$

$$= \frac{1}{a_0}\left\{1 - \frac{Q}{a_0} + \frac{Q^2}{a_0^2} - \frac{Q^3}{a_0^3} + \cdots\right\}$$

for all x in $\mathfrak{B}$. We have now only to apply 161, 1.

3. Suppose $\qquad P = a_m x^m + a_{m+1} x^{m+1} + \cdots \qquad a_m \neq 0.$

To reduce this case to the former, we remark that

$$P = x^m Q$$

where

$$Q = a_m + a_{m+1} x + \cdots$$

Then

$$\frac{1}{P} = \frac{1}{x^m} \cdot \frac{1}{Q}.$$

But $1/Q$ has been treated in 2.

164. 1. Although the reasoning in 161 affords us a method of determining the coefficients in the development of the quotient of two power series, there is a more expeditious method applicable also to many other problems, called the method of *undetermined coefficients*. It rests on the hypothesis that $f(x)$ can be developed in a power series in a certain interval about some point, let us say the origin. Having assured ourselves on this head, we set

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

where the a's are undetermined coefficients. We seek enough relations between the a's to determine as many of them as we need. The spirit of the method will be readily grasped by the aid of the following examples.

Let us first prove the following theorem, which will sometimes shorten our labor.

2. *If* $$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots; \quad -R \leq x \leq R, \tag{1}$$

is an even function, the right-hand side can contain only even powers of x; if $f(x)$ is odd, only odd powers occur on the right.

For if f is even, $$f(x) = f(-x). \tag{2}$$

But $$f(-x) = a_0 - a_1 x + a_2 x^2 - \cdots \tag{3}$$

Subtracting 3) from 1), we have by 2)

$$0 = 2(a_1 x + a_3 x^3 + a_5 x^5 + \cdots)$$

for all x near the origin. Hence by 160, 2

$$a_1 = a_3 = a_5 = \cdots = 0.$$

The second part of the theorem is similarly proved.

165. *Example 1.* $$f(x) = \tan x.$$

Since $$\tan x = \frac{\sin x}{\cos x},$$

and $$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

we have $$\tan x = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots} = \frac{P}{1+Q}. \tag{1}$$

Since $\cos x > 0$ in any interval $\mathfrak{B} = \left(-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right)$, $\delta > 0$, it follows that $$|Q| < 1 \quad \text{in } \mathfrak{B}.$$

Thus by 163, 2, $\tan x$ can be developed in a power series about the origin valid in $\mathfrak{B}$. We thus set

$$\tan x = a_1 x + a_3 x^3 + a_5 x^5 + \cdots \tag{2}$$

since $\tan x$ is an odd function. From 1), 2) we have, clearing fractions,

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = (a_1 x + a_3 x^3 + \cdots)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

$$= a_1 x + \left(a_3 - \frac{a_1}{2!}\right)x^3 + \left(a_5 - \frac{a_3}{2!} + \frac{a_1}{4!}\right)x^5$$

$$+ \left(a_7 - \frac{a_5}{2!} + \frac{a_3}{4!} - \frac{a_1}{6!}\right)x^7 + \left(a_9 - \frac{a_7}{2!} + \frac{a_5}{4!} - \frac{a_3}{6!} + \frac{a_1}{8!}\right)x^9 + \cdots$$

Comparing coefficients on each side of this equation gives

$$a_1 = 1.$$

$$a_3 - \frac{a_1}{2!} = -\frac{1}{3!}. \qquad \therefore\ a_3 = \frac{1}{3}.$$

$$a_5 - \frac{a_3}{2!} + \frac{a_1}{4!} = \frac{1}{5!}. \qquad \therefore\ a_5 = \frac{2}{15}.$$

$$a_7 - \frac{a_5}{2!} + \frac{a_3}{4!} - \frac{a_1}{6!} = -\frac{1}{7!}. \qquad \therefore\ a_7 = \frac{17}{315}.$$

$$a_9 - \frac{a_7}{2!} + \frac{a_5}{4!} - \frac{a_3}{6!} + \frac{a_1}{8!} = \frac{1}{9!}. \qquad \therefore\ a_9 = \frac{62}{2835}.$$

Thus
$$\tan x = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7 + \tfrac{62}{2835}x^9 + \cdots \tag{3}$$

valid in $\left(-\frac{\pi}{2}^*, \frac{\pi}{2}^*\right)$.

Example 2.
$$f(x) = \operatorname{cosec} x = \frac{1}{\sin x}$$

$$= \frac{1}{x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)} = \frac{1}{xP} = \frac{1}{x(1-Q)}.$$

Since
$$Q = 1 - \frac{\sin x}{x},$$

we see that
$$|Q| < 1$$

when x is in $\mathfrak{B} = (-\pi + \delta, \pi - \delta)$, $\delta > 0$. Thus $xf(x) = 1/P$ can be developed in a power series in $\mathfrak{B}$. As $f(x)$ is an odd function, $xf(x)$ is even, hence its development contains only even powers of x. Thus we have

$$xf(x) = a_0 + a_2 x^2 + a_4 x^4 + \cdots$$

Hence

$$1=(a_0+a_2x^2+\cdots)\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\cdots\right)$$

$$=a_0+\left(a_2-\frac{a_0}{3!}\right)x^2+\left(a_4-\frac{a_2}{3!}+\frac{a_0}{5!}\right)x^4$$

$$+\left(a_6-\frac{a_4}{3!}+\frac{a_2}{5!}-\frac{a_0}{7!}\right)x^6+\left(a_8-\frac{a_6}{3!}+\frac{a_4}{5!}-\frac{a_2}{7!}+\frac{a_0}{9!}\right)x^8+\cdots$$

Comparing like coefficients gives

$$a_0=1.$$

$$a_2-\frac{a_0}{3!}=0. \qquad \therefore\ a_2=\frac{1}{6}.$$

$$a_4-\frac{a_2}{3!}+\frac{a_0}{5!}=0. \qquad \therefore\ a_4=\frac{7}{360}.$$

$$a_6-\frac{a_4}{3!}+\frac{a_2}{5!}-\frac{a_0}{7!}=0. \qquad \therefore\ a_6=\frac{31}{3\cdot 7!}.$$

Thus

$$\frac{1}{\sin x}=\frac{1}{x}+\frac{1}{6}x+\frac{7}{360}x^3+\frac{31}{3\cdot 7!}x^5+\cdots \tag{4}$$

valid in $(-\pi^*,\ \pi^*)$.

166. Let $$F(x)=f_1(x)+f_2(x)+\cdots$$
where $$f_n(x)=a_{n0}+a_{n1}x+a_{n2}x^2+\cdots \qquad n=1,\ 2\cdots$$

Let the adjoint series

$$\alpha_{n0}+\alpha_{n1}\xi+\alpha_{n2}\xi^2+\cdots$$

converge for $\xi=R$ and have ϕ_n as sums for this value of ξ.

Let $$\Phi=\phi_1+\phi_2+\cdots$$

converge. Then F converges uniformly in $\mathfrak{A}=(-R,\ R)$ and F may be developed as a power series, valid in $\mathfrak{A}$, by summing by columns the double series

$$\begin{aligned}&a_{10}+a_{11}x+a_{12}x^2+\cdots\\ +\,&a_{20}+a_{21}x+a_{22}x^2+\cdots\\ +\,&a_{30}+a_{31}x+a_{32}x^2+\cdots\\ +\,&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{aligned} \tag{1}$$

F converges uniformly in $\mathfrak{A}$. For as $|x| \leq \xi$,

$$|f_n(x)| \leq \alpha_{n0} + \alpha_{n1}|x| + \alpha_{n2}|x|^2 + \cdots$$
$$\leq \alpha_{n0} + \alpha_{n1}\xi + \alpha_{n2}\xi^2 + \cdots = \phi_n.$$

We now apply 136, 2 as $\Sigma\phi_n$ is convergent for $\xi = R$.

To prove the latter part of the theorem we observe that

$$\begin{aligned}&\alpha_{10} + \alpha_{11}R + \alpha_{12}R^2 + \cdots\\ &+ \alpha_{20} + \alpha_{21}R + \alpha_{22}R^2 + \cdots\\ &+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\end{aligned}$$

is convergent, since summing it by rows it has Φ as sum. Thus the double series 1) converges absolutely for $|x| \leq \xi$, by 123, 2. Thus the series 1) may be summed by columns by 130, 1 and has $F(x)$ as sum, since 1) has F as sum on summing by rows.

167. *Example.*

$$F(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{1 + a^n x} = \Sigma f_n(x) \qquad a > 1.$$

This series we have seen converges in $\mathfrak{A} = (0, b)$, b positive and arbitrarily large.

Since it is impossible to develop the $f_n(x)$ in a power series about the origin which will have a common interval of convergence, let us develop F in a power series about $x_0 > 0$.

We have

$$\frac{1}{1 + a^n x} = \frac{1}{1 + a^n x_0} \frac{1}{1 + \dfrac{a^n(x - x_0)}{1 + a^n x_0}}$$

$$= \frac{1}{1 + a^n x_0}\left\{1 - \frac{a^n(x - x_0)}{1 + a^n x_0} + \frac{a^{2n}(x - x_0)^2}{(1 + a^n x_0)^2} - \cdots\right\}$$

$$= A_{n0} + A_{n1}(x - x_0) + A_{n2}(x - x_0)^2 + \cdots$$

where
$$A_{n,\kappa} = \frac{(-1)^\kappa a^{n\kappa}}{(1 + a^n x_0)^{\kappa+1}}.$$

Thus F give rise to the double series

$$\begin{aligned} D = A'_{00} + A'_{01}(x-x_0) + A'_{02}(x-x_0)^2 + \cdots \\ + A'_{10} + A'_{11}(x-x_0) + A'_{12}(x-x_0)^2 + \cdots \\ + \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

where
$$A'_{n\kappa} = \frac{(-1)^n}{n!} A_{n\kappa}.$$

The adjoint series to $f_n(x)$ is, setting $\xi = |x - x_0|$,

$$\phi_n(\xi) = \frac{1}{n!}\left(\frac{1}{1+a^n x_0} + \frac{a^n \xi}{(1+a^n x_0)^2} + \frac{a^{2n}\xi^2}{(1+a^n x_0)^3} + \cdots\right).$$

This is convergent if

$$\frac{a^n \xi}{1+a^n x_0} < 1 \quad \text{or if} \quad \xi < x_0,$$

that is, if
$$0 < x < 2\,x_0,$$

For any x such that $x_0 \leq x < 2\,x_0$, $\xi = x - x_0$.

Then for such an x

$$\phi_n = \frac{1}{n!}\,\frac{1}{1 + a^n(2\,x_0 - x)}$$

and the corresponding series

$$\Phi = \Sigma \phi_n$$

is evidently convergent, since $\phi_n < \frac{1}{n!}$.

We may thus sum D by columns; we get

$$F(x) = \sum_{\kappa=0}^{\infty} B_\kappa (x - x_0)^\kappa \tag{1}$$

where
$$B_\kappa = \sum_{n=0}^{\infty} \frac{(-1)^{n+\kappa}}{n!} \frac{a^{n\kappa}}{(1+a^n x_0)^{\kappa+1}}.$$

The relation 1) is valid for $0 < x < 2\,x_0$.

168. *Inversion of a Power Series.*

Let the series
$$v = b_0 + b_1 t + b_2 t^2 + \cdots \tag{1}$$
have $b_1 \neq 0$, and let it converge for $t = t_0$. If we set
$$t = x t_0, \qquad u = \frac{v - b_0}{b_1 t_0},$$
it goes over into a series of the form
$$u = x - a_2 x^2 - a_3 x^3 - \cdots \tag{2}$$
which converges for $x = 1$. Without loss of generality we may suppose that the original series 1) has the form 2) and converges for $x = 1$. We shall therefore take the given series to be 2). By I, 437, 2 the equation 2) defines uniquely a function x of u which is continuous about the point $u = 0$, and takes on the value $x = 0$, for $u = 0$.

We show that this function x may be developed in a power series in u, valid in some interval about $u = 0$.

To this end let us set
$$x = u + c_2 u^2 + c_3 u^3 + \cdots \tag{3}$$
and try to determine the coefficient c, so that 3) satisfies 2) formally. Raising 3) to successive powers, we get
$$\begin{aligned} x^2 &= u^2 + 2c_2 u^3 + (c_2^2 + 2c_3)u^4 + (2c_4 + 2c_2 c_3)u^5 + \cdots \\ x^3 &= u^3 + 3c_2 u^4 + (3c_2^2 + 3c_3)u^5 + \cdots \\ x^4 &= u^4 + 4c_2 u^5 + \cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{4}$$
Putting these in 2) it becomes
$$\begin{aligned} u = u &+ (c_2 - a_2)u^2 + (c_3 - 2a_2 c_2 - a_3)u^3 \\ &+ (c_4 - a_2(c_2^2 + 2c_3) - 3a_3 c_2 - a_4)u^4 \\ &+ (c_5 - 2a_2(c_4 + c_2 c_3) - 3a_3(c_2^2 + c_3) - 4a_4 c_2 - a_5)u^5 \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{5}$$
Equating coefficients of like powers of u on both sides of this equation gives
$$\begin{aligned} c_2 &= a_2 \\ c_3 &= 2a_2 c_2 + a_3 \\ c_4 &= a_2(c_2^2 + 2c_3) + 3a_3 c_2 + a_4 \\ c_5 &= 2a_2(c_4 + c_2 c_3) + 3a_3(c_2^2 + c_3) + 4a_4 c_2 + a_5. \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{6}$$

This method enables us thus to determine the coefficient c in 3) such that this series when put in 2) *formally* satisfies this relation. We shall call the series 3) where the coefficients c have the values given in 6), the *inverse series* belonging to 2).

Suppose now the inverse series 3) converges for some $u_0 \neq 0$; can we say it satisfies 2) for values of u near the origin? The answer is, Yes. For by 161, 3, we may sum by columns the double series which results by replacing in the right side of 2)

$$x, \quad x^2, \quad x^3, \quad \cdots$$

by their values in 3), 4). But when we do this, the right side of 2) goes over into the right side of 5), all of whose coefficients $= 0$ by 6) except the first.

We have therefore only to show that the inverse series converges for some $u \neq 0$. To show this we make use of the fact that 2) converges for $x = 1$. Then $a_n \doteq 0$, and hence

$$|a_n| < \text{some } \alpha \qquad n = 2, 3, \cdots \tag{7}$$

On the other hand, the relations 6) show that

$$c_n = f_n(a_2, a_3, \cdots a_n) \tag{8}$$

is a polynomial with integral positive coefficients. In 8) let us replace $a_2, a_3 \cdots$ by α, getting

$$\gamma_n = f_n(\alpha, \alpha, \cdots \alpha). \tag{9}$$

Obviously $$|c_n| < \gamma_n. \tag{10}$$

Let us now replace all the a's in 2) by α; we get the geometric series

$$u = x - \alpha x^2 - \alpha x^3 - \alpha x^4 - \cdots \tag{11}$$

$$= x - \frac{\alpha x^2}{1 - x}. \tag{12}$$

The inverse series belonging to 11) is

$$x = u + \gamma_2 u^2 + \gamma_3 u^3 + \gamma_4 u^4 + \cdots \tag{13}$$

where obviously the γ's are the functions 9).

We show now that 11) is convergent about $u = 0$. For let us solve 12); we get

$$x = \frac{1 + u + \sqrt{1 - 2(2\alpha + 1)u + u^2}}{2(1 + \alpha)}. \tag{14}$$

Let us set $1 - 2(2\alpha + 1)u + u^2 = 1 - v$. For u near $u = 0$, $|v| < 1$. Then by the Binomial Theorem

$$\sqrt{1 - v} = 1 + d_1 v + d_2 v^2 + \cdots$$

Replacing v by its value in u, this becomes a power series in u which holds for u near the origin, by 161, 3. Thus 14) shows that x can be developed in a power series about the origin. Thus 13) converges about $u = 0$. But then by 10) the inverse series 3) converges in some interval about $u = 0$.

We may, therefore, state the theorem:

Let
$$u = b + b_1 x + b_2 x^2 + b_3 x^3 + \cdots \quad , \quad b_1 \neq 0, \tag{15}$$

converge about the point $x = 0$. Then this relation defines x as a function of u which admits the development

$$x = (u - b)\left\{\frac{1}{b_1} + a_1(u - b) + a_2(u - b)^2 + \cdots\right\}$$

about the point $u = b$. The coefficients a may be obtained from 15) *by the method of undetermined coefficients.*

Example. We saw that

$$u = \log\,(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \tag{1}$$

If we set

$$u = x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \tag{2}$$

we have
$$a_2 = -\tfrac{1}{2} \quad , \quad a_3 = \tfrac{1}{3} \quad , \quad a_4 = -\tfrac{1}{4} \quad , \quad a_5 = \tfrac{1}{5} \cdots$$

If we invert 2), we get

$$x = u + c_2 u^2 + c_3 u^3 + \cdots$$

where c's are given by 6) in 168. Thus

$$-c_2 = -\tfrac{1}{2}. \qquad \therefore\ c_2 = \tfrac{1}{2}.$$

$$-c_3 = 2(-\tfrac{1}{2})(\tfrac{1}{2}) + \tfrac{1}{3} = -\tfrac{1}{6}. \qquad \therefore\ c_3 = \tfrac{1}{6}.$$

$$-c_4 = -\tfrac{1}{2}(\tfrac{1}{4} + 2\cdot\tfrac{1}{6}) + 3\cdot\tfrac{1}{3}\cdot\tfrac{1}{2} - \tfrac{1}{4} = -\tfrac{1}{24}. \qquad \therefore\ c_4 = \tfrac{1}{24}.$$

$$-c_5 = 2(-\tfrac{1}{2})(\tfrac{1}{24} + \tfrac{1}{2}\cdot\tfrac{1}{6}) + 3\cdot\tfrac{1}{3}(\tfrac{1}{4} + \tfrac{1}{6}) + 4(-\tfrac{1}{4})(\tfrac{1}{2}) + \tfrac{1}{5}$$
$$= -\tfrac{1}{120}. \qquad \therefore\ c_5 = \tfrac{1}{120}.$$

Thus we get

$$x = u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \frac{u^5}{5!} + \cdots \tag{3}$$

But from 1) we have

$$1 + x = e^u = 1 + \frac{u}{1!} + \frac{u^2}{3!} + \cdots$$

which agrees with 3).

Taylor's Development

169. 1. We have seen, I, 409, that if $f(x)$ together with its first n derivatives are continuous in $\mathfrak{A} = (a < b)$, then

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{n-1!}f^{(n-1)}(a)$$
$$+ \frac{h^n}{n!}f^{(n)}(a + \theta h) \tag{1}$$

where

$$a \leq a + h \leq b \quad , \quad 0 < \theta < 1.$$

Consider the infinite power series in h.

$$T = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \cdots \tag{2}$$

We call it the *Taylor's series belonging to* $f(x)$. The first n terms of 1) and 2) are the same. Let us set

$$R_n = \frac{h^n}{n!}f^{(n}(a + \theta h). \tag{3}$$

We observe that R_n is a function of n, h, a and an unknown variable θ lying between 0 and 1.

We have

$$f(a+h) = T_n + R_n.$$

From this we conclude at once:

If 1°, $f(x)$ *and its derivatives of every order are continuous in* $\mathfrak{A} = (a, b)$, *and* 2°

$$\lim R_n = \lim \frac{h^n}{n!}f^{(n)}(a + \theta h) = 0 \quad , \quad n = \infty, \tag{4}$$

$$a \leq a + h \leq b \qquad 0 < \theta < 1.$$

Then

$$f(a+h)=f(a)+\frac{h}{1\,!}f'(a)+\frac{h^2}{2\,!}f''(a)+\cdots \tag{5}$$

The above theorem is called *Taylor's theorem;* and the equation 5) is the development of $f(x)$ in the interval $\mathfrak{A}$ by *Taylor's series.*

Another form of 5) *is*

$$f(x)=f(a)+\frac{(x-a)}{1\,!}f'(a)+\frac{(x-a)^2}{2\,!}f''(a)+\cdots \tag{6}$$

When the point a is the origin, that is, when $a=0$, 5) or 6) gives

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2\,!}f''(0)+\cdots \tag{7}$$

This is called *Maclaurin's development* and the right side of 7) *Maclaurin's series.* It is of course only a special case of Taylor's development.

2. Let us note the content of Taylor's Theorem. It says:

If 1° $f(x)$ can be developed in this form in the interval $\mathfrak{A}=(a<b)$;

2° if $f(x)$ and all its derivatives are known at the point $x=a$;

then the value of f and all its derivatives are known at every point x within $\mathfrak{A}$.

The remarkable feature about this result is that the 2° condition requires a knowledge of the values of $f(x)$ in an interval $(a, a+\delta)$ as small as we please. Since the values that a function of a real variable takes on in a part of its interval as $(a<c)$, have no effect on the values that $f(x)$ may have in the rest of the interval $(c<b)$, the condition 1° must impose a condition on $f(x)$ which obtains throughout the whole interval $\mathfrak{A}$.

170. *Let $f(x)$ be developable in a power series about the point a,* viz. *let*

$$f(x)=a_0+a_1(x-a)+a_2(x-a)^2+\cdots \tag{1}$$

Then

$$a_n=\frac{f^{(n)}(a)}{n\,!} \qquad n=0,1,\cdots \tag{2}$$

i.e. *the above series is Taylor's series.*

For differentiating 1) n times, we get

$$f^{(n)}(x) = n!\,a_n + \frac{n+1!}{2!}a_{n+1}(x-a) + \cdots$$

Setting here $x = a$, we get 2).

The above theorem says that if $f(x)$ can be developed in a power series about $x = a$, this series can be no other than Taylor's series.

171. 1. *Let $f^{(n)}(x)$ exist and be numerically less than some constant M for all $a \le x \le b$, and for every n. Then $f(x)$ can be developed in Taylor's series for all x in (a, b).*

For then
$$|R_n| < M\frac{h^n}{n!}.$$

But obviously
$$\lim_{n=\infty}\frac{h^n}{n!} = 0.$$

2. The application of the preceding theorem gives at once:

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad (1$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad (2$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad (3$$

which are *valid for every x.*

Since
$$a^x = e^{x\log a}, \qquad a > 0,$$
we have
$$a^x = 1 + x\frac{\log a}{1!} + x^2\frac{\log^2 a}{2!} + \cdots \quad (4$$
valid for all x and $a > 0$.

172. 1. *To develop* $(1+x)^\mu$ *and* $\log(1+x)$ we need another expression of the remainder R_n due to Cauchy. We shall conduct our work so as to lead to a very general form for R_n.

From 169, 1 we have

$$R_n = f(x) - f(a) - (x-a)f'(a) - \cdots - \frac{(x-a)^{n-1}}{n-1!}f^{(n-1)}(a).$$

We introduce the auxiliary function defined over (a, b).

$$g(t) = f(t) + f'(t)(x-t) + \cdots + f^{(n-1)}(t)\frac{(x-t)^{n-1}}{n-1!}. \quad (1$$

Then

$$g(x) = f(x)$$

and

$$g(a) = f(a) + f'(a)(x-a) + \cdots + f^{(n-1)}(a)\frac{(x-a)^{n-1}}{n-1!}.$$

Hence

$$R_n = g(x) - g(a). \quad (2$$

If we differentiate 1), we find the terms cancel in pairs, leaving

$$g'(t) = \frac{(x-t)^{n-1}}{n-1!} f^{(n)}(t). \quad (3$$

We apply now Cauchy's theorem, I, 448, introducing another arbitrary auxiliary function $G(x)$ which satisfies the conditions of that theorem.

Then

$$\frac{g(x)-g(a)}{G(x)-G(a)} = \frac{g'(c)}{G'(c)}, \qquad a < c < x.$$

Using 2) and 3), we get, since $x = a + h$,

$$R_n = \frac{G(a+h)-G(a)}{G'(a+\theta h)} \frac{h^{n-1}(1-\theta)^{n-1}}{n-1!} f^{(n)}(a+\theta h) \quad (4$$

where $0 < \theta < 1$.

2. If we set

$$G(x) = (b-x)^\mu, \qquad \mu \neq 0,$$

we have a function which satisfies our conditions. Then 4) becomes

$$R_n = \frac{h^n(1-\theta)^{n-\mu}}{n-1!\,\mu} f^{(n)}(a+\theta h), \quad (5$$

a formula due to *Schlömilch* and *Roche*.

For $\mu = 1$, this becomes

$$R_n = \frac{h^n(1-\theta)^{n-1}}{n-1!} f^{(n)}(a+\theta h),$$

which is *Cauchy's* formula.

For $\mu = n$, we get from 5)

$$R_n = \frac{h^n}{n\,!} f^{(n)}(a + \theta h),$$

which is Lagrange's formula already obtained.

173. 1. *We consider now the development of*

$$(1 + x)^\mu \qquad x \gtreqless - 1 \quad , \quad \mu \text{ arbitrary.}$$

The corresponding Taylor's series is

$$T = 1 + \frac{\mu}{1} x + \frac{\mu \cdot \mu - 1}{1 \cdot 2} x^2 + \frac{\mu \cdot \mu - 1 \cdot \mu - 2}{1 \cdot 2 \cdot 3} x^3 + \cdots$$

We considered this series in 99, where we saw that:

T converges for $|x| < 1$ and diverges for $|x| > 1$.

When $x = 1$, T converges only when $\mu > - 1$; when $x = - 1$, T converges only when $\mu \gtreqless 0$.

We wish to know when

$$(1 + x)^\mu = 1 + \frac{\mu}{1} x + \frac{\mu \cdot \mu - 1}{1 \cdot 2} x^2 + \cdots \tag{1}$$

The cases when T diverges are to be thrown out at once. Consider in succession the cases that T converges. We have to investigate when $\lim R_n = 0$.

Case 1°. $0 < |x| < 1$. It is convenient to use here Cauchy's form of the remainder. This gives

$$R_n = (1 - \theta)^{n-1} \frac{\mu \cdot \mu - 1 \cdot \cdots \mu - n + 1}{1 \cdot 2 \cdot \cdots n} x^n (1 + \theta x)^{\mu - n}$$

$$= \frac{1}{n} S_n U_n W_n,$$

setting

$$S_n = \frac{\mu \cdot \mu - 1 \cdot \cdots \mu - n + 1}{1 \cdot 2 \cdot \cdots n - 1} x^n,$$

$$U_n = (1 + \theta x)^{\mu - 1},$$

$$W_n = \left(\frac{1 - \theta}{1 + \theta x} \right)^{n-1}.$$

Now in W_n, $\qquad 1 - \theta < 1 + \theta x,$

hence $\lim W_n = 0$.

In U_n,
$$|1+\theta x|<1+|x|,$$
which is finite. Hence U_n is $<$ some constant M.

To show that $\lim S_n = 0$, we make use of the fact that the series T converges for the values of x under consideration. Thus for every μ
$$\lim \frac{\mu \cdot \mu - 1 \cdot \cdots \mu - n + 2}{1 \cdot 2 \cdot \cdots n - 1} x^{n-1} = 0,$$
since the limit of the n^{th} term of a convergent series is 0. In this formula replace μ by $\mu - 1$, then
$$\lim \frac{\mu - 1 \cdot \mu - 2 \cdot \cdots \mu - n + 1}{1 \cdot 2 \cdot \cdots n - 1} x^{n-1} = \lim \frac{S_n}{\mu x} = 0.$$
Hence
$$\lim S_n = 0.$$
Thus
$$\lim R_n = 0.$$
Hence 1) is valid for $|x|<1$.

Case 2. $x=1$, $\mu > -1$. We employ here Lagrange's form of the remainder, which gives
$$R_n = \frac{\mu \cdot \mu - 1 \cdot \cdots \mu - n + 1}{1 \cdot 2 \cdot \cdots n}(1+\theta)^{\mu - n}$$
$$= U_n W_n,$$
setting
$$U_n = \frac{\mu \cdot \mu - 1 \cdot \cdots \mu - n + 1}{1 \cdot 2 \cdot \cdots n},$$
$$W_n = (1+\theta)^{\mu - n}.$$
Consider W_n. Since n increases without limit, $\mu - n$ becomes and remains negative. As $\theta > 0$
$$\lim W_n = 0.$$
For U_n, we use I, 143. This shows at once that
$$\lim U_n = 0.$$
Hence
$$\lim R_n = 0$$
and 1) is valid in this case, *i.e.* for $x = 1$, $\mu > -1$.

Case 3. $x = -1$, $\mu \gtreqless 0$. We use here for $\mu > 0$ the Schlömilch-Roche form of the remainder 172, 5). We set $a = 0$, $h = -1$ and get

$$R_n = (-1)^n \frac{(1-\theta)^{n-\mu}}{n-1!\,\mu}\mu \cdot \mu - 1 \cdot \cdots \mu - n + 1 \cdot (1-\theta)^{\mu - n}$$

$$= (-1)^n \frac{\mu - 1 \cdot \mu - 2 \cdot \cdots \mu - n + 1}{1 \cdot 2 \cdot \cdots n - 1}.$$

Applying I, 143, we see that $\lim R_n = 0$.

Hence 1) is valid here if $\mu > 0$.

When $\mu = 0$ equation 1) is evidently true, since both sides reduce to 1.

Summing up, we have the theorem:

The development of $(1+x)^\mu$ in Taylor's series is valid when $|x| < 1$ for all μ. When $x = +1$ it is necessary that $\mu > -1$; when $x = -1$ it is necessary that $\mu \gtreqless 0$.

2. We note the following formulas obtained from 1), setting $x = 1$ and -1.

$$2^\mu = 1 + \frac{\mu}{1} + \frac{\mu \cdot \mu - 1}{1 \cdot 2} + \frac{\mu \cdot \mu - 1 \cdot \mu - 2}{1 \cdot 2 \cdot 3} + \cdots \qquad \mu > -1.$$

$$0 = 1 - \frac{\mu}{1} + \frac{\mu \cdot \mu - 1}{1 \cdot 2} - \frac{\mu \cdot \mu - 1 \cdot \mu - 2}{1 \cdot 2 \cdot 3} + \cdots \qquad \mu > 0.$$

174. 1. We develop now $\log(1+x)$. The corresponding Taylor's series is

$$T = 1 + \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

We saw, 89, Ex. 2, that T converges when and only when $|x| < 1$ or $x = 1$.

Let $0 \leq x \leq 1$. We use Lagrange's remainder, which gives here

$$R_n = \frac{(-1)^{n-1}x^n}{n(1+\theta x)^n}.$$

Thus

$$|R_n| < \frac{1}{n}.$$

Hence

$$\lim R_n = 0.$$

Let $-1 < x < 0$. We use here Cauchy's remainder, which gives, setting $x = -\xi$, $0 < \xi < 1$,

$$|R_n| = \xi^n \cdot \frac{1}{1-\theta\xi} \cdot \left(\frac{1-\theta}{1-\theta\xi}\right)^{n-1}$$
$$= S_n U_n W_n,$$

if

$$S_n = \xi^n,$$
$$U_n = \frac{1}{1-\theta\xi},$$
$$W_n = \left(\frac{1-\theta}{1-\theta\xi}\right)^{n-1}.$$

Evidently
$$\lim S_n = 0.$$

Also
$$U_n < \frac{1}{1-\xi}.$$

Finally
$$\lim W_n = 0 \qquad \text{since } \frac{1-\theta}{1-\theta\xi} < 1.$$

We can thus sum up in the theorem :

Taylor's development of $\log(1+x)$ *is valid when and only when* $|x| < 1$ *or* $x = 1$. That is, for such values of x

$$\log(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

2. We note the following special case :

$$\tfrac{1}{1} - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots = \log 2.$$

The series on the left we have already met with.

175. We add for completeness the development of the following functions for which it can be shown that $\lim R_n = 0$.

$$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots \qquad (1$$

which is valid for $(-1, 1)$.

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad (2$$

which is valid for $(-1^*, 1)$.

$$\log(x + \sqrt{1+x^2}) = x - \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots \qquad (3$$

which is valid for $(-1^*, 1^*)$.

176. We wish now to call attention to various false notions which are prevalent regarding the development of a function in Taylor's series.

Criticism 1. It is commonly supposed, if the Taylor's series T belonging to a function $f(x)$ is convergent, that then

$$f(x) = T.$$

That this is not always true we proceed to illustrate by various examples.

Example 1. For $f(x)$ take Cauchy's function, I, 335,

$$C(x) = \lim_{n=\infty} e^{\frac{-1}{x^2+\frac{1}{n}}}.$$

For $x \neq 0$ $\quad C(x) = e^{-\frac{1}{x^2}}$; for $x = 0$ $\quad C(x) = 0$.

1° *derivative.* For $x \neq 0$, $\quad C'(x) = \frac{2}{x^3} C(x)$.

For $x = 0$, $\quad C'(0) = \lim_{h=0} \frac{C(h) - C(0)}{h} = \lim \frac{e^{-\frac{1}{h^2}}}{h} = 0.$

2° *derivative.* $x \neq 0$, $\quad C''(x) = C(x)\left\{\frac{4}{x^6} - \frac{6}{x^4}\right\}$.

$x = 0$, $\quad C''(0) = \lim \frac{C'(h) - C'(0)}{h} = \lim \frac{2}{h^4} e^{-\frac{1}{h^2}} = 0.$

3° *derivative.* $x \neq 0$, $\quad C'''(x) = C(x)\left\{\frac{8}{x^9} - \frac{36}{x^7} + \frac{24}{x^5}\right\}$.

$x = 0$, $\quad C'''(0) = \lim \frac{C''(0)}{h} = 0.$

In general we have :

$x \neq 0$, $\quad C^{(n)}(x) = C(x)\left\{\frac{2^n}{x^{3n}} + \text{terms of lower degree}\right\}$.

$x = 0$, $\quad C^{(n)}(0) = 0.$

Thus the corresponding Taylor's series is

$$T = C(0) + \frac{x}{1!} C'(0) + \frac{x^2}{2!} C''(0) + \cdots$$
$$= 0 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 + \cdots$$

That is, T is convergent for every x, but vanishes identically. It is thus obvious that $C(x)$ cannot be developed about the origin in Taylor's series.

Example 2. Because the Taylor's series about the origin belonging to $C(x)$ vanishes identically, the reader may be inclined to regard this example with suspicion, yet without reason.

Let us consider therefore the following function,

$$f(x) = C(x) + e^x = C(x) + g(x).$$

Then $f(x)$ and its derivatives of every order are continuous.

Since

$$f^{(n)}(x) = C^{(n)}(x) + g^{(n)}(x) \qquad n = 1, 2 \cdots$$

and

$$C^{(n)}(0) = 0$$

we have

$$f^{(n)}(0) = 1.$$

Hence Taylor's development for $f(x)$ about the origin is

$$T = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

This series is convergent, but it does not converge to the right value since

$$T = e^x.$$

177. 1. *Example 3.* The two preceding examples leave nothing to be desired from the standpoint of rigor and simplicity. They involve, however, a function, namely, $C(x)$, which is not defined in the usual way; it is therefore interesting to have examples of functions defined in one of the ordinary everyday ways, *e.g.* as infinite series. Such examples have been given by *Pringsheim.*

The infinite series

$$F(x) = \sum_0^\infty \frac{(-1)^n}{n!} \frac{1}{1 + a^n x} = \Sigma f_n(x) \qquad a > 1 \qquad (1$$

defines, as we saw, 155, 2, a function in the interval $\mathfrak{A} = (0, b)$, $b > 0$ but otherwise arbitrary, which has derivatives in $\mathfrak{A}$ of every order, viz.:

$$F^{(\lambda)}(x) = \Sigma f_n^{(\lambda)}(x) = (-1)^\lambda \lambda! \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{a^{n\lambda}}{(1 + a^n x)^{\lambda+1}}. \qquad (2$$

The Taylor's series about the origin for $F(x)$ is

$$T(x)=\sum_{\lambda=0}^{\infty}\frac{x^\lambda}{\lambda\,!}F^{(\lambda)}(0) \quad ; \quad \lambda\,!=1 \text{ for } \lambda=0,$$

and by 2)

$$\frac{F^{(\lambda)}(0)}{\lambda!}=(-1)^\lambda\sum_{n=0}^{\infty}\frac{(-1)^n}{n\,!}a^{n\lambda}=\frac{(-1)^\lambda}{e^{a^\lambda}}.$$

Hence

$$T(x)=\sum_{\lambda=0}^{\infty}\frac{(-1)^\lambda}{e^{a^\lambda}}x^\lambda=\Sigma(-1)^\lambda t_\lambda. \tag{3}$$

As $t_\lambda \gtreqless 0$ and $\lim t_\lambda=0$, $t_{\lambda+1}<t_\lambda$, this series is an alternate series for any x in $\mathfrak{A}$. Hence T converges in $\mathfrak{A}$.

2. Readers familiar with the elements of the theory of functions of a complex variable will know without any further reasoning that our Taylor's series T given in 3) cannot equal the given function F in any interval $\mathfrak{A}$, however small b is taken. In fact, $F(x)$ is an analytic function for which the origin is an essentially singular point, since F has the poles $-\frac{1}{a^n}$ $n=1, 2, 3 \cdots$, whose limiting point is 0.

3. To show by elementary means that $F(x)$ cannot be developed about the origin in a Taylor's series is not so simple. We prove now, however, with *Pringsheim*:

If we take $a \gtreqless \left(\frac{e+1}{e-1}\right)^2=4.68 \cdots$, $T(x)$ *does not equal* $F(x)$ *throughout any interval* $\mathfrak{A}=(0, b)$, *however small* $b>0$ *is taken.*

We show 1° that if $F(x)=T(x)$ throughout $\mathfrak{A}$, this relation is true in $\mathfrak{B}=(0, 2\,b^*)$.

In fact let $$0<x_0<b.$$

By 161, 4 we can develop T about x_0, getting a relation

$$T(x)=\sum_0^\infty C_\kappa(x-x_0)^\kappa \tag{1}$$

valid for all x sufficiently near x_0. On the other hand, we saw in 167 that

$$F(x)=\sum_0^\infty B_\kappa(x-x_0)^\kappa \tag{2}$$

is also valid for $0<x<2\,x_0$. But by hypothesis, the two power series 1) and 2) are equal for points near x_0. Hence they are

equal for $0 \leq x < 2x_0$. As we can take x_0 as near b as we choose, $F = T$ in $\mathfrak{B}$.

By repeating the operation often enough, we can show that $F = T$ in any interval $(0, B)$ where $B > 0$ is arbitrarily large.

To prove our theorem we have now only to show $F \neq T$ for some one $x > 0$.

Since

$$F(x) = \left(\frac{1}{1+x} - \frac{1}{1+ax}\right) + \left(\frac{1}{2!}\frac{1}{1+a^2x} - \frac{1}{3!}\frac{1}{1+a^3x}\right) + \cdots$$

we have

$$F(x) > \frac{1}{1+x} - \frac{1}{1+ax} = G(x).$$

On the other hand

$$T(x) = \frac{1}{e} - \left(\frac{x}{e^a} - \frac{x^2}{e^{a^2}}\right) - \left(\qquad \right) - \cdots$$

Hence

$$T(x) < \frac{1}{e}.$$

To find a value of x for which $G \geq \frac{1}{e}$ take $x = a^{-\frac{1}{2}}$. For this value of x

$$G = \frac{a^{\frac{1}{2}} - 1}{a^{\frac{1}{2}} + 1}.$$

Observe that G considered as a function of a is an increasing function. For

$$a = \left(\frac{e+1}{e-1}\right)^2, \quad G = \frac{1}{e}.$$

Hence $F > T$ for $x \geq a^{-\frac{1}{2}}$.

178. *Criticism 2.* It is commonly thought if $f(x)$ and its derivatives of every order are continuous in an interval $\mathfrak{A}$, that then the corresponding Taylor's series is convergent in $\mathfrak{A}$.

That this is not always so is shown by the following example, due to *Pringsheim*.

It is easy to see that

$$F(x) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} \frac{1}{1+a^{2n}x} \qquad a > 1$$

converges for every $x \geq 0$, and has derivatives of every order for these values of x, viz.:

$$F^{(\lambda)}(x) = (-1)^\lambda \lambda! \sum_{n=0}^{\infty} \frac{a^{2n\lambda}}{(2n)!} \frac{1}{(1+a^{2n}x)^{\lambda+1}}.$$

Taylor's series about the origin is

$$T=\frac{1}{2}\sum_{\lambda=0}^{\infty}(-1)^\lambda(e^{a^\lambda}+e^{-a^\lambda})x^\lambda$$

since

$$F^{(\lambda)}(0)=(-1)^\lambda\lambda!\sum_{n=0}^{\infty}\frac{a^{2\lambda n}}{(2n)!}=(-1)^\lambda\lambda!\frac{(e^{a^\lambda}+e^{-a^\lambda})}{2}.$$

The series T is divergent for $x>0$, as is easily seen.

179. *Criticism 3.* It is commonly thought if $f(x)$ and all its derivatives vanish for a certain value of x, say for $x=a$, that then $f(x)$ vanishes identically. One reasons thus:

The development of $f(x)$ about $x=a$ is

$$f(x)=f(a)+\frac{x-a}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots$$

As f and all its derivatives vanish at a, this gives

$$f(x)=0+0\cdot(x-a)+0\cdot(x-a)^2+\cdots$$
$$=0 \text{ whatever } x \text{ is.}$$

There are two tacit assumptions which invalidate this conclusion.

First, one assumes because f and all its derivatives exist and are finite at $x=a$, that therefore $f(x)$ can be developed in Taylor's series. An example to the contrary is Cauchy's function $C(x)$. We have seen that $C(x)$ and all its derivatives are 0 at $x=0$, yet $C(x)$ is not identically 0; in fact C vanishes only once, viz. at $x=0$.

Secondly, suppose $f(x)$ were developable in Taylor's series in a certain interval $\mathfrak{A}=(a-h, a+h)$. Then f is indeed 0 throughout $\mathfrak{A}$, but we cannot infer that it is therefore 0 outside $\mathfrak{A}$. In fact, from Dirichlet's definition of a function, the values that f has in $\mathfrak{A}$ nowise interferes with our giving f any other values we please outside of $\mathfrak{A}$.

180. 1. *Criticism 4.* Suppose $f(x)$ can be developed in Taylor's series at a, so that

$$f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\cdots=T$$

for $\qquad \mathfrak{A}=(a<b).$

Since Taylor's series T is a power series, it converges not only in $\mathfrak{A}$, but also within $\mathfrak{B} = (2a - b, a)$. It is commonly supposed that $f(x) = T$ also in $\mathfrak{B}$. A moment's reflection shows such an assumption is unjustified without further conditions on $f(x)$.

2. *Example.* We construct a function by the method considered in I, 333, viz.

$$f(x) = \lim_{n=\infty} \frac{(1+x)^n \cos x + 1 + \sin x}{1 + (1+x)^n}. \tag{1}$$

Then $f(x) = \cos x$, in $\mathfrak{A} = (0, 1)$

$= 1 + \sin x$, within $\mathfrak{B} = (0, -1)$.

We have therefore as a development in Taylor's series valid in $\mathfrak{A}$,

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = T.$$

It is obviously not valid within $\mathfrak{B}$, although T converges in $\mathfrak{B}$.

3. We have given in 1) an arithmetical expression for $f(x)$. Our example would have been just as conclusive if we had said:

Let $f(x) = \cos x$ in $\mathfrak{A}$,

and $= 1 + \sin x$ within $\mathfrak{B}$.

181. 1. *Criticism 5.* The following error is sometimes made. Suppose Taylor's development

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots \tag{1}$$

valid in $\mathfrak{A} = (a < b)$.

It may happen that T is convergent in a larger interval $\mathfrak{B} = (a < B)$.

One must not therefore suppose that 1) is also valid in $\mathfrak{B}$.

2. *Example.*

Let $f(x) = e^x$ in $\mathfrak{A} = (a, b)$,

and $= e^x + \sin(x - b)$ in $\mathfrak{B} = (b, B)$.

Then Taylor's development

$$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{1}$$

is valid for $\mathfrak{A}$. The series T converging for every x converges in $\mathfrak{B}$ but 1) is not valid for $\mathfrak{B}$.

182. Let $f(x)$ have finite derivatives of every order in $\mathfrak{A} = (a < b)$. In order that $f(x)$ can be developed in the Taylor's series

$$f(x) = f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots \tag{1}$$

valid in the interval $\mathfrak{A}$ we saw that it is necessary and sufficient that

$$\lim_{n=\infty} R_n = 0.$$

But R_n is not only a function of the independent variable h, but of the unknown variable θ which lies within the interval $(0, 1)$ and is a function of n and h.

Pringsheim has shown how the above condition may be replaced by the following one in which θ is an independent variable.

For the relation 1) *to be valid for all h such that $0 \leq h < H$, it is necessary and sufficient that Cauchy's form of the remainder*

$$R_n(h, \theta) = \frac{(1-\theta)^{n-1}h^n}{n-1!} f^{(n)}(a + \theta h),$$

the h and θ being independent variables, converge uniformly to zero for the rectangle D whose points (h, θ) satisfy

$$0 \leq h < H$$

$$0 \leq \theta \leq 1.$$

1° *It is sufficient.* For then there exists for each $\epsilon > 0$ an m such that

$$| R_n(h, \theta) | < \epsilon \qquad n \geqq m$$

for every point (h, θ) of D.

Let us fix h; then $| R_n | < \epsilon$ no matter how θ varies with n.

2° *It is necessary.* Let h_0 be an arbitrary but fixed number in $\mathfrak{A} = (0, H^*)$.

We have only to show that, from the existence of 1), for $h \leq h_0$, it follows that

$$R_n(h, \theta) \doteq 0$$

uniformly in the rectangle D, defined by

$$0 \leq h \leq h_0 \quad , \quad 0 \leq \theta \leq 1.$$

The demonstration depends upon the fact that $R_n(h, \theta)$ is h times the n^{th} term $f_n(\alpha, k)$ of the development of $f'(x)$ about the point $a+\alpha$. In fact let $h = \alpha + k$. Then by 158

$$f'(a+h) = f'(a+\alpha+k) = f'(a+\alpha) + \cdots + \frac{k^{n-1}}{n-1!} f^{(n)}(a+\alpha) + \cdots$$

whose n^{th} term is

$$f_n(\alpha, k) = \frac{k^{n-1}}{n-1!} f^{(n)}(a+\alpha).$$

Let $\alpha = \theta h$, then

$$R_n(h, \alpha) = h f_n(\alpha, k)$$

as stated.

The image Δ_0, of D_0 is the half of a square of side h_0, below the diagonal.

To show that R_n converges uniformly to 0 in D_0 we have only to show that

$$f_n(\alpha, k) \doteq 0 \qquad \text{uniformly in } \Delta_0. \tag{2}$$

To this end we have from 1) for all t in $\mathfrak{A}$

$$f'(a+t) = f'(a) + t f''(a) + \frac{t^2}{2!} f'''(a) + \cdots \tag{3}$$

Its adjoint

$$G(t) = |f'(a)| + t|f''(a)| + \cdots \tag{4}$$

also converges in $\mathfrak{A}$.

By 161, 4 we can develop 4) about $t = \alpha$, which gives

$$G(\alpha, k) = G(\alpha) + kG'(\alpha) + \cdots + \frac{k^{n-1}}{n-1!} G^{(n-1)}(\alpha) + \cdots$$

But obviously $G(\alpha, k)$ is continuous in Δ_0, and evidently all its terms are also continuous there. Therefore by 149, 3,

$$\frac{k^{n-1}}{n-1!} G^{(n-1)}(\alpha) \doteq 0 \qquad \text{uniformly in } \Delta_0. \tag{5}$$

But if we show that

$$|f^{(n)}(a+\alpha)| \leq G^{(n-1)}(a) \tag{6}$$

it follows from 5) that 2) is true. Our theorem is then established.

To prove 6) we have from 1)

$$f^{(n)}(a+\alpha)=f^{(n)}(a)+\alpha f^{(n+1)}(a)+\frac{\alpha^2}{2!}f^{(n+2)}(a)+\cdots \tag{7}$$

and from 4)

$$G^{(n-1)}(\alpha)=|f^{(n)}(a)|+\alpha\left|f^{(n+1)}(a)\right|+\frac{\alpha^2}{2!}\left|f^{(n+2)}(a)\right|+\cdots \tag{8}$$

The comparison of 7), 8) proves 6).

Circular and Hyperbolic Functions

183. 1. We have defined the circular functions as the length of certain lines; from this definition their elementary properties may be deduced as is shown in trigonometry.

From this geometric definition we have obtained an arithmetical expression for these functions. In particular

$$\sin x=\frac{x}{1!}-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots \tag{1}$$

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots \tag{2}$$

valid for every x.

As an interesting and instructive exercise in the use of series we propose now to develop some of the properties of these functions purely from their definition as infinite series. Let us call these series respectively S and C.

Let us also define $\tan x=\dfrac{\sin x}{\cos x}$, $\sec x=\dfrac{1}{\cos x}$, etc.

2. To begin, we observe that both S and C converge absolutely for every x, as we have seen. They therefore define continuous one-valued functions for every x. Let us designate them by the usual symbols

$$\sin x \quad , \quad \cos x.$$

We could just as well denote them by any other symbols, as

$$\phi(x) \quad , \quad \psi(x).$$

3. Since

$$S=0 \quad , \quad C=1 \quad \text{for } x=0,$$

we have

$$\sin 0=0 \quad , \quad \cos 0=1.$$

4. Since S involves only odd powers of x, and C only even powers,

$\sin x$ is an *odd*, $\cos x$ is an *even* function.

5. Since S and C are power series which converge for every x, they have derivatives of every order. In particular

$$\frac{dS}{dx} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = C.$$

$$\frac{dC}{dx} = -\frac{x}{1} + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots = -S.$$

Hence
$$\frac{d\sin x}{dx} = \cos x \quad , \quad \frac{d\cos x}{dx} = -\sin x. \qquad (3$$

6. To get the *addition theorem*, let an index as x, y attached to S, C indicate the variable which occurs in the series. Then

$$S_xC_y = x - \left(\frac{x^3}{3!} + \frac{xy^2}{2!}\right) + \left(\frac{x^5}{5!} + \frac{x^3y^2}{3!2!} + \frac{xy^4}{1!4!}\right)$$
$$- \left(\frac{x^7}{7!} + \frac{x^5}{5!}\frac{y^2}{2!} + \frac{x^3}{3!}\frac{x^4}{4!} + \frac{xy^6}{6!}\right) + \cdots$$

$$C_xS_y = y - \left(\frac{y^3}{3!} + \frac{x^2y}{2!}\right) + \left(\frac{y^5}{5!} + \frac{y^3}{3!}\frac{x^2}{2!} + \frac{x^4y}{4!1!}\right)$$
$$- \left(\frac{y^7}{7!} + \frac{y^5}{5!}\frac{x^2}{2!} + \frac{y^3}{3!}\frac{x^4}{4!} + \frac{y}{1!}\frac{x^6}{6!}\right) + \cdots$$

Adding,

$$S_xC_y + C_xS_y = x + y - \frac{1}{3!}\left\{x^3 + \binom{3}{1}x^2y + \binom{3}{1}xy^2 + y^3\right\}$$
$$+ \frac{1}{5!}\left\{x^5 + \binom{5}{1}x^4y + \binom{5}{2}x^3y^2 + \binom{5}{3}x^2y^3 + \binom{5}{4}xy^4 + y^5\right\} + \cdots$$
$$= \frac{x+y}{1!} - \frac{(x+y)^3}{3!} + \frac{(x+5)^5}{5!} - \cdots$$
$$= S_{x+y}.$$

Thus for every x, y

$$\sin(x+y) = \sin x \cos y + \cos x \sin y.$$

In the same way we find the addition formula for $\cos x$.

7. We can get now the important relation

$$\sin^2 x + \cos^2 x = 1 \tag{4}$$

directly from the addition theorem. Let us, however, find it by aid of the series. We have

$$S^2 = \frac{x^2}{1} - x^4\left(\frac{1}{3!} + \frac{1}{3!}\right) + x^6\left(\frac{1}{5!} + \frac{1}{3!}\frac{1}{3!} + \frac{1}{5!}\right)$$

$$- x^8\left(\frac{1}{7!} + \frac{1}{3!}\frac{1}{5!} + \frac{1}{5!}\frac{1}{3!} + \frac{1}{7!}\right) + \cdots$$

$$C^2 = 1 - x^2\left(\frac{1}{2!} + \frac{1}{2!}\right) + x^4\left(\frac{1}{4!} + \frac{1}{2!}\frac{1}{2!} + \frac{1}{4!}\right)$$

$$- x^6\left(\frac{1}{6!} + \frac{1}{4!}\frac{1}{2!} + \frac{1}{2!}\frac{1}{4!} + \frac{1}{6!}\right)$$

$$+ x^8\left(\frac{1}{8!} + \frac{1}{6!}\frac{1}{2!} + \frac{1}{4!}\frac{1}{4!} + \frac{1}{6!}\frac{1}{2!} + \frac{1}{8!}\right) \cdots$$

Hence

$$S^2 + C^2 = 1 - \frac{x^2}{2!}\left(1 - \binom{2}{1} + 1\right) + \frac{x^4}{4!}\left(1 - \binom{4}{1} + \binom{4}{2} - \binom{4}{3} + 1\right)$$

$$- \frac{x^6}{6!}\left(1 - \binom{6}{1} + \binom{6}{2} - \binom{6}{3} + \binom{6}{4} - \binom{6}{5} + 1\right) + \cdots$$

Now by I, 96, $\quad 1 - \binom{m}{1} + \binom{m}{2} - \binom{m}{3} + \cdots = 0.$

Thus

$$S^2 + C^2 = \sin^2 x + \cos^2 x = 1.$$

8. In 2 we saw $\sin x$, $\cos x$ were continuous for x; 4) shows that they are limited and indeed that they lie between ± 1.

For the left side of 4) is the sum of two positive numbers and thus neither can be greater than the right side.

9. Let us study the graph of $\sin x$, $\cos x$, which we shall call Σ and Γ, respectively.

Since $\sin x = 0$, $\dfrac{d \sin x}{dx} = \cos x = 1$, for $x = 0$, Σ cuts the x-axis at O under an angle of 45 degrees.

Similarly we see $y=1$ for $x=0$. Γ crosses the y-axis there and is parallel to the x-axis.

Since
$$S = x\left(1 - \frac{x^2}{2\cdot 3}\right) + \frac{x^5}{5!}\left(1 - \frac{x^2}{6\cdot 7}\right) + \cdots$$
and each parenthesis is positive for $0 < x^2 < 6$,
$$\sin x > 0 \qquad \text{for } 0 < x < \sqrt{6} = 2.449\cdots$$

Since
$$C = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\left(1 - \frac{x^2}{5\cdot 6}\right) + \frac{x^8}{8!}\left(1 - \frac{x^2}{9\cdot 10}\right) + \cdots$$
we see
$$\cos x > 0 \qquad \text{for } 0 \leq x < \sqrt{2} = 1.414\cdots$$

Since
$$C = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\left(1 - \frac{x^2}{7\cdot 8}\right) - \frac{x^{10}}{10!}\left(1 - \frac{x^2}{11\cdot 12}\right) - \cdots$$
$$\cos x < 0 \qquad \text{for } x = 2.$$

Since $D_x \cos x = -\sin x$ and $\sin x > 0$ for $0 < x < \sqrt{6}$, we see $\cos x$ is a decreasing function for these values of x. As it is continuous and >0 for $x = \sqrt{2}$, but <0 for $x = 2$, $\cos x$ vanishes once and only once in $(\sqrt{2}, 2)$.

This root, uniquely determined, of $\cos x$ we denote by $\frac{\pi}{2}$. As a first approximation, we have
$$\sqrt{2} < \frac{\pi}{2} < 2.$$

From 4) we have $\sin^2 \frac{\pi}{2} = 1$. As we saw $\sin x > 0$ for $x < \sqrt{6}$, we have
$$\sin \frac{\pi}{2} = +1.$$

Thus $\sin x$ increases constantly from 0 to 1 while $\cos x$ decreases from 1 to 0 in the interval $\left(0, \frac{\pi}{2}\right) = I_1$. We thus know how $\sin x$, $\cos x$ behave in I_1.

From the addition theorem
$$\sin\left(\frac{\pi}{2} + x\right) = \sin\frac{\pi}{2}\cos x + \cos\frac{\pi}{2}\sin x = \cos x.$$
$$\cos\left(\frac{\pi}{2} + x\right) = \cos\frac{\pi}{2}\cos x - \sin\frac{\pi}{2}\sin x = -\sin x.$$

Knowing how $\sin x$, $\cos x$ march in I_1, these formulæ tell us how they march in $I_2 = \left(\frac{\pi}{2}, \pi\right)$.

From the addition theorem,

$$\sin(\pi + x) = -\sin x, \qquad \cos(\pi + x) = -\cos x.$$

Knowing how $\sin x$, $\cos x$ march in $(0, \pi)$, these formulæ inform us about their march in $(0, 2\pi)$.

The addition theorem now gives

$$\sin(x + 2\pi) = \sin x, \qquad \cos(x + 2\pi) = \cos x.$$

Thus the functions $\sin x$, $\cos x$ are periodic and have 2π as period.

The graph of $\sin x \cos x$ for negative x is obtained now by recalling that $\sin x$ is odd and $\cos x$ is even.

10. As a first approximation of π we found

$$\sqrt{2} < \frac{\pi}{2} < 2.$$

By the aid of the development given 159, 3

$$\operatorname{arctg} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \tag{5}$$

we can compute π as accurately as we please.

In fact, from the addition theorem we deduce readily

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \qquad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Hence

$$\tan\frac{\pi}{4} = 1.$$

This in 5) gives *Leibnitz's formula*,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

The convergence of this series is extremely slow. In fact by 81, 3 we see that the error committed in stopping the summation at the n^{th} term is not greater than $\frac{1}{2n-1}$. How much less the error is, is not stated. Thus to be sure of making an error less than $\frac{1}{10^m}$ it would be necessary to take $\frac{1}{2}(10^m + 2)$ terms.

11. To get a more rapid means of computation, we make use of the addition theorem.

To start with, let
$$\alpha = \operatorname{arctg} \tfrac{1}{5}.$$

Then 5) gives
$$\alpha = \frac{1}{5} - \frac{1}{3}\frac{1}{5^3} + \frac{1}{5}\frac{1}{5^5} - \frac{1}{7}\frac{1}{5^7} + \cdots \tag{6}$$

a rapidly converging series.

The error E_α committed in breaking off the summation at the n^{th} term is
$$E_\alpha < \frac{1}{2n-1}\frac{1}{5^{2n-1}}.$$

By virtue of the formula for duplicating the argument
$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha},$$
we have
$$\tan 2\alpha = \tfrac{5}{12}.$$

Similarly
$$\tan 4\alpha = \tfrac{120}{119}.$$

Let
$$\beta = 4\alpha - \frac{\pi}{4}. \tag{7}$$

The addition theorem gives
$$\tan\beta = \frac{\tan 4\alpha - 1}{1 + \tan 4\alpha} = \frac{1}{239}.$$

Then 5) gives
$$\beta = \frac{1}{239} - \frac{1}{3}\frac{1}{239^3} + \frac{1}{5}\frac{1}{239^5} - \cdots \tag{8}$$

also a very rapidly converging series.

We find for the error
$$E_\beta < \frac{1}{2n-1}\frac{1}{239^{2n-1}}.$$

The formula 7) in connection with 6) and 8) gives $\frac{\pi}{4}$. The error on breaking off the summation with the n^{th} term is
$$E = E_\alpha + E_\beta < \frac{1}{2n+1}\left(\frac{1}{5^{2n+1}} + \frac{1}{239^{2n+1}}\right).$$

184. *The Hyperbolic Functions.* Closely related with the circular functions are the hyperbolic functions. These are defined by the equations

$$\sinh x = \frac{e^x - e^{-x}}{2}. \tag{1}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}. \tag{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \quad \operatorname{cosech} x = \frac{1}{\sinh x}.$$

Since

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

we have

$$\sinh x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \tag{3}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \tag{4}$$

valid for every x. From these equations we see at once:

$$\sinh(-x) = -\sinh x \quad ; \quad \cosh(-x) = \cosh x.$$

$$\sinh 0 = 0. \qquad \cosh 0 = 1.$$

$$\frac{d}{dx}\sinh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \cosh x. \tag{5}$$

$$\frac{d}{dx}\cosh x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sinh x. \tag{6}$$

Let us now look at the graph of these functions. Since $\sinh x$, $\cosh x$ are continuous functions, their graph is a continuous curve. For $x > 0$, $\sinh x > 0$ since each term in 3) is > 0. The relation 4) shows that $\cosh x$ is positive for every x.

If $x' > x > 0$, $\sinh x' > \sinh x$, since each term in 3) is greater for x' than for x. The same may be seen from 5).

Evidently from 3), 4)

$$\lim_{x=+\infty} \sinh x = +\infty \quad , \quad \lim_{x=+\infty} \cosh x = +\infty .$$

At $x = 0$, $\cosh x$ has a minimum, and $\sinh x$ cuts the x-axis at 45°.

For $x \geq 0$, $\cosh x > \sinh x$ since

$$e^x + e^{-x} > e^x - e^{-x}.$$

The two curves approach each other asymptotically as $x \doteq +\infty$. For the difference of their ordinates is e^{-x} which $\doteq 0$ as $x \doteq +\infty$.

The addition theorem is easily obtained from that of e^x. In fact

$$\sinh x \cosh y = \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2}$$

$$= \tfrac{1}{4}(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}).$$

Similarly

$$\cosh x \sinh y = \tfrac{1}{4}(e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y}).$$

Hence

$$\sinh x \cosh y + \cosh x \sinh y = \tfrac{1}{2}(e^{x+y} - e^{-(x+y)}) = \sinh (x+y).$$

Similarly we find

$$\cosh (x+y) = \cosh x \cosh y + \sinh x \sinh y.$$

In the same way we may show that

$$\cosh^2 x - \sinh^2 x = 1.$$

The Hypergeometric Function

185. This function, although known to Wallis, Euler, and the earlier mathematicians, was first studied in detail by Gauss. It may be defined by the following power series in x:

$$F(\alpha, \beta, \gamma;\ x) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha \cdot \alpha+1 \cdot \beta \cdot \beta+1}{1 \cdot 2 \cdot \gamma \cdot \gamma+1} x^2$$

$$+ \frac{\alpha \cdot \alpha+1 \cdot \alpha+2 \cdot \beta \cdot \beta+1 \cdot \beta+2}{1 \cdot 2 \cdot 3 \cdot \gamma \cdot \gamma+1 \cdot \gamma+2} x^3 + \cdots$$

The numbers α, β, γ are called *parameters*. We observe that α, β enter symmetrically, also when $\alpha = 1$, $\beta = \gamma$ it reduces to the geometric series. Finally let us note that γ cannot be zero or a negative integer, for then all the denominators after a certain term $= 0$.

The convergence of the series F was discussed in 100. The main result obtained there is that F converges absolutely for all $|x| < 1$, whatever values the parameters have, excepting of course γ a negative integer or zero.

186. For special values of the parameters, F reduces to elementary functions in the following cases:

1. If α or β is a negative integer $-n$, F is a polynomial of degree n.

2. $$F(1, 1, 2; -x) = \frac{1}{x}\log(1+x). \tag{1}$$

For
$$F(1, 1, 2, -x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots$$

Also
$$\log(1+x) = x\left(1 - \frac{x}{2} + \frac{x^2}{3} - \cdots\right).$$

The relation 1) is now obvious.

Similarly we have
$$F(1, 1, 2; x) = \frac{-1}{x}\log(1-x).$$
$$F(\tfrac{1}{2}, 1, \tfrac{3}{2}, x^2) = \frac{1}{2x}\log\frac{1+x}{1-x}.$$

3. $$F(-\alpha, \beta, \beta; x) = 1 - \frac{\alpha}{1}x + \frac{\alpha \cdot \alpha - 1}{1 \cdot 2}x^2 - \cdots$$
$$= (1-x)^\alpha.$$

4. $$xF(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, x^2) = \arcsin x.$$

5. $$xF(\tfrac{1}{2}, 1, \tfrac{3}{2}, -x^2) = \arctan x.$$

6. $$\lim_{\alpha=+\infty} F\left(\alpha, 1, 1, \frac{x}{\alpha}\right) = e^x. \tag{2}$$

For
$$F\left(\alpha, 1, 1, \frac{x}{\alpha}\right) = 1 + \frac{\alpha \cdot 1}{1 \cdot 1} \cdot \frac{x}{\alpha} + \frac{\alpha \cdot \alpha+1 \cdot 1 \cdot 2}{1 \cdot 2 \cdot 1 \cdot 2}\left(\frac{x}{\alpha}\right)^2$$
$$+ \frac{\alpha \cdot \alpha+1 \cdot \alpha+2}{1 \cdot 2 \cdot 3}\,\frac{1 \cdot 2 \cdot 3}{1 \cdot 2 \cdot 3}\left(\frac{x}{\alpha}\right)^3 + \cdots$$
$$= 1 + \frac{x}{1!} + \left(1 + \frac{1}{\alpha}\right)\frac{x^2}{2!} + \left(1 + \frac{1}{\alpha}\right)\left(1 + \frac{2}{\alpha}\right)\frac{x^3}{3!} + \cdots \tag{3}$$

Let $0 < G < \beta$. Then

$$F\left(\beta, 1, 1;\ \frac{G}{\beta}\right) = 1 + \frac{G}{1!} + \left(1 + \frac{1}{\beta}\right)\frac{G^2}{2!} + \left(1 + \frac{1}{\beta}\right)\left(1 + \frac{2}{\beta}\right)\frac{G^3}{3!} + \cdots \quad (4$$

is convergent since its argument is numerically < 1. Comparing 3), 4) we see each term of 3) is numerically $\leq$ the corresponding term of 4) for any $|x| \leq G$ and any $\alpha \geq \beta$. Thus the series 3) considered as a function of α is uniformly convergent in the interval $(\beta + \infty)$ by 136, 2; and hereby x may have any value in $(-G, G)$. Applying now 146, 4 to 3) and letting $\alpha \doteq +\infty$, we see 3) goes over into 2).

7.
$$\lim_{\alpha=+\infty} xF\left(\alpha, \alpha, \frac{3}{2};\ -\frac{x^2}{4\alpha^2}\right) = \sin x. \quad (5$$

For

$$xF\left(\alpha, \alpha, \frac{3}{2};\ -\frac{x^2}{4\alpha^2}\right) = x - \frac{x^3}{3!} + \left(1 + \frac{1}{\alpha}\right)^2\frac{x^5}{5!} - \left(1 + \frac{1}{\alpha}\right)^2\left(1 + \frac{2}{\alpha}\right)^2\frac{x^7}{7!} + \cdots$$

Let $x = G > 0$ and $\alpha = G$. Then

$$GF\left(G, G, \frac{3}{2};\ \frac{1}{4}\right) = G + \frac{G^3}{3!} + \left(1 + \frac{1}{G}\right)^2\frac{G^5}{5!} + \cdots$$

is convergent by 185. We may now reason as in 6.

8. Similarly we may show:

$$\lim_{\alpha=+\infty} F\left(\alpha, \alpha, \frac{1}{2};\ -\frac{x^2}{4\alpha^2}\right) = \cos x.$$

$$\lim_{\alpha=+\infty} F\left(\alpha, \alpha, \frac{3}{2},\ \frac{x^2}{4\alpha^2}\right) = \sinh x.$$

$$\lim_{\alpha=+\infty} F\left(\alpha, \alpha, \frac{1}{2},\ \frac{x^2}{4\alpha^2}\right) = \cosh x.$$

187. *Contiguous Functions.* Consider two F functions

$$F(\alpha, \beta, \gamma;\ x) \quad , \quad F(\alpha', \beta', \gamma';\ x).$$

If α differs from α' by unity, these two functions are said to be *contiguous*. The same holds for β, and also for γ. Thus to $F(\alpha\beta\gamma x)$ correspond 6 contiguous functions,

$$F(\alpha \pm 1, \beta \pm 1, \gamma \pm 1;\ x).$$

Between F and two of its contiguous functions exists a linear relation. As the number of such pairs of contiguous functions is

$$\frac{6 \cdot 5}{1 \cdot 2} = 15,$$

there are 15 such linear relations. Let us find one of them. We set

$$Q_n = \frac{\alpha+1 \cdot \alpha+2 \cdot \cdots \alpha+n-1 \cdot \beta \cdot \beta+1 \cdot \cdots \beta+n-2}{1 \cdot 2 \cdot \cdots n \cdot \gamma \cdot \gamma+1 \cdot \cdots \gamma+n-1}.$$

Then the coefficient of x^n in $F(\alpha\beta\gamma x)$ is

$$\alpha(\beta+n-1)Q_n;$$

in $F(\alpha+1, \beta, \gamma, x)$ it is

$$(\alpha+n)(\beta+n-1)Q_n;$$

in $F(\alpha, \beta, \gamma-1, x)$ it is

$$\frac{\alpha(\beta+n-1)(\gamma+n-1)}{\gamma-1}Q_n.$$

Thus the coefficient of x^n in

$$(\gamma-\alpha-1)F(\alpha, \beta, \gamma, x) + \alpha F(\alpha+1, \beta, \gamma, x) \\ +(1-\gamma)F(\alpha, \beta, \gamma-1, x)$$

is 0. This being true for each n, we have

$$(\gamma-\alpha-1)F(\alpha, \beta, \gamma, x) + \alpha F(\alpha+1, \beta, \gamma, x) \\ +(1-\gamma)F(\alpha, \beta, \gamma-1, x) = 0. \quad (1$$

Again, the coefficient of x^n in $F(\alpha, \beta-1, \gamma, x)$ is $\alpha(\beta-1)Q_n$; in $xF(\alpha+1, \beta, \gamma, x)$ it is $n(\gamma+n-1)Q_n$.

Hence using the above coefficients, we get

$$(\gamma-\alpha-\beta)F(\alpha, \beta, \gamma, x) + \alpha(1-x)F(\alpha+1, \beta, \gamma, x) \\ +(\beta-\gamma)F(\alpha, \beta-1, \gamma, x) = 0. \quad (2$$

From these two we get others by elimination or by permuting the first two parameters, which last does not alter the value of the function $F(\alpha\beta\gamma x)$.

Thus permuting α, β in 1) gives

$$(\gamma-\beta-1)F(\alpha, \beta, \gamma, x) + \beta F(\alpha, \beta+1, \gamma, x) \\ +(1-\gamma)F(\alpha, \beta, \gamma-1, x) = 0. \quad (3$$

Eliminating $F(\alpha, \beta, \gamma - 1, x)$ from 1), 3) gives

$$(\beta - \alpha)F(\alpha, \beta, \gamma, x) + \alpha F(\alpha + 1, \beta, \gamma, x) - \beta F(\alpha, \beta + 1, \gamma, x) = 0. \qquad (4$$

Permuting α, β in 2) gives

$$(\gamma - \alpha - \beta)F(\alpha, \beta, \gamma, x) + \beta(1 - x)F(\alpha, \beta + 1, \gamma, x) + (\alpha - \gamma)F(\alpha - 1, \beta, \gamma, x) = 0. \qquad (5$$

From 3), 5) let us eliminate $F(\alpha, \beta + 1, \gamma, x)$, getting

$$(\alpha - 1 - (\gamma - \beta - 1)x)F(\alpha, \beta, \gamma, x) + (\gamma - \alpha)F(\alpha - 1, \beta, \gamma, x) + (1 - \gamma)(1 - x)F(\alpha, \beta, \gamma - 1, x) = 0. \qquad (6$$

In 1) let us replace α by $\alpha - 1$ and γ by $\gamma + 1$; we get

$$(\gamma - \alpha + 1)F(\alpha - 1, \beta, \gamma + 1, x) + (\alpha - 1)F(\alpha, \beta, \gamma + 1, x) - \gamma F(\alpha - 1, \beta, \gamma, x) = 0. \qquad \text{(a)}$$

In 6) let us replace γ by $\gamma + 1$; we get

$$(\alpha - 1 - (\gamma - \beta)x)F(\alpha, \beta, \gamma + 1, x) + (\gamma + 1 - \alpha)F(\alpha - 1, \beta, \gamma + 1, x) - \gamma(1 - x)F(\alpha, \beta, \gamma, x) = 0. \quad \text{(b)}$$

Subtracting (b) from (a), eliminates $F(\alpha - 1, \beta, \gamma + 1, x)$ and gives

$$\gamma(1 - x)F(\alpha\beta\gamma x) - \gamma F(\alpha - 1, \beta, \gamma, x) + (\gamma - \beta)xF(\alpha, \beta, \gamma + 1, x) = 0. \qquad (7$$

From 6), 7) we can eliminate $F(\alpha - 1, \beta, \gamma, x)$, getting

$$\gamma\{\gamma - 1 + (\alpha + \beta + 1 - 2\gamma)x\}F(\alpha, \beta, \gamma, x) + (\gamma - \alpha)(\gamma - \beta)xF(\alpha, \beta, \gamma + 1, x) + \gamma(1 - \gamma)(1 - x)F(\alpha, \beta, \gamma - 1, x) = 0. \qquad (8$$

In this manner we may proceed, getting the remaining seven.

188. *Conjugate Functions.* From the relations between contiguous functions we see that a linear relation exists between any three functions

$$F(\alpha, \beta, \gamma, x) \qquad F(\alpha', \beta', \gamma', x) \qquad F(\alpha'', \beta'', \gamma'', x)$$

whose corresponding parameters differ only by integers. Such functions are called *conjugate*.

For let p, q, r be any three integers. Consider the functions

$$F(\alpha\beta\gamma x),\qquad F(\alpha+1,\beta,\gamma,x)\cdots F(\alpha+p,\beta,\gamma,x),$$
$$F(\alpha+p,\beta+1,\gamma,x),\ F(\alpha+p,\beta+2,\gamma,x)\cdots F(\alpha+p,\beta+q,\gamma,x),$$
$$F(\alpha+p,\beta+q,\gamma+1,x), F(\alpha+p,\beta+q,\gamma+2,x)\cdots F(\alpha+p,\beta+q,\gamma+r,x).$$

We have $p+q+r+1$ functions, and any 3 consecutive ones are contiguous. There are thus $p+q+r-1$ linear relations between them. We can thus by elimination get a linear relation between any three of these functions.

189. *Derivatives.* We have

$$F'(\alpha,\beta,\gamma,x)=\sum_{n=1}^{\infty} n\,\frac{\alpha\cdot\alpha+1\cdots\alpha+n-1\cdot\beta\cdot\beta+1\cdots\beta+n-1}{1\cdot 2\cdots n\cdot\gamma\cdot\gamma+1\cdots\gamma+n-1}\,x^{n-1}$$
$$=\sum_{n=0}^{\infty}(n+1)\,\frac{\alpha\cdot\alpha+1\cdots\alpha+n\cdot\beta\cdot\beta+1\cdots\beta+n}{1\cdot 2\cdots n+1\cdot\gamma\cdot\gamma+1\cdots\gamma+n}\,x^{n}$$
$$=\frac{\alpha\beta}{\gamma}\sum_{0}^{\infty}(n+1)\,\frac{\alpha+1\cdots\alpha+n\cdot\beta+1\cdots\beta+n}{1\cdot 2\cdots n+1\cdot\gamma+1\cdots\gamma+n}\,x^{n}$$
$$=\frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1,\gamma+1,x).$$

Hence

$$F''(\alpha,\beta,\gamma,x)=\frac{\alpha\cdot\beta}{\gamma}F'(\alpha+1,\beta+1,\gamma+1,x)$$
$$=\frac{\alpha\cdot\alpha+1\cdot\beta\cdot\beta+1}{\gamma\cdot\gamma+1}F(\alpha+2,\beta+2,\gamma+2,x)$$

and so on for the higher derivatives. We see they are conjugate functions.

190. *Differential Equation for F.* Since F, F', F'' are conjugate functions, a linear relation exists between them. It is found to be

$$x(x-1)F''+\{(\alpha+\beta+1)x-\gamma\}F'+\alpha\beta F=0. \qquad (1$$

To prove the relation let us find the coefficient of x^n on the left side of 1). We set

$$P_n=\frac{\alpha\cdot\alpha+1\cdots\alpha+n-1\cdot\beta\cdot\beta+1\cdots\beta+n-1}{1\cdot 2\cdots n\cdot\gamma\cdot\gamma+1\cdots\gamma+n-1}.$$

The coefficient of x^n in x^2F'' is

$$n(n-1)P_n,$$

in $-xF''$ it is

$$-\frac{n(\alpha+n)(\beta+n)}{\gamma+n}P_n,$$

in $(\alpha+\beta+1)xF'$ it is

$$n(\alpha+\beta+1)P_n,$$

in $-\gamma F'$ it is

$$-\gamma\frac{(\alpha+n)(\beta+n)}{\gamma+n}P_n,$$

in $\alpha\beta F$ it is

$$\alpha\beta P_n.$$

Adding all these gives the coefficient of x^n in the left side of 1). We find it is 0.

191. *Expression of $F(\alpha\beta\gamma x)$ as an Integral.*

We show that for $|x|<1$,

$$B(\beta, \gamma-\beta)\cdot F(\alpha\beta\gamma x)=\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha}du \quad (1$$

where $B(p, q)$ is the Beta function of I, 692,

$$B(p, q)=\int_0^1 u^{p-1}(1-u)^{q-1}du.$$

For by the Binomial Theorem

$$(1-xu)^{-\alpha}=1+\frac{\alpha}{1}xu+\frac{\alpha\cdot\alpha+1}{1\cdot 2}x^2u^2+\frac{\alpha\cdot\alpha+1\cdot\alpha+2}{1\cdot 2\cdot 3}x^3u^3+\cdots$$

for $|xu|<1$. Hence

$$J=\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-xu)^{-\alpha}du$$

$$=\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}du+\frac{\alpha\cdot x}{1}\int_0^1 u^{\beta}(1-u)^{\gamma-\beta-1}$$

$$+\frac{\alpha\cdot\alpha+1}{1\cdot 2}\cdot x^2\int_0^1 u^{\beta+1}(1-u)^{\gamma-\beta-1}du+\cdots$$

$$=B(\beta, \gamma-\beta)+\alpha xB(\beta+1, \gamma-\beta)$$

$$+\frac{\alpha\cdot\alpha+1}{1\cdot 2}x^2B(\beta+2, \gamma-\beta)+\cdots \quad (2$$

Now from I, 692, 10)

$$B(\beta+1, \gamma-\beta)=\frac{\beta}{\gamma} B(\beta, \gamma-\beta).$$

Hence

$$B(\beta+2, \gamma-\beta)=\frac{\beta+1}{\gamma+1} B(\beta+1, \gamma-\beta)=\frac{\beta \cdot \beta+1}{\gamma \cdot \gamma+1} B(\beta, \gamma-\beta)$$

etc. Putting these values in 2) we get 1).

192. *Value of $F(\alpha, \beta, \gamma, x)$ for $x=1$.*

We saw that the F series converges absolutely for $x=1$ if $\alpha+\beta-\gamma<0$. The value of F when $x=1$ is particularly interesting. As it is now a function of α, β, γ only, we may denote it by $F(\alpha, \beta, \gamma)$. The relation between this function and the Γ function may be established, as Gauss showed, by means of 187, 8) viz.:

$$\gamma\{\gamma-1+(\alpha+\beta+1-2\gamma)x\} F(\alpha\beta\gamma x)$$
$$+(\gamma-\alpha)(\gamma-\beta)xF(\alpha, \beta, \gamma+1, x)$$
$$+\gamma(1-\gamma)(1-x)F(\alpha, \beta, \gamma-1, x)=0. \quad (1$$

Assuming that

$$\alpha+\beta-\gamma<0, \quad (2$$

we see that the first and second terms are convergent for $x=1$; but we cannot say this in general for the third, as it is necessary for this that $\alpha+\beta-(\gamma-1)<0$. We can, however, show that

$$L\lim_{x=1}(1-x)F(\alpha, \beta, \gamma-1, x)=0, \quad (3$$

supposing 2) to hold. For if $|x|<1$,

$$F(\alpha, \beta, \gamma-1, x)=a_0+a_1x+a_2x^2+\cdots \quad (4$$

Now by 100, this series also converges for $x=-1$. Thus

$$\lim_{n=\infty} a_n=0. \quad (5$$

From 4) we have

$$(1-x)F(\alpha, \beta, \gamma-1, x)=a_0+(a_1-a_0)x+(a_2-a_1)x^2+\cdots$$

Let the series on the right be denoted by $G(x)$. As $G_{n+1}(1)=a_n$, we see $G(1)$ is a convergent series, by 5), whose sum is 0. But then by 147, 6, $G(x)$ is continuous at $x=1$. Hence

$$L\lim_{x=1} G(x)=G(1)=0,$$

and this establishes 3). Thus passing to the limit $x = 1$ in 1) gives

$$\gamma(\alpha+\beta-\gamma)F(\alpha, \beta, \gamma) + (\gamma-\alpha)(\gamma-\beta)F(\alpha, \beta, \gamma+1) = 0,$$

or
$$F(\alpha, \beta, \gamma) = \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma(\gamma-\alpha-\beta)} F(\alpha, \beta, \gamma+1).$$

Replacing γ by $\gamma + 1$, this gives

$$F(\alpha, \beta, \gamma+1) = \frac{(\gamma+1-\alpha)(\gamma+1-\beta)}{(\gamma+1)(\gamma+1-\alpha-\beta)} F(\alpha, \beta, \gamma+2),$$

etc. Thus in general

$$F(\alpha, \beta, \gamma) = \frac{(\gamma-\alpha)(\gamma+1-\alpha)\cdots(\gamma+n-1-\alpha)\cdot(\gamma-\beta)(\gamma+1-\beta)\cdots(\gamma+n-1-\beta)}{\gamma(\gamma+1)\cdots(\gamma+n-1)(\gamma-\alpha-\beta)(\gamma-\alpha-\beta+1)\cdots(\gamma-\alpha-\beta+n-1)} \cdot F(\alpha, \beta, \gamma+n).$$

Gauss sets now

$$\Pi(n, x) = \frac{n!\, n^x}{(x+1)(x+2)\cdots(x+n)}.$$

Hence the above relation becomes

$$F(\alpha\beta\gamma) = \frac{\Pi(n, \gamma-1)\Pi(n, \gamma-\alpha-\beta-1)}{\Pi(n, \gamma-\alpha-1)\Pi(n, \gamma-\beta-1)} F(\alpha, \beta, \gamma+n). \quad (6$$

Now
$$\lim_{n=\infty} F(\alpha, \beta, \gamma+n) = 1. \quad (7$$

For the series

$$F(\alpha, \beta, \gamma) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} + \frac{\alpha \cdot \alpha+1 \cdot \beta \cdot \beta+1}{1 \cdot 2 \cdot \gamma \cdot \gamma+1} + \cdots \quad (8$$

converges absolutely when 2) holds. Hence

$$1 + \frac{|\alpha| \cdot |\beta|}{1 \cdot G} + \frac{|\alpha| \cdot |\alpha+1| \cdot |\beta| \cdot |\beta+1|}{1 \cdot 2 \cdot G \cdot G+1} + \cdots \quad (9$$

is convergent. Now each term in 8) is numerically $\leq$ the corresponding term in 9) for any $\gamma \geq G$. Hence 8) converges uniformly about the point $\gamma = +\infty$. We may therefore apply 146, 4. As each term of 8) has the limit 0 as $\gamma \doteq +\infty$, the relation 7) is established.

We shall show in the next chapter that

$$\lim_{n=\infty} \Pi(n, x)$$

exists for all x different from a negative integer. Gauss denotes it by $\Pi(x)$; as we shall see,

$$\Gamma(x) = \Pi(x-1) \quad , \quad \text{for } x > 0.$$

Letting $n \doteq \infty$, 6) gives

$$F(\alpha, \beta, \gamma) = \frac{\Pi(\gamma-1)\Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1)\Pi(\gamma-\beta-1)}.$$

We must of course suppose that

$$\gamma, \qquad \gamma-\alpha, \qquad \gamma-\beta, \qquad \gamma-\alpha-\beta,$$

are not negative integers or zero, as otherwise the corresponding Π or F function are not defined.

Bessel Functions

193. 1. The infinite series

$$J_n(x) = x^n \sum_{s=0}^{\infty} (-1)^s \frac{x^{2s}}{2^{n+2s} s!\,(n+s)!} \qquad n = 0, 1, 2 \cdots \qquad (1$$

converges for every x. For the ratio of two successive terms of the adjoint series is

$$\frac{|x|^2}{2^2(s+1)(n+s+1)}$$

which $\doteq 0$ as $s \doteq \infty$ for any given x.

The series 1) thus define functions of x which are everywhere continuous. They are called Bessel functions of order

$$n = 0, 1, 2 \cdots$$

In particular we have

$$J_0(x) = 1 - \frac{x^2}{2 \cdot 2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \qquad (2$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \cdots \qquad (3$$

Since 1) is a power series, we may differentiate it termwise and get

$$J_n'(x) = \sum_0^{\infty} \frac{(-1)^s(2s+n)}{2^{n+2s} s!\,(n+s)!} x^{2s+n-1}. \qquad (4$$

2. The following linear relation exists between three consecutive Bessel functions:

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \qquad n > 0. \tag{5}$$

For
$$J_{n-1} = \frac{x^{n-1}}{2^{n-1}(n-1)!} + \sum_{s=1}^{\infty} (-1)^s \frac{x^{2s+n-1}}{2^{n+2s-1} s!(n-1+s)!}. \tag{6}$$

$$J_{n+1} = -\sum_{s=1}^{\infty} (-1)^s \frac{x^{2s+n-1}}{2^{n+2s-1}(s-1)!(n+s)!}. \tag{7}$$

Hence

$$\begin{aligned}
&J_{n-1} + J_{n+1} \\
&= \frac{x^{n-1}}{2^{n-1}(n-1)!} + \sum_1^{\infty} (-1)^s \frac{x^{2s+n-1}}{2^{n+2s-1}} \left\{ \frac{1}{s!(n-1+s)!} - \frac{1}{(s-1)!(n+s)!} \right\} \\
&= \frac{x^{n-1}}{2^{n-1}(n-1)!} + n \sum_1^{\infty} (-1)^s \frac{x^{2s+n-1}}{2^{n+2s-1} s!(n+s)!} \\
&= \frac{n}{x} \sum_0^{\infty} (-1)^s \frac{x^{2s+n}}{2^{n+2s-1} s!(n+s)!} \\
&= \frac{2n}{x} J_n(x).
\end{aligned}$$

3. We show next that

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \qquad n > 0. \tag{8}$$

For subtracting 7) from 6) gives

$$\begin{aligned}
J_{n-1} - J_{n+1} &= \frac{x^{n-1}}{2^{n-1}(n-1)!} + \sum_1^{\infty} (-1)^s \frac{x^{2s+n-1}}{2^{n+2s-1}} \cdot \frac{n+2s}{s!(n+s)!} \\
&= \sum_0^{\infty} (-1)^s \frac{(n+2s) x^{2s+n-1}}{2^{n+2s-1} s!(n+s)!} \\
&= 2J_n'.
\end{aligned}$$

From 8) we get, on replacing J_{n+1} by its value as given by 5):

$$J_n'(x) = -\frac{n}{x} J_n(x) + J_{n-1}(x), \qquad n > 0. \tag{9}$$

From 5) we also get

$$J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x) \qquad n > 0. \tag{10}$$

4. The Bessel function J_n satisfies the following linear homogeneous differential equation of the 2° order:

$$J_n'' + \frac{1}{x} J_n' + \left(1 - \frac{n^2}{x^2}\right) J_n = 0. \tag{11}$$

This may be shown by direct differentiation of 1) or more simply thus: Differentiating 9) gives

$$J_n'' = \frac{n}{x^2}J_n - \frac{n}{x}J_n' + J_{n-1}'. \tag{12}$$

Equation 10) gives

$$J_{n-1}' = \frac{n-1}{x}J_{n-1} - J_n.$$

Replacing here J_{n-1} by its value as given by 9), we get

$$J_{n-1}' = \frac{n-1}{x}J_n' + \left(\frac{n(n-1)}{x^2} - 1\right)J_n.$$

Putting this in 12) gives 11).

5. $$e^{x\frac{u-u^{-1}}{2}} = \sum_{-\infty}^{\infty} u^n J_n(x) \tag{13}$$

for any x, and for $u \neq 0$.

For

$$\begin{aligned} e^{x\frac{u-u^{-1}}{2}} &= e^{\frac{1}{2}xu}e^{-\frac{1}{2}\frac{x}{u}} \\ &= \left\{1 + \frac{xu}{2} + \frac{x^2u^2}{2^2 \cdot 2!} + \frac{x^3u^3}{2^3 \cdot 3!} + \cdots\right\} \\ &\qquad \cdot \left\{1 - \frac{x}{2u} + \frac{x^2}{2^2 \cdot 2!\, u^2} - \cdots\right\}. \end{aligned}$$

Now for any x and for any $u \neq 0$, the series in the braces are absolutely convergent. Their product may therefore be written in the form

$$\begin{aligned} &\left(1 - \frac{x^2}{2^2} + \left(\frac{x}{2}\right)^4 \frac{1}{2!\,2!} - \cdots\right) \\ &+ u\left(\frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 + \frac{1}{3!\,2!}\left(\frac{x}{2}\right)^5 - \cdots\right) \\ &- u^{-1}\left(\frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 + \frac{1}{3!\,2!}\left(\frac{x}{2}\right)^5 - \cdots\right) \\ &+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &+ u^n\left(\qquad\qquad\qquad\qquad\qquad\qquad \right) \\ &(-1)^n u^{-n}\left(\qquad\qquad\qquad\qquad\qquad \right) \\ &+ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &= J_0(x) + uJ_1(x) + u^2J_2(x) + \cdots \\ &- \frac{J_1(x)}{u} + \frac{J_2(x)}{u^2} - \cdots \end{aligned}$$

194. 1. *Expression of $J_n(x)$ as an Integral.*

$$J_n(x) = \frac{x^n}{2^n\sqrt{\pi}} \cdot \frac{1}{\Gamma\left(\frac{2n+1}{2}\right)} \int_0^\pi \cos(x\cos\phi)\sin^{2n}\phi d\phi. \qquad (1$$

For

$$\cos u = \sum_0^\infty (-1)^s \frac{u^{2s}}{(2s)!}.$$

Hence

$$\cos(x\cos\phi) = \sum_0^\infty \frac{(-1)^s}{(2s)!} x^{2s}\cos^{2s}\phi$$

and thus

$$\cos(x\cos\phi)\sin^{2n}\phi = \sum_0^\infty \frac{(-1)^s}{(2s)!} x^{2s}\cos^{2s}\phi\sin^{2n}\phi.$$

As this series converges uniformly in $(0, \pi)$ for any value of x, we may integrate termwise, getting

$$\int_0^\pi \cos(x\cos\phi)\sin^{2n}\phi d\phi = \sum_0^\infty \frac{(-1)^s}{(2s)!} x^{2s} \int_0^\pi \cos^{2s}\phi\sin^{2n}\phi d\phi$$

$$= \sum_0^\infty \frac{(-1)^s}{(2s)!} x^{2s} B\left(\frac{2s+1}{2}, \frac{2n+1}{2}\right) \text{ by I, 692}$$

$$= \sum_0^\infty \frac{(-1)^s}{(2s)!} x^{2s} \frac{\Gamma\left(\frac{2s+1}{2}\right)\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(s+n+1)} \text{ by I, 692.}$$

We shall show in 225, 6, that

$$\Gamma\left(\frac{2s+1}{2}\right) = \frac{1\cdot 3\cdot 5\cdot \cdots 2s-1}{2^s}\sqrt{\pi}.$$

Thus the last series above

$$= \sqrt{\pi}\cdot\Gamma\left(\frac{2n+1}{2}\right)\sum_0^\infty \frac{(-1)^s}{(2s)!}\frac{1\cdot 3\cdot 5\cdot \cdots (2s-1)}{2^s(n+s)!}x^{2s}.$$

Thus

$$\frac{x^n}{2^n\sqrt{\pi}\,\Gamma\left(\frac{2n+1}{2}\right)}\int_0^\pi \cos(x\cos\phi)\sin^{2n}\phi d\phi$$

$$= \sum_0^\infty \frac{(-1)^s x^{2s+n}}{2^{2s+n} s!(n+s)!} = J_n(s).$$

CHAPTER VII

INFINITE PRODUCTS

195. 1. Let $\{a_{\iota_1 \cdots \iota_s}\}$ be an infinite sequence of numbers, the indices $\iota = (\iota_1 \cdots \iota_s)$ ranging over a lattice system $\mathfrak{L}$ in s-way space. The symbol

$$P = \prod_{\mathfrak{L}} a_{\iota_1 \cdots \iota_s} = \prod_{\mathfrak{L}} a_\iota \tag{1}$$

is called an *infinite product*. The numbers a_ι are its *factors*. Let P_μ denote the product of all the factors in the rectangular cell R_μ. If

$$\lim_{\mu=\infty} P_\mu \tag{2}$$

is finite or definitely infinite, we call it the *value of P*. It is customary to represent a product and its value by the same letter when no ambiguity will arise.

When the limit 2) is finite and $\neq 0$ or when one of the factors $= 0$, we say P is *convergent*, otherwise P is *divergent*.

We shall denote by $\overline{P}_\mu$ the product obtained by setting all the factors $a_\iota = 1$, whose indices ι lie in the cell R_μ. We call this the *co-product of P_μ*.

The products most often occurring in practice are of the type

$$P = a_1 \cdot a_2 \cdot a_3 \cdot \cdots = \prod_1^\infty a_n. \tag{3}$$

The factor P_μ is here replaced by

$$P_m = a_1 \cdot a_2 \cdot \cdots a_m$$

and the co-product $\overline{P}_\mu$ by

$$\overline{P}_m = a_{m+1} \cdot a_{m+2} \cdot a_{m+3} \cdot \cdots$$

Another type is

$$P = \prod_{n=-\infty}^{+\infty} a_n. \tag{4}$$

The products 3), 4) are *simple*, the product 1) is *s-tuple*. The products 3), 4) may be called *one-way* and *two-way simple* products when necessary to distinguish them.

2. Let

$$P = \tfrac{1}{1} \cdot \tfrac{1}{2} \cdot \tfrac{2}{3} \cdot \tfrac{3}{4} \cdot \tfrac{4}{5} \cdots$$

Obviously the product $P = 0$, as

$$P_n = \frac{1}{n} \doteq 0.$$

Hence $P = 0$, although no factor is zero. Such products are called *zero products*. Now we saw in I, 77 that the product of a finite number of factors cannot vanish unless one of its factors vanishes. For this reason zero products hold an exceptional position and will not be considered in this work. We therefore have classed them among the divergent products. In the following theorems relative to convergence, we shall suppose, for simplicity, that *there are no zero factors*.

196. 1. *For $P = \Pi a_{\iota_1 \dots \iota_s}$ to converge it is necessary that each $\overline{P}_\mu$ is convergent. If one of these $\overline{P}_\mu$ converges, P is convergent and*

$$P = P_\mu \cdot \overline{P}_\mu.$$

The proof is obvious.

2. *If the simple product $P = a_1 \cdot a_2 \cdot a_3 \cdots$ is convergent, its factors finally remain positive.*

For, when P is convergent, $|P_n| >$ some positive number, for $n >$ some m. If now the factors after a_m were not all positive, P_n and P_ν could have opposite signs $\nu > n$, however large n is taken. Thus P_n has no limit.

197. 1. To investigate the convergence or divergence of an infinite product $P = \Pi a_{\iota_1 \dots \iota_s}$ when $a_\iota > 0$, it is often convenient to consider the series

$$L = \sum_{\mathfrak{L}} \log a_{\iota_1 \dots \iota_s} = \Sigma\, l_{\iota_1 \dots \iota_s},$$

called the *associate logarithmic series*. Its importance in this connection is due to the following theorem :

The infinite product P with positive factors and the infinite series L converge or diverge simultaneously. When convergent, $P = e^L$, $L = \log P$.

For

$$\log P_\mu = L_\mu, \tag{1}$$

$$P_\mu = e^{L_\mu}. \tag{2}$$

If P is convergent, P_μ converges to a finite limit $\neq 0$. Hence L_μ is convergent by 1). If L_μ is convergent, P_μ converges to a finite limit $\neq 0$ by 2).

2. *Example 1.*

$$P = \Pi\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}} = \Pi a_n \qquad n = 1, 2, \cdots$$

is convergent for every x.

For, however large $|x|$ is taken and then fixed, we can take m so large that

$$1+\frac{x}{n} > 0 \qquad n > m.$$

Instead of P we may therefore consider $\overline{P}_m$.

Then
$$\overline{L}_m = \sum_{m+1}^{\infty}\left\{-\frac{x}{n}+\log\left(1+\frac{x}{n}\right)\right\}. \tag{3}$$

But by I, 413

$$\log\left(1+\frac{x}{n}\right) = \frac{x}{n} + M_n\frac{x^2}{n^2}, \quad |M_n| < M.$$

Hence
$$\overline{L}_m = \sum_{m+1} M_n x^2 \cdot \frac{1}{n^2}$$

which is convergent.

The product P occurs in the expression of $\sin x$ as an infinite product.

Let us now consider the product

$$Q = \Pi\left(1+\frac{x}{n}\right)e^{-\frac{x}{n}} \qquad n = \pm 1, \pm 2, \cdots$$

The associate logarithmic series L is a two-way simple series. We may break it into two parts L', L'', the first extended over positive n, the second over negative n. We may now reason on these as we did on the series 3), and conclude that Q converges for every x.

3. *Example 2.*

$$G = \frac{1}{x}\prod_1^\infty \frac{\left(1+\frac{1}{n}\right)^x}{1+\frac{x}{n}}$$

is convergent for any x different from

$$0, -1, -2, -3, \cdots$$

For let p be taken so large that $|x| < p$. We show that the co-product

$$\overline{G}_p = \prod_{p+1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^x}{1+\frac{x}{n}}$$

converges for this x. The corresponding logarithmic series is

$$L = \sum_{p+1}^{\infty} \left\{ x \log\left(1+\frac{1}{n}\right) - \log\left(1+\frac{x}{n}\right) \right\}$$

$$= \sum_{p+1}^{\infty} \left\{ \frac{x}{n} - \log\left(1+\frac{x}{n}\right) \right\} - x \sum_{p+1}^{\infty} \left\{ \frac{1}{n} - \log\left(1+\frac{1}{n}\right) \right\}.$$

As each of the series on the right converges, so does L. Hence G converges for this value of x.

198. 1. When the associate logarithmic series

$$L = \Sigma \log a_{\iota_1 \cdots \iota_s} \quad , \quad a_\iota > 0$$

is convergent,

$$\lim_{|\iota|=\infty} \log a_{\iota_1 \cdots \iota_s} = 0, \qquad \text{by 121, 1,}$$

and therefore

$$\lim_{|\iota|=\infty} a_{\iota_1 \cdots \iota_s} = 1.$$

For this reason it is often convenient to write the factors $a_{\iota_1 \cdots \iota_s}$ of an infinite product P in the form $1 + b_{\iota_1 \cdots \iota_s}$. When P is written in the form

$$P = \Pi(1 + b_{\iota_1 \cdots \iota_s}),$$

we shall say it is written in its *normal form*. The series

$$\Sigma b_{\iota_1 \cdots \iota_s} = \Sigma b_\iota$$

we shall call the *associate normal series of* P.

2. *The infinite product*

$$P = \Pi(1 + a_{\iota_1 \cdots \iota_s}) \quad , \quad a_\iota > -1,$$

and its associate normal series

$$A = \Sigma a_{\iota_1 \cdots \iota_s},$$

converge or diverge simultaneously.

For P and
$$L = \Sigma \log (1 + a_\iota)$$
converge or diverge simultaneously by 197. But A and L converge or diverge simultaneously by 123, 4.

3. *If the simple product* $P = a_1 \cdot a_2 \cdot a_3 \cdots$ *is convergent,* $a_n \doteq 1$.

For by 196, 2 the factors a_n finally become > 0, say for $n \geq m$. Hence by 197, 1 the series
$$\sum_{n=m}^{\infty} \log a_n \qquad a_n > 0$$
is convergent. Hence $\log a_n \doteq 0$. $\therefore a_n \doteq 1$.

199. *Let* $R_{\lambda_1} < R_{\lambda_2} < \cdots \qquad |\lambda| \doteq \infty$ *be a sequence of rectangular cells. Then if* P *is convergent,*
$$P = P_{\lambda_1} + \sum_1^{\infty} (P_{\lambda_{n+1}} - P_{\lambda_n}).$$

For P is a telescopic series and
$$P_{\lambda_{\nu+1}} = P_{\lambda_1} + \sum_1^{\nu} (P_{\lambda_{n+1}} - P_{\lambda_n}).$$

200. 1. Let $P = \Pi(1 + a_{\iota_1 \dots \iota_s})$.

We call
$$\mathfrak{P} = \Pi(1 + \alpha_{\iota_1 \dots \iota_s}) \quad , \quad \alpha_\iota = |a_\iota|$$
the *adjoint of* P, and write
$$\mathfrak{P} = \text{Adj } P.$$

2. *P converges, if its adjoint is convergent.* We show that
$$\epsilon > 0, \ \lambda, \qquad |P_\mu - P_\nu| < \epsilon \qquad \mu, \nu > \lambda.$$

Since $\mathfrak{P}$ is convergent,
$$\mathfrak{P}_{\lambda_1} + \sum_1^{\infty} (\mathfrak{P}_{\lambda_n} - \mathfrak{P}_{\lambda_{n-1}})$$
is also convergent by 199. Hence
$$0 \leq \mathfrak{P}_\nu - \mathfrak{P}_\mu < \epsilon \qquad \lambda < \mu < \nu.$$

But $P_\nu - P_\mu$ is an integral rational function of the a's with *positive* coefficients. Hence
$$|P_\nu - P_\mu| \leq \mathfrak{P}_\nu - \mathfrak{P}_\mu. \tag{1}$$

3. When the adjoint of P converges, we say P is *absolutely* convergent.

The reader will note that absolute convergence of infinite products is defined quite differently from that of infinite series. At first sight one would incline to define the adjoint of

$$P = \Pi a_{\iota_1 \cdots \iota_s}$$

to be

$$\mathfrak{P} = \Pi \mid a_{\iota_1 \cdots \iota_s} \mid.$$

With this definition the fundamental theorem 2 would be false. For let

$$P = \Pi(-1)^n;$$

its adjoint would be, by this definition,

$$\mathfrak{P} = 1 \cdot 1 \cdot 1 \cdot \cdots.$$

Now $\mathfrak{P}_n = 1$. $\therefore \mathfrak{P}$ is convergent. On the other hand, $P_n = (-1)^n$ and this has no limit, as $n \doteq \infty$. Hence P is divergent.

4. *In order that* $P = \Pi(1 + a_{\iota_1 \cdots \iota_s})$ *converge absolutely, it is necessary and sufficient that*

$$\Sigma\, a_{\iota_1 \cdots \iota_s}$$

converges absolutely.

Follows at once from 198, 2.

Example.

$$\prod_1^\infty \left(1 - \frac{x^2}{n^2}\right)$$

converges absolutely for every x.

For

$$\sum \frac{x^2}{n^2} = x^2 \sum \frac{1}{n^2}$$

is convergent.

201. 1. Making use of the reasoning similar to that employed in 124, we see that with each multiple product

$$P = \Pi a_{\iota_1 \cdots \iota_s}$$

are associated an infinite number of simple products

$$Q = \Pi a_n,$$

and conversely.

We have now the following theorems:

2. *If an associate simple product Q is convergent, so is P, and $P = Q$.*

For since Q is convergent, we may assume that all the a's are > 0 by 196, 2. Then

$$Q = e^{\Sigma \log a_n} \quad \text{by 197, 1,}$$

$$= e^{\Sigma \log a_{\iota_1 \cdots \iota_s}} \quad \text{by 124, 3,}$$

$$= P \quad \text{by 197, 1.}$$

3. *If the associate simple product Q is absolutely convergent, so is P.*

For let

$$P = \Pi(1 + a_{\iota_1 \cdots \iota_s}),$$

$$Q = \Pi(1 + a_n).$$

Since Q is absolutely convergent,

$$\Pi(1 + \alpha_n) \quad , \quad \alpha_n = |a_n|$$

is convergent. Hence $\Pi(1 + \alpha_{\iota_1 \cdots \iota_s})$ is convergent by 2.

4. *Let $P = \Pi(1 + a_{\iota_1 \cdots \iota_m})$ be absolutely convergent. Then each associate simple product $Q = \Pi(1 + a_n)$ is absolutely convergent and $P = Q$.*

For since P is absolutely convergent,

$$\Sigma \alpha_{\iota_1 \cdots \iota_s}$$

converges by 200, 4. But then by 124, 5

$$\Sigma \alpha_n$$

is convergent. Hence Q is absolutely convergent.

5. *If $P = \Pi a_{\iota_1 \cdots \iota_s}$ is absolutely convergent, the factors $a_{\iota_1 \cdots \iota_s} > 0$ if they lie outside of some rectangular cell R_μ.*

For since P converges absolutely, any one of its simple associate products $Q = \Pi a_n$ converges. But then $a_n > 0$ for $n > m$, by 198, 3. Thus $a_{\iota_1 \cdots \iota} > 0$ if ι lies outside of some R_μ.

6. From 5 it follows that in demonstrations regarding absolutely convergent products, we may take all the factors > 0, without loss of generality.

For
$$P = P_\mu \cdot \overline{P}_\mu ;$$
and all the factors of $\overline{P}_\mu$ are > 0, if μ is sufficiently large. This we shall feel at liberty to do, without further remark.

7.
$$A = \Pi(1 + a_{\iota_1 \cdots \iota_s}) \qquad a_\iota > 0$$
and
$$L = \Sigma \log(1 + a_{\iota_1 \cdots \iota_s})$$
converge or diverge simultaneously.

For if A is convergent,
$$\Sigma a_{\iota_1 \cdots \iota_s}$$
is convergent by 200, 4. But then L is convergent by 123, 4. The converse follows similarly.

202. 1. As in 124, 10 we may form from a given m-tuple product
$$A = \Pi a_{\iota_1 \cdots \iota_m}$$
as infinite number of conjugate n-tuple products
$$B = \Pi b_{j_1 \cdots j_n}$$
where $a_\iota = b_j$ if ι and j are corresponding lattice points in the two systems.

We have now:

2. *If A is absolutely convergent, so is B, and $A = B$.*

For by 201, 6, without loss of generality, we may take all the factors > 0.

Then
$$\begin{aligned} A &= e^{\Sigma \log a_{\iota_1 \cdots \iota_m}} \\ &= e^{\Sigma \log a_\mu} \\ &= e^{\Sigma \log a_{j_1 \cdots j_n}} \\ &= B. \end{aligned}$$

3. *Let*
$$A = \Pi a_{\iota_1 \cdots \iota_m}$$
be an absolutely convergent m-tuple product.

Let
$$B = \Pi b_{j_1 \cdots j_n}$$
be any p-tuple product formed of a part of or all the factors of A. Then B is absolutely convergent.

For $\qquad \Sigma \log \alpha_\iota$ is convergent.

Hence $\qquad \Sigma \log \beta_j$ is.

Arithmetical Operations

203. *Absolutely convergent products are commutative, and conversely.*

For let
$$A = \Pi a_{\iota_1 \cdots \iota_m}$$
be absolutely convergent. Then its associate simple product
$$\mathfrak{A} = \Pi a_n$$
is absolutely convergent and $A = \mathfrak{A}$, by 201, 4. Let us now rearrange the factors of A, getting the product B. To it corresponds a simple associate series $\mathfrak{B}$ and $B = \mathfrak{B}$. But $\mathfrak{A} = \mathfrak{B}$ since $\mathfrak{A}$ is absolutely convergent. Hence $A = B$.

Conversely, let A be commutative. Then all the factors $a_{\iota_1 \cdots \iota_m}$ finally become > 0. For if not, let
$$R_1 < R_2 < \cdots \doteq \infty \tag{1}$$
be a sequence of rectangular cells such that any point of $\mathfrak{R}_m$ lies in some cell. We may arrange the factors a_ι such that the partial products corresponding to 1),
$$A_1 \quad , \quad A_2 \quad , \quad A_3 \cdots$$
have opposite signs alternately. Then A is not convergent, which is a contradiction. We may therefore assume all the a's > 0. Then
$$A = e^{\Sigma \log a_{\iota \cdots \iota_m}}$$
remains unaltered however the factors on the left are rearranged. Hence
$$\Sigma \log a_{\iota_1 \cdots \iota_m}$$
is commutative and therefore absolutely convergent by 124, 8. Hence the associate simple series
$$\mathfrak{A} = \Sigma \log a_n = \Sigma \log (1 + b_n)$$
is absolutely convergent by 124, 5. Hence
$$\Sigma \beta_n$$
is convergent and therefore A is absolutely convergent.

204. 1. *Let*

$$A = \Pi a_{\iota_1 \cdots \iota_s}$$

be absolutely convergent. Then the s-tuple iterated product

$$B = \prod_{\iota_1'} \prod_{\iota_2'} \cdots \prod_{\iota_s'} a_{\iota_1 \cdots \iota_s}$$

is absolutely convergent and $A = B$ where $\iota_1' \cdots \iota_s'$ is a permutation of $\iota_1, \iota_2 \cdots \iota_s$.

For by 202, 3 all the products of the type

$$\prod_{\iota_{s-1}\iota_s} a_{\iota_1 \cdots \iota_s} \qquad \prod_{\iota_s} a_{\iota_1 \cdots \iota_s}$$

are absolutely convergent, and by I, 324

$$\prod_{\iota_{s-1}\iota_s} = \prod_{\iota_{s-1}} \prod_{\iota_s}.$$

Similarly the products of the type

$$\prod_{\iota_{s-1}\iota_{s-2}\iota_{s-3}}$$

are absolutely convergent and hence

$$\prod_{\iota_{s-1}\iota_{s-2}\iota_s} = \prod_{\iota_{s-2}} \prod_{\iota_{s-1}} \prod_{\iota_s}.$$

In this way we continue till we reach A and B.

2. We may obviously generalize 1 as follows:

Let

$$A = \Pi a_{\iota_1 \cdots \iota_s}$$

be absolutely convergent. Let us establish a 1 to 1 correspondence between the lattice system $\mathfrak{L}$ over which $\iota = (\iota_1 \cdots \iota_s)$ ranges, and the lattice system $\mathfrak{M}$ over which

$$j = (j_{11} j_{12} \cdots j_{21} j_{22} \cdots j_{r1} j_{r2} \cdots j_{rp})$$

ranges. Then the p-tuple iterated product

$$B = \prod_1 \cdot \prod_2 \cdot \cdots \prod_r a_{j_{11} j_{12} \cdots j_{rp}}$$

is absolutely convergent, and

$$A = B.$$

3. An important special case of 2 is the following:

Let
$$A = \Pi a_n \quad , \quad n = 1, 2, \cdots$$
converge absolutely. Let us throw the a_n *into the rectangular array*
$$\begin{array}{ll} a_{11} \;, & a_{12} \cdots \\ a_{21} \;, & a_{22} \cdots \\ \cdot \quad \cdot & \cdot \quad \cdot \quad \cdot \\ a_{r1} \;, & a_{r2} \cdots \end{array}$$
Then
$$B_1 = \Pi a_{1n} \quad , \quad B_2 = \Pi a_{2n} \cdots$$
converge absolutely, and
$$A = B_1 B_2 \cdots B_r .$$

4. *The convergent infinite product*
$$P = (1 + a_1)(1 + a_2) \cdots$$
is associative.

For let
$$m_1 < m_2 < \cdots \doteq \infty.$$
Let
$$1 + b_1 = (1 + a_1) \cdots (1 + a_{m_1})$$
$$1 + b_2 = (1 + a_{m_1+1}) \cdots (1 + a_{m_2})$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

We have to show that
$$Q = (1 + b_1)(1 + b_2) \cdots$$
is convergent and $P = Q$.

This, however, is obvious. For
$$Q_n = (1 + b_1) \cdots (1 + b_n) = (1 + a_1) \cdots (1 + a_\nu)$$
$$= P_\nu \qquad \nu = m_1 + \cdots + m_n.$$

But when $n \doteq \infty$ so does ν.

Hence
$$\lim Q_n = \lim P_n .$$

Remark. We note that $m_{m+1} - m_m$ may $\doteq \infty$ with n.

205. *Let* $A = \Pi a_{\iota_1 \cdots \iota_s}$, $B = \Pi b_{\iota_1 \cdots \iota_s}$

be convergent. Then

$$C = \Pi a_\iota \cdot b_\iota \quad , \quad D = \Pi \frac{a_\iota}{b_\iota}$$

are convergent and

$$C = A \cdot B \quad , \quad D = \frac{A}{B}.$$

Moreover if A, B are absolutely convergent, so are C, D.

Let us prove the theorem regarding C; the rest follows similarly. We have

$$C_\mu = A_\mu \cdot B_\mu.$$

Now by hypothesis $A_\mu \doteq A$, $B_\mu \doteq B$ as $\mu \doteq \infty$.
Hence

$$C_\mu \doteq A \cdot B.$$

To show that C is absolutely convergent when A, B are, let us write $a_\iota = 1 + \mathfrak{a}_\iota$, $b_\iota = 1 + \mathfrak{b}_\iota$ and set $|\mathfrak{a}_\iota| = \alpha_\iota$, $|\mathfrak{b}_\iota| = \beta_\iota$.
Since A, B converge absolutely,

$$\Sigma \log (1 + \alpha_\iota) \quad , \quad \Sigma \log (1 + \beta_\iota)$$

are convergent. Hence

$$\Sigma \{\log (1 + \alpha_\iota) + \log (1 + \beta_\iota)\} = \Sigma \log (1 + \alpha_\iota)(1 + \beta_\iota)$$

is absolutely convergent. Hence C is absolutely convergent by 201, 7.

206. *Example.* The following infinite products occur in the theory of elliptic functions:

$$\begin{aligned} Q_1 &= \Pi (1 + q^{2n}) \\ Q_2 &= \Pi (1 + q^{2n-1}) \qquad n = 1, 2, \cdots \\ Q_3 &= \Pi (1 - q^{2n-1}). \end{aligned}$$

They are absolutely convergent for all $|q| < 1$.
For the series

$$\Sigma |q^{2n}| \quad , \quad \Sigma |q^{2n-1}|$$

are convergent. We apply now 200, 4.

As an exercise let us prove the important relation

$$P = Q_1 Q_2 Q_3 = 1.$$

For by 205,
$$P = \Pi(1+q^{2n})(1+q^{2n-1})(1-q^{2n-1})$$
$$= \Pi(1+q^{2n})(1-q^{4n-2})$$

Now all integers of the type $2n$, are of the type $4n-2$ or $4n$. Hence by 204, 3,
$$\Pi(1-q^{2n}) = \Pi(1-q^{4n})\,\Pi(1-q^{4n-2}),$$
or
$$\Pi(1-q^{4n-2}) = \frac{\Pi(1-q^{2n})}{\Pi(1-q^{4n})}.$$

Thus
$$P = \Pi\frac{(1+q^{2n})(1-q^{2n})}{1-q^{4n}} = \Pi\frac{1-q^{4n}}{1-q^{4n}}$$
$$= 1.$$

Uniform Convergence

207. *In the limited or unlimited domain* $\mathfrak{A}$, *let*
$$L = \Sigma \log f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m) \quad , \quad f_\iota > 0$$
be uniformly convergent and limited. Then
$$F = \Pi f_{\iota_1 \cdots \iota_s}$$
is uniformly convergent in $\mathfrak{A}$.

For
$$F_\lambda = e^{L_\lambda}.$$

Now $L_\lambda \doteq L$ uniformly. Hence by 144, 1, F is uniformly convergent.

208. *If the adjoint of*
$$F = \Pi(1 + f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m))$$
is uniformly convergent in $\mathfrak{A}$ (*finite or infinite*), F *is uniformly convergent.*

For if the adjoint product,
$$\mathfrak{P} = \Pi(1 + \phi_{\iota_1 \cdots \iota_s}),$$
is uniformly convergent, we have
$$|\mathfrak{P}_\mu - \mathfrak{P}_\nu| < \epsilon \qquad \mu, \nu > \lambda$$
for any x in $\mathfrak{A}$.

But as already noticed in 200, 2, 1)

$$|P_\mu - P_\nu| \leq |\mathfrak{P}_\mu - \mathfrak{P}_\nu|.$$

Hence F is uniformly convergent.

209. *The product*

$$F = \Pi(1 + f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m))$$

is uniformly convergent in the limited or unlimited domain $\mathfrak{A}$, *if*

$$\Phi = \Sigma\phi_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m) \quad , \quad \phi_\iota = |f_\iota|$$

is limited and uniformly convergent in $\mathfrak{A}$.

For by 138, 2 the series

$$L = \Sigma \log(1 + \phi_\iota)$$

is uniformly convergent and limited in $\mathfrak{A}$. Then by 207, the adjoint of F is uniformly convergent, and hence by 208, F is.

210. *Let*

$$F(x_1 \cdots x_m) = \Pi f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)$$

be uniformly convergent at $x = a$. *If each* f_ι *is continuous at* a, F *is also continuous at* a.

This is a corollary of 147, 1.

211. 1. *Let* $G = \Sigma|f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)|$ *converge in the limited complete domain* $\mathfrak{A}$ *having* a *as a limiting point. Let* G *and each* f_ι *be continuous at* a. *Then*

$$F(x_1 \cdots x_m) = \Pi(1 + f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m))$$

is continuous at a.

For by 149, 4, G is uniformly convergent. Then by 209, F is uniformly convergent, and therefore by 210, F is continuous.

2. *Let* $G = \Sigma|f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)|$ *converge in the limited complete domain* $\mathfrak{A}$, *having* $x = a$ *as limiting point. Let*

$$\lim_{x=a} f_\iota = a_\iota \quad , \quad \lim_{x=a} G = \Sigma a_\iota.$$

Then

$$\lim_{x=a} \Pi(1 + f_{\iota_1 \cdots \iota_s}(x_1 \cdots x_m)) = \Pi(1 + a_{\iota_1 \cdots \iota_s}). \qquad (1$$

For by 149, 5, G is uniformly convergent at $x=a$. It is also limited near $x=a$. Thus by 209,

$$\Pi(1+f_\iota)$$

is uniformly convergent at a. To establish 1) we need now only to apply 146, 1.

212. 1. Let
$$F=\Pi f_{\iota_1 \cdots \iota_s}(x) \quad , \quad f_\iota > 0 \tag{1}$$
converge in $\mathfrak{A}=(a, a+\delta)$. Then
$$\log F = L = \Sigma \log f_\iota. \tag{2}$$

If we can differentiate this series termwise in $\mathfrak{A}$ we have
$$\frac{d}{dx}\log F = \sum \frac{f'_\iota}{f_\iota}. \tag{3}$$

Thus to each infinite product 1) of this kind corresponds an infinite series 3). Conditions for termwise differentiation of the series 2) are given in 153, 155, 156. Other conditions will be given in Chapter XVI.

2. *Example.* Let us consider the infinite product
$$\theta(x) = 2\,q^{\frac{1}{4}} Q \sin \pi x \prod_1^\infty (1 - 2\,q^{2n} \cos 2\,\pi x + q^{4n}) \tag{1}$$
which occurs in the elliptic functions.

Let us set
$$1 - u_n = 1 - 2\,q^{2n} \cos 2\,\pi x + q^{4n}.$$
Then
$$|\,u_n\,| \le 2\,|\,q\,|^{2n} + |\,q\,|^{4n}.$$

Thus if $|\,q\,| < 1$, the product 1) is absolutely convergent for any x. It is uniformly convergent for any x and for $|\,q\,| \le r < 1$.

If it is permissible to differentiate termwise the series obtained by taking the logarithm of both sides of 1), we get
$$\frac{\theta'(x)}{\theta(x)} = \pi \cot \pi x + 4\,\pi \sin 2\,\pi x \sum_1^\infty \frac{q^{2n}}{1 - 2\,q^{2n} \cos 2\,\pi x + q^{4n}}. \tag{2}$$

If we denote the terms under the Σ sign in 2) by v_n we have
$$|\,v_n\,| \le \frac{|\,q\,|^{2n}}{|1 - 2\,q^{2n} + q^{4n}\,|} = a_n.$$

Now the series Σa_n converges if $|q|<1$. For setting $b_n = |q^{2n}|$, the series Σb_n is convergent in this case. Moreover,

$$\lim_{n=\infty} \frac{a_n}{b_n} = 1.$$

Thus we may differentiate termwise.

The Circular Functions

213. 1. Sin x *and* cos x *as Infinite Products.*

From the addition theorem

$$\sin(mx + x) = \sin(m+1)x = \sin mx \cos x + \cos mx \sin x$$

$m = 1, 2, 3 \cdots$ we see that for an odd n

$$\sin nx = a_0 \sin^n x + a_1 \sin^{n-1} x + \cdots + a_{n-1} \sin x$$

where the coefficients a are integers. If we set $t = \sin x$, we get

$$\sin nx = F_n(t) = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t. \qquad (1$$

Now F_n being a polynomial of degree n, it has n roots. They are

$$0, \quad \pm \sin\frac{\pi}{n}, \quad \pm \sin\frac{2\pi}{n}, \quad \cdots \pm \sin\frac{n-1}{2}\frac{\pi}{n},$$

corresponding to the values of x which make $\sin nx = 0$. Thus

$$F_n(t) = a_0 t\left(t - \sin\frac{\pi}{n}\right)\left(t + \sin\frac{\pi}{n}\right)\cdots$$

$$= a_0 t\left(t^2 - \sin^2\frac{\pi}{n}\right)\cdots\left(t^2 - \sin^2\frac{n-1}{n}\cdot\frac{\pi}{2}\right). \qquad (2$$

Dividing through by

$$\sin^2\frac{\pi}{n}\sin^2\frac{2\pi}{n}\cdots\sin^2\frac{n-1}{n}\frac{\pi}{2}$$

and denoting the new constant factor by α, 1), 2) give

$$\sin nx = \alpha \sin x\left[1 - \frac{\sin^2 x}{\sin^2\frac{\pi}{n}}\right]\cdots\left[1 - \frac{\sin^2 x}{\sin^2\frac{n-1}{n}\frac{\pi}{2}}\right]. \qquad (3$$

To find α we observe that this equation gives

$$\frac{\sin nx}{\sin x} = \alpha \left[1 - \frac{\sin^2 x}{\sin^2 \frac{\pi}{n}}\right] \cdots$$

Letting $x \doteq 0$ we now get $\alpha = n$. Thus putting this value of α in 3), and replacing x by $\frac{x}{n}$, we have finally

$$\sin x = n \sin \frac{x}{n} P(x, n) \tag{4}$$

where

$$P(x, n) = \Pi \left[1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}}\right] \qquad r = 1, 2, \cdots \frac{n-1}{2}.$$

We note now that as $n \doteq \infty$,

$$n \sin \frac{x}{n} = x \frac{\sin \frac{x}{n}}{\frac{x}{n}} \doteq x.$$

Similarly

$$\frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}} \doteq \frac{x^2}{r^2 \pi^2}.$$

It seems likely therefore that if we pass to the limit $n = \infty$ in 4), we shall get

$$\sin x = x P(x) \tag{5}$$

where

$$P(x) = \prod_1^\infty \left(1 - \frac{x^2}{r^2 \pi^2}\right).$$

The correctness of 5) is easily shown.

Let us set

$$L(x, n) = \log P(x, n) = \Sigma \log \left[1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}}\right]. \tag{6}$$

$$L(x) = \log P(x) = \Sigma \log \left(1 - \frac{x^2}{r^2 \pi^2}\right).$$

We observe that

$$\lim_{n=\infty} P(x, n) = \lim_{n=\infty} e^{L(x,n)} = e^{L(x)} = P(x)$$

provided

$$\lim_{n=\infty} L(x, n) = L(x). \tag{7}$$

We have thus only to prove 7). Let us denote the sum of the first m terms in 6) by $L_m(x, n)$ and the sum of the remaining by $\overline{L}_m(x, n)$. Then

$$|L(x,n)-L(x)| \le |L_m(x,n)-L_m(x)| + |\overline{L}_m(x,n)| + |\overline{L}_m(x)|. \tag{8}$$

Since for $0 < x < \frac{\pi}{2}$,

$$\frac{x}{2} < \sin x < x,$$

we have

$$\frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}} > \frac{\frac{x^2}{4n^2}}{\frac{\pi^2 r^2}{n^2}} = \frac{x^2}{4\pi^2 r^2},$$

and hence for an m_1 so large that $\frac{|x|}{m_1} < 1$, we have,

$$-\log\left[1 - \frac{\sin^2 \frac{x}{n}}{\sin^2 \frac{r\pi}{n}}\right] < -\log\left(1 - \frac{x^2}{4\pi^2 r^2}\right), \qquad r > m.$$

But the series

$$\sum_{r=m_1}^{\infty} \log\left(1 - \frac{x^2}{4\pi^2 r^2}\right)$$

is convergent. Hence for a sufficiently large m

$$|\overline{L}_m(x, n)| < \frac{\epsilon}{3} \quad , \quad |\overline{L}_m(x)| < \frac{\epsilon}{3}.$$

Now giving m this fixed value, obviously for all $n >$ some ν the first term on the right of 8) is $< \epsilon/3$, and thus 7) holds.

2. In algebra we learn that every polynomial

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

can be written as a product

$$a_n(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n),$$

where α_1, $\alpha_2 \cdots$ are its roots. Now

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \tag{9}$$

is the limit of a polynomial, viz. the first n terms of 9). It is natural to ask, Can we not express $\sin x$ as the limit of a product which vanishes at the zeros of $\sin x$? That this can be done we have just shown in 1.

3. If we set $x = \pi/2$ in 5), it gives,

$$1 = \frac{\pi}{2}\Pi\left(1 - \frac{1}{4r^2}\right) = \frac{\pi}{2}\Pi\frac{(2r-1)(2r+1)}{2r\cdot 2r}.$$

Hence
$$\frac{\pi}{2} = \Pi\frac{2r\cdot 2r}{(2r-1)(2r+1)} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdots}, \tag{10}$$

a formula due to *Wallis*.

4. From 5) we can get another expression for $\sin x$, viz.:

$$\sin x = x\Pi\left(1 - \frac{x}{r\pi}\right)e^{\frac{x}{r\pi}} \qquad r = \pm 1, \pm 2, \cdots \tag{11}$$

For the right side is convergent by 197, 2. If now we group the factors in pairs, we have

$$\left(1 - \frac{x}{r\pi}\right)e^{\frac{x}{r\pi}}\left(1 + \frac{x}{r\pi}\right)e^{-\frac{x}{r\pi}} = 1 - \frac{x^2}{r^2\pi^2}.$$

This shows that the products in 5) and 11) are equal.

5. From 5) or 11) we have

$$\sin x = \lim_{n=\infty} P_n(x) = \lim x \prod_{s=-n}^{n}{}' \frac{x + s\pi}{s\pi} \tag{12}$$

where the dash indicates that $s = 0$ is excluded.

214. *We now show that*

$$\cos x = \prod_1^\infty \left(1 - \frac{4x^2}{(2n-1)^2\pi^2}\right). \tag{1}$$

To this end we use the relation

$$\sin 2x = 2 \sin x \cos x.$$

Hence

$$\cos x = \frac{1}{2} \cdot \frac{2x}{x} \frac{\Pi\left(1 - \frac{4x^2}{n^2\pi^2}\right)}{\Pi\left(1 - \frac{x^2}{n^2\pi^2}\right)} = \frac{\Pi\left(1 - \frac{4x^2}{(2m)^2\pi^2}\right)\left(1 - \frac{4x^2}{(2m-1)^2\pi^2}\right)}{\Pi\left(1 - \frac{x^2}{n^2\pi^2}\right)}$$

$$= \frac{\Pi\left(1 - \frac{x^2}{m^2\pi^2}\right)}{\Pi\left(1 - \frac{x^2}{n^2\pi^2}\right)} \cdot \Pi\left(1 - \frac{4x^2}{(2m-1)^2\pi^2}\right)$$

from which 1) is immediate.

From 1) we have, as in 213, 4,

$$\cos x = \Pi\left(1 - \frac{2x}{(2n-1)\pi}\right)e^{\frac{2x}{(2n-1)\pi}} \qquad n = 0, \pm 1, \pm 2, \cdots \tag{2}$$

215. From the expression of $\sin x$, $\cos x$ as infinite products, their *periodicity* is readily shown. Thus from 213, 12)

$$\sin x = \lim_{n=\infty} P_n(x).$$

But

$$\frac{P_n(x+\pi)}{P_n(x)} = \frac{x + (n+1)\pi}{x - n\pi} \doteq -1 \quad , \quad \text{as } n \doteq \infty.$$

Hence

$$\lim P_n(x + \pi) = -\lim P_n(x),$$

or

$$\sin(x + \pi) = -\sin x.$$

Hence

$$\sin(x + 2\pi) = \sin x$$

and thus $\sin x$ admits the period 2π.

216. 1. *Infinite Series for* $\tan x$, $\operatorname{cosec} x$, *etc.*

If $0 < x < \pi$, all the factors in the product 213, 5) are positive. Thus

$$\log \sin x = \log x + \sum_1^\infty \log\left(1 - \frac{x^2}{s^2\pi^2}\right) \quad , \quad 0 < x < \pi. \tag{1}$$

Similarly 214, 1) gives

$$\log \cos x = \sum_1^\infty \log\left(1 - \frac{4x^2}{(2s-1)^2\pi^2}\right) \quad , \quad 0 \leq x < \frac{\pi}{2}. \tag{2}$$

To get formulæ having a wider range we have only to square the products 213, 5) and 214, 1). We then get

$$\log \sin^2 x = \log x^2 + \sum_1^\infty \log\left(1 - \frac{x^2}{s^2\pi^2}\right)^2, \tag{3}$$

valid for any x such that $\sin x \neq 0$; and

$$\log \cos^2 x = \sum_1^\infty \log\left(1 - \frac{4x^2}{(2s-1)^2\pi^2}\right)^2, \tag{4}$$

valid for any x such that $\cos x \neq 0$.

If we differentiate 3), 4) we get

$$\cot x = \frac{1}{x} + 2\sum_1^\infty \frac{x}{x^2 - s^2\pi^2}, \tag{5}$$

$$\tan x = 2\sum_1^\infty \frac{x}{\left(\frac{2s-1}{2}\right)^2\pi^2 - x^2} \tag{6}$$

valid as in 3), 4).

Remark. The relations 5), 6) exhibit $\cot x$, $\tan x$ as a series of rational functions whose poles are precisely the poles of the given functions. They are analogous to the representation in algebra of a fraction as the sum of partial fractions.

2. *To get developments of* $\sec x$, $\operatorname{cosec} x$, we observe that

$$\operatorname{cosec} x = \tan \tfrac{1}{2}x + \cot x.$$

Hence

$$\operatorname{cosec} x = 2\sum_1^\infty \frac{\frac{1}{2}x}{\left(\frac{2s-1}{2}\right)^2\pi^2 - \frac{x^2}{4}} + \frac{1}{x} - 2\sum_1^\infty \frac{x}{s^2\pi^2 - x^2}$$

$$= \frac{1}{x} + \sum \frac{4x}{(2s-1)^2\pi^2 - x^2} - 2\sum \frac{x}{s^2\pi^2 - x^2}$$

$$= \frac{1}{x} + \sum_1^\infty \frac{(-1)^{s-1} 2x}{s^2\pi^2 - x^2},$$

valid for $x \neq \pm s\pi$.

3. *To get* sec x, we observe that

$$\operatorname{cosec}\left(\frac{\pi}{2}-x\right)=\sec x.$$

Now
$$\operatorname{cosec} x=\frac{1}{x}+\sum_1^\infty(-1)^{s-1}\left\{\frac{1}{s\pi-x}-\frac{1}{s\pi+x}\right\}.$$

Hence

$$\operatorname{cosec}\left(\frac{\pi}{2}-x\right)=\frac{1}{\frac{\pi}{2}-x}+\sum_1^\infty(-1)^{s-1}\left\{\frac{1}{s\pi-\frac{\pi}{2}+x}-\frac{1}{s\pi+\frac{\pi}{2}-x}\right\}=S.$$

Let us regroup the terms of S, forming the series

$$T=\left\{\frac{1}{\frac{\pi}{2}-x}+\frac{1}{\frac{\pi}{2}+x}\right\}-\left\{\frac{1}{\frac{3\pi}{2}-x}+\frac{1}{\frac{3\pi}{2}+x}\right\}+\cdots$$

As
$$|S_n-T_n|=\frac{1}{\left|\frac{2n-1}{2}\pi-x\right|}\doteq 0,$$

we see that T is convergent and $=S$. Thus

$$\sec x=\sum_1^\infty(-1)^{s-1}\frac{(2s-1)\pi}{\left(\frac{2s-1}{2}\right)^2\pi^2-x^2},$$

valid for all x such that $\cos x\neq 0$.

217. As an exercise let us show the periodicity of $\cot x$ from 216, 5). We have

$$\cot x=\lim F_n(x)=\lim_{n=\infty}\sum_{s=-n}^{n}\frac{1}{x+s\pi}\qquad x\neq s\pi.$$

Now
$$F_n(x+\pi)=F_n(x)+\frac{1}{x+(n+1)\pi}-\frac{1}{x-n\pi}.$$

Letting $n\doteq\infty$ we see that

$$\lim F_n(x+\pi)=\lim F_n(x)$$

and hence

$$\cot(x+\pi)=\cot x.$$

218. *Development of* log sin x, tan x, *etc., in power series.* From 216, 1)

$$\log \frac{\sin x}{x} = \sum_1^\infty \log\left(1 - \frac{x^2}{s^2\pi^2}\right). \tag{1}$$

If we give to $\frac{\sin x}{x}$ its limiting value 1 as $x \doteq 0$, the relation 1) holds for $|x| < \pi$.

Now for $|x| < \pi$

$$-\log\left(1 - \frac{x^2}{s^2\pi^2}\right) = \frac{x^2}{s^2\pi^2} + \frac{1}{2}\frac{x^4}{s^4\pi^4} + \cdots$$

Thus

$$-\log\frac{\sin x}{x} = \frac{x^2}{\pi^2} + \frac{1}{2}\frac{x^4}{\pi^4} + \frac{1}{3}\frac{x^6}{\pi^6} + \cdots$$

$$+ \frac{x^2}{2^2\pi^2} + \frac{1}{2}\frac{x^4}{2^4\pi^4} + \frac{1}{3}\frac{x^6}{2^6\pi^6} + \cdots$$

$$+ \frac{x^2}{3^2\pi^2} + \frac{1}{2}\frac{x^4}{3^4\pi^4} + \frac{1}{3}\frac{x^6}{3^6\pi^6} + \cdots$$

$$+ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

provided we sum this double series by rows. But since the series is a positive term series, we may sum by columns, by 129, 2. Doing this we get

$$-\log\frac{\sin x}{x} = H_2\frac{x^2}{\pi^2} + \tfrac{1}{2} H_4\frac{x^4}{\pi^4} + \tfrac{1}{3} H_6\frac{x^6}{\pi^6} + \cdots \tag{2}$$

where

$$H_n = \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots$$

The relation 2) *is valid for* $|x| < \pi$.

In a similar manner we find

$$-\log\cos x = G_2\frac{2^2x^2}{\pi^2} + \tfrac{1}{2} G_4\frac{2^4x^4}{\pi^4} + \tfrac{1}{3} G_6\frac{2^6x^6}{\pi^6} + \cdots \tag{3}$$

valid for $|x| < \frac{\pi}{2}$. Here

$$G_n = \frac{1}{1^n} + \frac{1}{3^n} + \frac{1}{5^n} + \cdots$$

The terms of G_n are a part of H_n. Obviously

$$G_n = \frac{2^n - 1}{2^n} H_n.$$

These coefficients put in 3) give

$$-\log\cos x = (2^2-1)H_2\frac{x^2}{\pi^2} + \tfrac{1}{2}(2^4-1)H_4\frac{x^4}{\pi^4} + \tfrac{1}{3}(2^6-1)H_6\frac{x^6}{\pi^6} + \cdots \quad (4$$

valid for $|x| < \frac{\pi}{2}$. If we differentiate 4) and 2), we get

$$\tan x = 2(2^2-1)H_2\frac{x}{\pi^2} + 2(2^4-1)H_4\frac{x^3}{\pi^4} + 2(2^6-1)H_6\frac{x^5}{\pi^6} + \cdots \quad (5$$

valid for $|x| < \frac{\pi}{2}$;

$$\cot x = \frac{1}{x} - 2H_2\frac{x}{\pi^2} - 2H_4\frac{x^3}{\pi^4} - 2H_6\frac{x^5}{\pi^6} - \cdots \quad (6$$

valid for $0 < |x| < \pi$.

Comparing 5) with the development of $\tan x$ given 165, 3) gives

$$\begin{aligned}
H_2 &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} = \frac{1}{6}\cdot\frac{2\pi^2}{2!} = B_1\cdot\frac{2\pi^2}{2!}.\\
H_4 &= \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90} = \frac{1}{30}\cdot\frac{2^3\pi^4}{4!} = B_3\cdot\frac{2^3\pi^4}{4!} \qquad (7\\
H_6 &= \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \cdots = \frac{\pi^6}{945} = \frac{1}{42}\cdot\frac{2^5\pi^6}{6!} = B_5\cdot\frac{2^5\pi^6}{6!}\\
H_8 &= \frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \cdots = \frac{\pi^8}{9450} = \frac{1}{30}\cdot\frac{2^7\pi^8}{8!} = B_7\cdot\frac{2^7\pi^8}{8!}\\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

Let us set

$$H_{2n} = \frac{2^{2n-1}\pi^{2n}}{(2n)!} B_{2n-1}. \quad (8$$

Then 5) gives

$$\tan x = \frac{2^2(2^2-1)}{2!} B_1 x + \frac{2^4(2^4-1)}{4!} B_3 x^3 + \frac{2^6(2^6-1)}{6!} B_5 x^5 + \cdots \quad (9$$

valid for $|x| < \frac{\pi}{2}$. The coefficients B_1, $B_3 \cdots$ are called *Bernouillian numbers.* From 7) we see

$$B_1 = \tfrac{1}{6}\ ,\quad B_3 = \tfrac{1}{30}\ ,\quad B_5 = \tfrac{1}{42}\ ,\quad B_7 = \tfrac{1}{30}.$$

From 6), 8) we get

$$\cot an\ x - \frac{1}{x} = -\sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} B_{2n-1} x^{2n-1} \tag{10}$$

valid for $0 < |x| < \pi$.

219. *Recursion formula for the Bernouillian Numbers.*

If we set $$f(x) = \tan x,$$
we have by Taylor's development

$$f(x) = xf'(0) + x^3 \frac{f'''(0)}{3!} + x^5 \frac{f^{\text{v}}(0)}{5!} + \cdots$$

where
$$\frac{f^{(2n-1)}(0)}{(2n-1)!} = \frac{2(2^{2n}-1)H_{2n}}{\pi^{2n}} = \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_{2n-1} \tag{1}$$

Now by I, 408,

$$f^{(2n-1)}(0) - \binom{2n-1}{2} f^{(2n-3)}(0) - \binom{2n-1}{4} f^{(2n-5)}(0) - \cdots = (-1)^{n-1}. \tag{2}$$

From 1), 2) we get

$$\frac{2^{2n-1}(2^{2n}-1)}{n} B_{2n-1} - \binom{2n-1}{2} \frac{2^{2n-3}(2^{2n-2}-1)}{n-1} B_{2n-3}$$
$$+ \binom{2n-1}{4} \frac{2^{2n-5}(2^{2n-4}-1)}{n-2} B_{2n-5} - \cdots = (-1)^{n-1}. \tag{3}$$

We have already found B_1, B_3, B_5, B_7; it is now easy to find successively:

$$B_9 = \tfrac{5}{66} \qquad B_{11} = \tfrac{691}{2730} \qquad B_{13} = \tfrac{7}{6} \qquad B_{15} = \tfrac{3617}{510}$$
$$B_{17} = \tfrac{43867}{798}, \quad B_{19} = \tfrac{174611}{330}.$$

Thus to calculate B_9, we have from 3)

$$\frac{2^9(2^{10}-1)}{5} B_9 - \frac{9\cdot 8}{1\cdot 2} \frac{2^7(2^8-1)}{4} \cdot \frac{1}{30} + \frac{9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4} \frac{2^5(2^6-1)}{3} \frac{1}{42}$$
$$- \frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3} \frac{2^3(2^4-1)}{2} \cdot \frac{1}{30} + 9\cdot 2(2^2-1)\cdot \frac{1}{6} = 1.$$

Thus

$$B_9 = \frac{5}{512\cdot 1023}\{1 - 9 + 168 - 2016 + 9792\}$$
$$= \frac{5\cdot 7936}{512\cdot 1023} = \frac{5}{66}.$$

The B *and* Γ *Functions*

220. In Volume I we defined the B and Γ functions by means of integrals:

$$B(u, v) = \int_0^\infty \frac{x^{u-1}dx}{(1+x)^{u+v}} \tag{1}$$

$$\Gamma(u) = \int_0^\infty e^{-x}x^{u-1}dx \tag{2}$$

which converge only when $u, v > 0$. Under this condition we saw that

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}. \tag{3}$$

We propose to show that $\Gamma(u)$ can be developed in the infinite product

$$G = \frac{1}{u}\prod_1^\infty \frac{\left(1+\frac{1}{n}\right)^u}{1+\frac{u}{n}}. \tag{4}$$

This product converges, as we saw, 197, 3, for any $u \neq 0, -1, -2, \cdots$ From 201, 7 and 207 it is obvious that G converges absolutely and uniformly at any point u different from these singular points. Thus the expression 4) has a wider domain of definition than that of 2). Since $G = \Gamma$, as we said, for $u > 0$, we shall extend the definition of the Γ function in accordance with 4), for negative u.

It frequently happens that a function $f(x)$ can be represented by different analytic expressions whose domains of convergence are different. For example, we saw 218, 9), that $\tan x$ can be developed in a power series

$$\tan x = \frac{2^2(2^2-1)}{1!}B_1 x + \frac{2^2(2^4-1)}{4!}B_3 x^3 + \cdots$$

valid for $|x| < \frac{\pi}{2}$. On the other hand,

$$\tan x = \frac{\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots} = \frac{\sin x}{\cos x}$$

and

$$\tan x = 2\sum_1^\infty \frac{x}{\left(\frac{2s-1}{2}\right)^2\pi^2 - x^2} \quad \text{by 216, 6)}$$

are analytic expressions valid for every x for which the function $\tan x$ is defined.

221. 1. Before showing that G and Γ have the same values for $u > 0$, let us develop some of the properties of the product G given in 220, 4). In the first place, we have, by 210:

The function $G(u)$ is continuous, except at the points $u = 0, -1, -2, \cdots$

Since the factors of 4) are all positive for $u > 0$, we see that

$G(u)$ is positive for $u > 0$.

2. *In the vicinity of the point $x = -m$, $\quad m = 0, 1, \cdots$*

$$G(u) = \frac{H(u)}{x+m}$$

where $H(u)$ is continuous near this point, and does not vanish at this point.

For

$$G(u) = \frac{\left(1+\frac{1}{m}\right)^u}{1+\frac{u}{m}} H(u)$$

where H is the infinite product G with one factor left out. As we may reason on H as we did on G, we see H converges at the point $x = -m$. Hence $H \neq 0$ at this point. But H also converges uniformly about this point; hence H is continuous about it.

222.

$$G = \lim_{n=\infty} \frac{1}{u} \frac{1 \cdot 2 \cdots (n-1)}{(u+1)(u+2)\cdots(u+n-1)} \cdot n^u. \tag{1}$$

To prove this relation, let us denote the product under the limit sign by P_n. We have

$$n^u = \left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdots\frac{n}{n-1}\right)^u = \left(1+\frac{1}{1}\right)^u\left(1+\frac{1}{2}\right)^u\cdots\left(1+\frac{1}{n-1}\right)^u.$$

Also

$$(u+1)(u+2)\cdots(u+n-1)=(n-1)!\left(1+\frac{u}{1}\right)\left(1+\frac{u}{2}\right)\cdots\left(1+\frac{u}{n-1}\right).$$

Thus $P_n = G_n$. But $G_n \doteq G$, hence P_n, is convergent and $G = \lim P_n$.

223. *Euler's Constant.* This is defined by the convergent series

$$C = \sum_1^\infty \left\{\frac{1}{n} - \log\left(1+\frac{1}{n}\right)\right\}.$$

It is easy to see at once that

$$C < \frac{1}{2}\sum\frac{1}{n^2} = \frac{1}{2}H_2 = \frac{\pi^2}{12} = .82\cdots$$ p 264

by 218, 7). By calculation it is found that

$$C = .577215\cdots$$

224. *Another expression of G is*

$$G = \frac{e^{-Cu}}{u\Pi\left(1+\frac{u}{n}\right)e^{-\frac{u}{n}}}, \quad n = 1, 2, \cdots \tag{1}$$

where C is the Eulerian constant.

For when $a > 0$, $\quad a^u = e^{u \log a}$.

Hence

$$G = \frac{1}{u}\Pi\frac{e^{u\log\left(1+\frac{1}{n}\right)}}{1+\frac{u}{n}}$$

$$= \frac{1}{u}\Pi\frac{e^{u\log\left(1+\frac{1}{n}\right)}e^{-\frac{u}{n}}}{\left(1+\frac{u}{n}\right)e^{-\frac{u}{n}}}$$

Now

$$\Pi e^{u\log\left(1+\frac{1}{n}\right)-\frac{u}{n}}$$

and

$$\Pi\left(1+\frac{u}{n}\right)e^{-\frac{u}{n}}$$

are convergent. Hence

$$G = \frac{\Pi e^{u\left[\log\left(1+\frac{1}{n}\right)-\frac{1}{n}\right]}}{u\Pi\left(1+\frac{u}{n}\right)e^{-\frac{u}{n}}}$$

from which 1) follows at once, using 223.

225. *Further Properties of G.*

1. $$G(u+1) = uG(u). \tag{1}$$

Let us use the product

$$P_n(u) = \frac{1}{u} \cdot \frac{(n-1)!}{(u+1)\cdots(u+n-1)} \cdot n^u$$

employed in 222. Then

$$P_n(u+1) = \frac{nuP_n(u)}{u+n}. \tag{2}$$

As

$$\frac{nu}{u+n} \doteq u \qquad \text{as } n \doteq \infty$$

we get 1) from 2) at once on passing to the limit.

2. $$G(u+n) = u(u+1)\cdots(u+n-1)G(u). \tag{3}$$

This follows from 1) by repeated applications.

3. $$G(n) = 1 \cdot 2 \cdot \cdots n-1 = (n-1)! \tag{4}$$

where n is a positive integer.

4. $$G(u)G(1-u) = \frac{\pi}{\sin \pi u}. \tag{5}$$

For

$$G(1-u) = -uG(-u) \qquad \text{by 1,}$$

$$= \frac{e^{Cu}}{\Pi\left(1-\frac{u}{n}\right)e^{\frac{u}{n}}}, \quad \text{by 224, 1).}$$

Hence

$$G(u)G(1-u) = \frac{1}{u} \cdot \frac{e^{-Cu}e^{Cu}}{\Pi\left(1+\frac{u}{n}\right)e^{-\frac{u}{n}} \cdot \left(1-\frac{u}{n}\right)e^{\frac{u}{n}}}$$

$$= \frac{1}{u}\frac{1}{\Pi\left(1-\frac{u^2}{n^2}\right)}.$$

We now use 213, 5).

Let us note that by virtue of 1, 2 the value of G is known for all $u > 0$, when it is known in the interval $(0, 1)$. By virtue of 5) G is known for $u < 0$ when its value is known for $u > 0$. Moreover the relation 5) shows the value of G is known in $(\frac{1}{2}, 1)$ when its value is known in $(0, \frac{1}{2})$.

As a result of this we see G is known when its values in the interval $(0, \frac{1}{2})$ are known; or indeed in any interval of length $\frac{1}{2}$.

Gauss has given a table of $\log G(u)$ for $1 \leq u \leq 1.5$ calculated to 20 decimal places. A four-place table is given in "A Short Table of Integrals" by *B. O. Peirce*, for $1 \leq u \leq 2$.

5. $$G(\tfrac{1}{2}) = \sqrt{\pi}. \tag{6}$$

For in 5) set $u = \frac{1}{2}$. Then

$$G^2(\tfrac{1}{2}) = \pi.$$

Hence $$G(\tfrac{1}{2}) = \pm\sqrt{\pi}.$$

We must take the plus sign here, since $G > 0$ when $u > 0$, by 221.

6. $$G\left(\frac{2n+1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \cdots 2n-1}{2^n} \cdot \sqrt{\pi} \tag{7}$$

where n is a positive integer.

For $$G\left(\frac{2n+1}{2}\right) = G\left(1 + \frac{2n-1}{2}\right) = \frac{2n-1}{2}\, G\left(\frac{2n-1}{2}\right), \text{ by 1.}$$

Similarly $$G\left(\frac{2n-1}{2}\right) = \frac{2n-3}{2}\, G\left(\frac{2n-3}{2}\right), \text{ etc.}$$

Thus $$G\left(\frac{2n+1}{2}\right) = \frac{2n-1}{2} \cdot \frac{2n-3}{2} \cdots \frac{3}{2} \cdot \frac{1}{2}\, G\left(\frac{1}{2}\right).$$

226. *Expressions for* $\log G(u)$, *and its Derivatives.*

From 224, 1) we have for $u > 0$,

$$L(u) = \log G(u) = -Cu - \log u + \sum_1^\infty \left\{\frac{u}{n} - \log\left(1 + \frac{u}{n}\right)\right\}. \tag{1}$$

Differentiating, we get

$$L' = -C - \frac{1}{u} + \sum_1^\infty \left\{\frac{1}{n} - \frac{1}{u+n}\right\}. \tag{2}$$

That this step is permissible follows from 155, 1.

We may write 2)

$$L' = -C + \sum_1^\infty \left\{ \frac{1}{n} - \frac{1}{u+n-1} \right\}. \tag{3}$$

That the relations 2), 3) hold for any $u \neq 0, -1, -2 \cdots$ follows by reasoning similar to that employed in 216. In general we have

$$L^{(r)} = (-1)^r (r-1)! \sum_1^\infty \frac{1}{(u+n-1)^r}, \quad r > 1. \tag{4}$$

In particular,

$$L'(1) = -C. \tag{5}$$

$$L^{(r)}(1) = (-1)^r (r-1)! \sum \frac{1}{n^r} = (-1)^r (r-1)!\, H_r.$$

227. *Development of* log $G(u)$ *in a Power Series.* If Taylor's development is valid about the point $u = 1$, we have

$$\log G(u) = L(u) = L(1) + \frac{u-1}{1!} L'(1) + \frac{(u-1)^2}{2!} L''(1) + \cdots;$$

or using 226, 5), and setting $u = 1 + x$,

$$\log G(1+x) = -Cx + \sum_{n=2}^\infty \frac{(-1)^n}{n} H_n x^n. \tag{1}$$

We show now *this relation is valid for* $-\frac{1}{2} \leq x \leq 1$, by proving that

$$R_s = \frac{x^s}{s!} L^{(s)}(1 + \theta x), \qquad 0 < \theta < 1$$

converges to 0, as $s \doteq \infty$.

For, if $0 \leq x \leq 1$, then

$$|R_s| \leq \frac{1}{s} \sum_1^\infty \frac{1}{n^s} \doteq 0.$$

Also if $-\frac{1}{2} \leq x \leq 0$,

$$|R_s| \leq \left\{ \left| \frac{x}{1+\theta x} \right|^s + \sum_2^\infty \frac{|x|^s}{|n + \theta x|^s} \right\} \cdot \frac{1}{s}$$

$$< \left\{ 1 + \sum_1^\infty \frac{1}{2^{s+1}(n-1)^{s+1}} \right\} \frac{1}{s} \doteq 0.$$

The relation 1) is really valid for $-1 < x \leq 1$, but for our purpose it suffices to know that it holds in $\mathfrak{A} = (-\frac{1}{2}, 1)$. Legendre

has shown how the series 1) may be made to converge more rapidly. We have for any x in $\mathfrak{A}$

$$\log(1+x) = x - \sum_2^\infty (-1)^n \frac{x^n}{n}.$$

This on adding and subtracting from 1) gives

$$\log G(1+x) = -\log(1+x) + (1-C)x + \sum_2^\infty (-1)^n (H_n - 1)\frac{x^n}{n}.$$

Changing here x into $-x$ gives

$$\log G(1-x) = -\log(1-x) - (1-C)x + \Sigma(H_n - 1)\frac{x^n}{n}.$$

Subtracting this from the foregoing gives

$$\log G(1+x) - \log G(1-x)$$
$$= -\log\frac{1+x}{1-x} + 2(1-C)x - \sum_1^\infty \frac{x^{2m+1}}{2m+1}(H_{2m+1} - 1).$$

From 225, 4

$$\log G(1+x) + \log G(1-x) = \log\frac{\pi x}{\sin \pi x}.$$

This with the preceding relation gives

$$\log G(1+x)$$
$$= (1-C)x - \tfrac{1}{2}\log\frac{1+x}{1-x} + \log\frac{\pi x}{\sin\pi x} - \tfrac{1}{2}\sum_1^\infty (H_{2m+1} - 1)\frac{x^{2m+1}}{2m+1} \qquad (2$$

valid in $\mathfrak{A}$.

This series converges rapidly for $0 \leq x \leq \frac{1}{2}$, and enables us to compute $G(u)$ in the interval $1 \leq u \leq \frac{3}{2}$. The other values of G may be readily obtained as already observed.

228. 1. *We show now with Pringsheim* that* $G(u) = \Gamma(u)$, *for* $u > 0$.

We have for $0 \leq u \leq 1$,

$$\Gamma(u+n) = \int_0^\infty e^{-x} x^{u+n-1} dx$$
$$= \int_0^n + \int_n^\infty$$

* *Math. Annalen*, vol. 31, p. 455.

Now for any x in the interval $(0, n)$,

$$x^u \leq n^u \quad , \quad x^u \geq x n^{u-1}$$

since $u \geq 0$ and $u - 1 \leq 0$.

Also for any x in the interval (n, ∞)

$$x^u \leq x n^{u-1} \quad , \quad x^u \geq n^u.$$

Hence

$$n^{u-1}\int_0^n e^{-x}x^n dx + n^u \int_n^\infty e^{-x}x^{n-1}dx < \Gamma(u+n)$$
$$< n^u \int_0^n e^{-x}x^{n-1}dx + n^{u-1}\int_n^\infty e^{-x}x^n dx.$$

Thus

$$\frac{\Gamma(u+n)}{n^u} < \int_0^n e^{-x}x^{n-1}dx + \frac{1}{n}\int_n^\infty e^{-x}x^n dx$$
$$< \int_0^n e^{-x}x^{n-1}dx + \frac{1}{n}\int_0^\infty e^{-x}x^n dx - \frac{1}{n}\int_0^n e^{-x}x^n dx.$$

Let us call these integrals A, B, C respectively.
We see at once that

$$B = \frac{\Gamma(n+1)}{n} = \frac{n!}{n} = (n-1)!$$

Also, integrating by parts,

$$A = \left[\frac{e^{-x}x^n}{n}\right]_0^n + \frac{1}{n}\int_0^n e^{-x}x^n dx = \frac{n^n}{ne^n} + C.$$

Thus
$$\frac{\Gamma(u+n)}{n^u} < (n-1)! + \frac{n^{n-1}}{e^n}.$$

Similarly
$$\frac{\Gamma(u+n)}{n^u} > (n-1)! - \frac{n^{n-1}}{e^n}.$$

Hence
$$\frac{\Gamma(u+n)}{(n-1)!\, n^u} = 1 + \theta_n q_n \qquad 0 < \theta_n < 1$$

where
$$q_n = \frac{n^{n-1}}{e^n(n-1)!} = \frac{n^n}{e^n n!}.$$

Now
$$e^n = 1 + \frac{n}{1!} + \frac{n^2}{2!} + \cdots > \frac{n^n}{n!}\left\{1 + \frac{n}{n+1} + \cdots \right\} = \frac{n^n}{n!}\nu_n.$$

But

$$\nu_n > 1 + \frac{n}{n+1} + \cdots + \frac{n^m}{(n+1)\cdots(n+m)} \quad , \quad \text{for any } m$$

$$> \frac{mn^m}{(n+1)\cdots(n+m)} = \frac{m}{\left(1+\frac{1}{n}\right)\cdots\left(1+\frac{m}{n}\right)} > \frac{m}{\left(1+\frac{m}{n}\right)^m}.$$

Let us take

$$n > m^2 \quad \text{or} \quad \frac{m}{n} < \frac{1}{m}.$$

Then

$$\nu_n > \frac{m}{\left(1+\frac{1}{m}\right)^m} > \frac{m}{e}.$$

Since m may be taken large at pleasure,

$$\lim \nu_n = \infty$$

and hence

$$\lim q_n = 0.$$

Thus

$$\lim_{n=\infty} \frac{\Gamma(u+n)}{(n-1)!\, n^u} = 1 \quad , \quad 0 \leq u \leq 1.$$

But from $\Gamma(u+1) = u\Gamma(u)$ we have

$$\frac{\Gamma(u+1+n)}{n^{u+1}(n-1)!} = \frac{u+n}{n} \frac{\Gamma(u+n)}{n^u(n-1)!} = 1$$

also, as $n \doteq \infty$. Thus the relation 1) holds for $1 \leq u \leq 2$, and in fact for any $u > 0$.

As

$$\Gamma(u+n) = u(u+1)\cdots(u+n-1)\Gamma(u),$$

we have

$$\Gamma(u) = \frac{\Gamma(u+n)}{u(u+1)\cdots(u+n-1)}.$$

Hence using 1),

$$\Gamma(u) = \frac{(n-1)!\, n^u}{u(u+1)\cdots(u+n-1)} \cdot \frac{\Gamma(u+n)}{(n-1)!\, n^u}.$$

Letting $n \doteq \infty$, we get $\Gamma(u) = G(u)$ for any $u > 0$, making use of 1) and 222, 1).

2. Having extended the definition of $\Gamma(u)$ to negative values of u, we may now take the relation

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \tag{2}$$

as a definition of the B function. This definition will be in accordance with 220, 1) for $u, v > 0$, and will define B for negative u, v when the right side of 2) has a value.

CHAPTER VIII

AGGREGATES

Equivalence

229. 1. Up to the present the aggregates we have dealt with have been point aggregates. We now consider aggregates in general. Any collection of well-determined objects, distinguishable one from another, and thought of as a whole, may be called an *aggregate* or *set*.

Thus the class of prime numbers, the class of integrable functions, the inhabitants of the United States, are aggregates.

Some of the definitions given for point aggregates apply obviously to aggregates in general, and we shall therefore not repeat them here, as it is only necessary to replace the term point by object or element.

As in point sets, $\mathfrak{A} = 0$ shall mean that $\mathfrak{A}$ embraces no elements.

Let $\mathfrak{A}$, $\mathfrak{B}$ be two aggregates such that each element a of $\mathfrak{A}$ is associated with some one element b of $\mathfrak{B}$, and conversely. We say that $\mathfrak{A}$ is *equivalent* to $\mathfrak{B}$ and write

$$\mathfrak{A} \sim \mathfrak{B}.$$

We also say $\mathfrak{A}$ and $\mathfrak{B}$ are in *one to one correspondence* or are *in uniform correspondence.* To indicate that a is associated with b in this correspondence we write

$$a \sim b.$$

2. *If* $\mathfrak{A} \sim \mathfrak{B}$ *and* $\mathfrak{B} \sim \mathfrak{C}$, *then* $\mathfrak{A} \sim \mathfrak{C}$.

For let $a \sim b$, $b \sim c$. Then we can set $\mathfrak{A}$, $\mathfrak{C}$ in uniform correspondence by setting $a \sim c$.

3. *Let*
$$\mathfrak{A} = \mathfrak{B} + \mathfrak{C} + \mathfrak{D} + \cdots$$
$$A = B + C + D + \cdots$$

If $\mathfrak{B} \sim B$, $\mathfrak{C} \sim C$, $\cdots$, *then* $\mathfrak{A} \sim A$.

For we can associate the elements of $\mathfrak{A}$ with those of A by keeping precisely the correspondence which exists between the elements of $\mathfrak{B}$ and B, of $\mathfrak{C}$ and C, etc.

Example 1.

$$\mathfrak{A} = 1, 2, 3, \cdots$$

$$\mathfrak{B} = a_1, a_2, a_3, \cdots$$

If we set $a_n \sim n$, $\mathfrak{A}$ and $\mathfrak{B}$ will stand in 1, 1 correspondence.

Example 2.

$$\mathfrak{A} = 1, 2, 3, 4, \cdots$$

$$\mathfrak{B} = 2, 4, 6, 8, \cdots$$

If we set n of $\mathfrak{A}$ in correspondence with $2n$ of $\mathfrak{B}$, $\mathfrak{A}$ and $\mathfrak{B}$ will be in uniform correspondence.

We note that $\mathfrak{B}$ is a part of $\mathfrak{A}$; we have thus this result: *An infinite aggregate may be put in uniform correspondence with a partial aggregate of itself.*

This is obviously impossible if $\mathfrak{A}$ is finite.

Example 3.

$$\mathfrak{A} = 1, 2, 3, 4, \cdots$$

$$\mathfrak{B} = 10^1, 10^2, 10^3, 10^4, \cdots$$

If we set $n \sim 10^n$, we establish a uniform correspondence between $\mathfrak{A}$ and $\mathfrak{B}$. We note again that $\mathfrak{A} \sim \mathfrak{B}$ although $\mathfrak{A} > \mathfrak{B}$.

Example 4. Let $\mathfrak{C} = \{\xi\}$, where, using the triadic system,

$$\xi = \cdot\, \xi_1 \xi_2 \xi_3 \cdots \qquad \xi_n = 0, 2$$

denote the Cantor set of I, 272. Let us associate with ξ the point

$$x = \cdot\, x_1 x_2 x_3 \cdots \tag{1}$$

where $x_n = 0$ when $\xi_n = 0$, and $= 1$ when $\xi_n = 2$ and read 1) in the dyadic system.

Then $\{x\}$ is the interval $(0, 1)$. Thus we have established a *uniform correspondence between $\mathfrak{C}$ and the points of a unit interval.*

In passing let us note that if $\xi < \xi'$ and x, x' are the corresponding points in $\{x\}$, then $x < x'$.

This example also shows that *we can set in uniform correspondence a discrete aggregate with the unit interval.*

We have only to prove that $\mathfrak{C}$ is discrete. To this end consider the set of intervals C marked heavy in the figure of I, 272. Ob-

viously we can select enough of these deleted intervals so that their lower content is as near 1 as we choose. Thus

$$\underline{\text{Cont}}\ C = 1.$$

As $\overline{\text{Cont}}\, C \leq 1$, C is metric and its content is 1. Hence $\mathfrak{C}$ is discrete.

230. 1. *Let* $\mathfrak{A} = \alpha + A$, $\mathfrak{B} = \beta + B$, *where* a, b *are elements of* $\mathfrak{A}$, $\mathfrak{B}$ *respectively. If* $\mathfrak{A} \sim \mathfrak{B}$, *then* $A \sim B$ *and conversely.*

For, since $\mathfrak{A} \sim \mathfrak{B}$, each element a of $\mathfrak{A}$ is associated with some one element b of $\mathfrak{B}$, and the same holds for $\mathfrak{B}$. If it so happens that $\alpha \sim \beta$, the uniform correspondence of A, B is obvious. If on the contrary $\alpha \sim b'$ and $\beta \sim a'$, the uniform correspondence between A, B can be established by setting $a' \sim b'$ and having the other elements in A, B correspond as in $\mathfrak{A} \sim \mathfrak{B}$.

2. We state as obvious the theorems:

No part $\mathfrak{B}$ *of a finite set* $\mathfrak{A}$ *can be* $\sim \mathfrak{A}$.

No finite part $\mathfrak{B}$ *of an infinite set* $\mathfrak{A}$ *can be* $\sim \mathfrak{A}$.

Cardinal Numbers

231. 1. We attach now to each aggregate $\mathfrak{A}$ an attribute called its *cardinal number*, which is defined as follows:

1° Equivalent aggregates have the same cardinal number.

2° If $\mathfrak{A}$ is $\sim$ to a part of $\mathfrak{B}$, but $\mathfrak{B}$ is not $\sim \mathfrak{A}$ or to any part of $\mathfrak{A}$, the cardinal number of $\mathfrak{A}$ is less than that of $\mathfrak{B}$, or the cardinal number of $\mathfrak{B}$ is greater than that of $\mathfrak{A}$. The cardinal number of $\mathfrak{A}$ may be denoted by the corresponding small letter $\mathfrak{a}$ or by Card $\mathfrak{A}$.

The cardinal number of an aggregate is sometimes called its *power* or *potency*.

If $\mathfrak{A}$ is a finite set, let it consist of n objects or elements. Then its cardinal number shall be n. The cardinal number of a finite set is said to be *finite*, otherwise *transfinite*. It follows from the preceding definition that all transfinite cardinal numbers are greater than any finite cardinal number.

2. It is a property of any two *finite* cardinal numbers $\mathfrak{a}$, $\mathfrak{b}$ that either

$$\mathfrak{a} = \mathfrak{b} \quad , \quad \text{or } \mathfrak{a} > \mathfrak{b} \quad , \quad \text{or } \mathfrak{a} < \mathfrak{b}. \tag{1}$$

This property has not yet been established for transfinite cardinal numbers. There is in fact a fourth alternative relative to $\mathfrak{A}$, $\mathfrak{B}$, besides the three involved in 1). For until the contrary has been shown, there is the possibility that:

No part of $\mathfrak{A}$ is $\sim \mathfrak{B}$, and no part of $\mathfrak{B}$ is $\sim \mathfrak{A}$.

The reader should thus guard against expressly or tacitly assuming that one of the three relations 1) must hold for *any* two cardinal numbers.

3. We note here another difference. If $\mathfrak{A}$, $\mathfrak{B}$ are finite without common element,

$$\text{Card}\,(\mathfrak{A} + \mathfrak{B}) > \text{Card}\,\mathfrak{A}. \tag{2}$$

Let now $\mathfrak{A}$ denote the positive even and $\mathfrak{B}$ the positive odd numbers. Obviously

$$\text{Card}\,(\mathfrak{A} + \mathfrak{B}) = \text{Card}\,\mathfrak{A} = \text{Card}\,\mathfrak{B}$$

and the relation 2) does not hold for these transfinite numbers.

4. We have, however, the following:

Let $\mathfrak{A} > \mathfrak{B}$, *then*

$$\text{Card}\,\mathfrak{A} \geq \text{Card}\,\mathfrak{B}.$$

For obviously $\mathfrak{B}$ is $\sim$ to a part of $\mathfrak{A}$, viz. $\mathfrak{B}$ itself.

5. This may be generalized as follows:

Let

$$\mathfrak{A} = \mathfrak{B} + \mathfrak{C} + \mathfrak{D} + \cdots$$

$$A = B + C + D + \cdots$$

If

$$\text{Card}\,\mathfrak{B} \leq \text{Card}\,B \quad , \quad \text{Card}\,\mathfrak{C} \leq \text{Card}\,C, \textit{ etc.},$$

then

$$\text{Card}\,\mathfrak{A} \leq \text{Card}\,A.$$

For from Card $\mathfrak{B} \leq$ Card B follows that we can associate in 1, 1 correspondence the elements of $\mathfrak{B}$ with a part or whole of B.

The same is true for $\mathfrak{C}$, C; $\mathfrak{D}$, D; $\cdots$

Thus we can associate the elements of $\mathfrak{A}$ with a part or the whole of A.

Enumerable Sets

232. 1. An aggregate which is equivalent to the system of positive integers $\mathfrak{J}$ or to a part of $\mathfrak{J}$ is *enumerable.*

Thus all finite aggregates are enumerable. The cardinal number attached to an infinite enumerable set is $\aleph_0$, *aleph zero.*

At times we shall also denote this cardinal by $\mathfrak{e}$, so that

$$\mathfrak{e} = \aleph_0.$$

2. *Every infinite aggregate $\mathfrak{A}$ contains an infinite enumerable set $\mathfrak{B}$.*

For let a_1 be an element of $\mathfrak{A}$ and

$$\mathfrak{A} = a_1 + \mathfrak{A}_1.$$

Then $\mathfrak{A}_1$ is infinite; let a_2 be one of its elements and

$$\mathfrak{A}_1 = a_2 + \mathfrak{A}_2.$$

Then $\mathfrak{A}_2$ is infinite, etc.

Then

$$\mathfrak{B} = a_1, a_2, \cdots$$

is a part of $\mathfrak{A}$ and forms an infinite enumerable set.

3. From this follows that

$\aleph_0$ *is the least transfinite cardinal number.*

233. *The rational numbers are enumerable.*

For any rational number may be written

$$r = \frac{m}{n} \tag{1}$$

where, as usual, m is relatively prime to n.

The equation

$$|m| + |n| = p \tag{2}$$

admits but a finite number of solutions for each value of

$$p = 2, 3, 4, \cdots$$

Each solution m, n of 2), these numbers being relatively prime, gives a rational number 1). Thus we get, *e.g.*

$$\begin{array}{llllll} p = 2 & , & \pm 1. & & & \\ p = 3 & , & \pm 2, & \pm \frac{1}{2}. & & \\ p = 4 & , & \pm 3, & \pm \frac{1}{3}. & & \\ p = 5 & , & \pm 4, & \pm \frac{1}{4} & , \pm \frac{3}{2} & . \pm \frac{2}{3}. \end{array}$$

Let us now arrange these solutions in a sequence, putting those corresponding to $p = q$ before those corresponding to $p = q + 1$. We get

$$r_1 \quad , \quad r_2 \quad , \quad r_3 \cdots \tag{3}$$

which is obviously enumerable.

234. *Let the indices* $\iota_1, \iota_2, \cdots \iota_p$ *range over enumerable sets. Then*

$$\mathfrak{A} = \{a_{\iota_1 \cdots \iota_p}\}$$

is enumerable.

For the equation

$$\nu_1 + \nu_2 + \cdots + \nu_p = n,$$

where the ν's are positive integers, admits but a finite number of solutions for each $n = p$, $p + 1$, $p + 2$, $p + 3 \cdots$ Thus the elements of

$$\mathfrak{B} = \{b_{\nu_1 \cdots \nu_p}\}$$

may be arranged in a sequence

$$b_1 \quad , \quad b_2 \quad , \quad b_3 \cdots$$

by giving to n successively the values p, $p + 1$, $\cdots$ and putting the elements $b_{\nu_1 \cdots \nu_p}$ corresponding to $n = q + 1$ after those corresponding to $n = q$.

Thus the set $\mathfrak{B}$ is enumerable. Consider now $\mathfrak{A}$. Since each index ι_m ranges over an enumerable set, each value of ι_m as ι'_m is associated with some positive integer as m' and conversely. We may now establish a 1, 1 correspondence between $\mathfrak{A}$ and $\mathfrak{B}$ by setting

$$b_{m'_1 m'_2 \cdots m'_p} \sim a_{\iota'_1 \iota'_2 \cdots \iota'_p}.$$

Hence $\mathfrak{A}$ is enumerable.

235. 1. *An enumerable set of enumerable aggregates form an enumerable aggregate.*

For let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ $\cdots$ be the original aggregates. Since they form an enumerable set, they can be arranged in the order

$$\mathfrak{A}_1 \quad , \quad \mathfrak{A}_2 \quad , \quad \mathfrak{A}_3 \quad , \quad \cdots \tag{1}$$

But each $\mathfrak{A}_m$ is enumerable; therefore its elements can be arranged in the order

$$a_{m1} \quad , \quad a_{m2} \quad , \quad a_{m3} \quad , \quad a_{m4} \quad , \quad \cdots$$

Thus the a-elements in 1) form a set

$$\{a_{mn}\} \qquad m, n, = 1, 2, \cdots$$

which is enumerable by 234.

2. *The real algebraic numbers form an enumerable set.*

For each algebraic number is a root of a uniquely determined irreducible equation of the form

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0,$$

the a's being rational numbers. Thus the totality of real algebraic numbers may be represented by

$$\{\rho_{n,\ a_1 a_2 \cdots a_n}\}$$

where the index n runs over the positive integers and $a_1 \cdots a_n$ range over the rational numbers.

3. *Let* $\mathfrak{A}$, $\mathfrak{B}$ *be two enumerable sets. Then*

$$\text{Card } \mathfrak{A} = \text{Card } \mathfrak{B} = \aleph_0.$$

$$\text{Card } (\mathfrak{A} + \mathfrak{B}) = \aleph_0.$$

And in general if $\mathfrak{A}_1$, $\mathfrak{A}_2 \cdots$ *are an enumerable set of enumerable aggregates,* $\quad \text{Card } (\mathfrak{A}_1, \mathfrak{A}_2, \cdots) = \aleph_0.$

This follows from 1.

236. *Every isolated aggregate* $\mathfrak{A}$, *limited or not, forms an enumerable set.*

For let us divide $\mathfrak{R}_m$ into cubes of side 1. Obviously these form an enumerable set C_1, $C_2 \cdots$. About each point a of $\mathfrak{A}$ in any C_n as center we describe a cube of side σ, so small that it contains no other point of $\mathfrak{A}$. This is possible since $\mathfrak{A}$ is isolated. There are but a finite number of these cubes in C_n of side $\sigma = \frac{1}{\nu}$, $\nu = 1, 2, 3, \cdots$ for each ν. Hence, by 235, 1, $\mathfrak{A}$ is enumerable.

237. 1. *Every aggregate of the first species* $\mathfrak{A}$, *limited or not, is enumerable.*

For let $\mathfrak{A}$ be of order n. Then

$$\mathfrak{A} = \mathfrak{A}_\iota + \mathfrak{A}'_p$$

where $\mathfrak{A}_\iota$ denotes the isolated points of $\mathfrak{A}$ and $\mathfrak{A}'_p$ the proper limiting points of $\mathfrak{A}$.

Similarly,

$$\mathfrak{A}'_p = \mathfrak{A}'_{p,\iota} + \mathfrak{A}''_p$$

$$\mathfrak{A}''_p = \mathfrak{A}''_{p,\iota} + \mathfrak{A}'''_p$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Thus,

$$\mathfrak{A} = \mathfrak{A}_\iota + \mathfrak{A}'_{p,\iota} + \mathfrak{A}''_{p,\iota} + \cdots + \mathfrak{A}^{(n)}_p.$$

But $\mathfrak{A}^{(n)}$ is finite and $\mathfrak{A}^{(n)}_p \leq \mathfrak{A}^{(n)}$.

Thus $\mathfrak{A}$ being the sum of $n+1$ enumerable sets, is enumerable.

2. *If $\mathfrak{A}'$ is enumerable, so is $\mathfrak{A}$.*

For as in 1,

$$\mathfrak{A} = \mathfrak{A}_\iota + \mathfrak{A}'_p$$

and

$$\mathfrak{A}'_p \leq \mathfrak{A}'.$$

238. 1. *Every infinite aggregate $\mathfrak{A}$ contains a part $\mathfrak{B}$ such that $\mathfrak{B} \sim \mathfrak{A}$.*

For let $\mathfrak{E} = (a_1, a_2, a_3 \cdots)$ be an infinite enumerable set in $\mathfrak{A}$, so that

$$\mathfrak{A} = \mathfrak{E} + \mathfrak{F}.$$

Let

$$\mathfrak{E} = a_1 + E.$$

To establish a uniform correspondence between E, $\mathfrak{E}$ let us associate a_n in $\mathfrak{E}$ with a_{n+1} in E. Thus $\mathfrak{E} \sim E$.

We now set

$$\mathfrak{B} = E + \mathfrak{F}.$$

Obviously $\mathfrak{A} \sim \mathfrak{B}$ since $E \sim \mathfrak{E}$, and the elements of $\mathfrak{F}$ are common to $\mathfrak{A}$ and $\mathfrak{B}$.

2. *If $\mathfrak{A} \sim \mathfrak{B}$ are infinite, each contains a part $\mathfrak{A}_1$, $\mathfrak{B}_1$ such that*

$$\mathfrak{A} \sim \mathfrak{B}_1 \quad , \quad \mathfrak{B} \sim \mathfrak{A}_1.$$

For by 1, $\mathfrak{A}$ contains a part $\mathfrak{A}_1$ such that $\mathfrak{A} \sim \mathfrak{A}_1$. Similarly, $\mathfrak{B}$ contains a part $\mathfrak{B}_1$ such that $\mathfrak{B} \sim \mathfrak{B}_1$. As $\mathfrak{A} \sim \mathfrak{B}$, we have the theorem.

239. 1. A theorem of great importance in determining whether two aggregates are equivalent is the following. It is the converse of 238, 2.

Let $\mathfrak{A}_1 < \mathfrak{A}$, $\mathfrak{B}_1 < \mathfrak{B}$. *If* $\mathfrak{A}_1 \sim \mathfrak{B}$ *and* $\mathfrak{B}_1 \sim \mathfrak{A}$, *then*

$$\mathfrak{A} \sim \mathfrak{B}.$$

In the correspondence $\mathfrak{A}_1 \sim \mathfrak{B}$, let $\mathfrak{A}_2$ be the elements of $\mathfrak{A}_1$ associated with $\mathfrak{B}_1$. Then

$$\mathfrak{A}_2 \sim \mathfrak{B}_1 \sim \mathfrak{A}$$

and hence

$$\mathfrak{A} \sim \mathfrak{A}_2. \tag{1}$$

But as $\mathfrak{A}_1 > \mathfrak{A}_2$, we would infer from 1) that also

$$\mathfrak{A} \sim \mathfrak{A}_1. \tag{2}$$

As $\mathfrak{A}_1 \sim \mathfrak{B}$ by hypothesis, the truth of the theorem follows at once from 2).

To establish 2) we proceed thus. In the correspondence 1), let $\mathfrak{A}_3$ be that part of $\mathfrak{A}_2$ which $\sim \mathfrak{A}_1$ in $\mathfrak{A}$. In the correspondence $\mathfrak{A}_1 \sim \mathfrak{A}_3$, let $\mathfrak{A}_4$ be that part of $\mathfrak{A}_3$ which $\sim \mathfrak{A}_2$ in $\mathfrak{A}_1$.

Continuing in this way, we get the indefinite sequence

$$\mathfrak{A} > \mathfrak{A}_1 > \mathfrak{A}_2 > \mathfrak{A}_3 > \cdots$$

such that

$$\mathfrak{A} \sim \mathfrak{A}_2 \sim \mathfrak{A}_4 \sim \cdots$$

$$\mathfrak{A}_1 \sim \mathfrak{A}_3 \sim \mathfrak{A}_5 \sim \cdots$$

Let now

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{C}_1 \quad , \quad \mathfrak{A}_1 = \mathfrak{A}_2 + \mathfrak{C}_2 \quad , \quad \cdots$$

$$\mathfrak{D} = Dv(\mathfrak{A}, \mathfrak{A}_1, \mathfrak{A}_2 \cdots) \geq 0.$$

Then

$$\mathfrak{A} = \mathfrak{D} + \mathfrak{C}_1 + \mathfrak{C}_2 + \mathfrak{C}_3 + \mathfrak{C}_4 + \cdots \tag{3}$$

and similarly

$$\mathfrak{A}_1 = \mathfrak{D} + \mathfrak{C}_2 + \mathfrak{C}_3 + \mathfrak{C}_4 + \mathfrak{C}_5 + \cdots$$

We note that we can also write

$$\mathfrak{A}_1 = \mathfrak{D} + \mathfrak{C}_3 + \mathfrak{C}_2 + \mathfrak{C}_5 + \mathfrak{C}_4 + \cdots \tag{4}$$

Now from the manner in which the sets $\mathfrak{A}_3, \mathfrak{A}_4 \cdots$ were obtained, it follows that

$$\mathfrak{C}_1 \sim \mathfrak{C}_3 \quad , \quad \mathfrak{C}_3 \sim \mathfrak{C}_5 \cdots \tag{5}$$

Thus the sets in 4) correspond uniformly to the sets directly above them in 3), and this establishes 1).

2. In connection with the foregoing proof, which is due to *Bernstein*, the reader must guard against the following error. It does not *in general* follow from

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{C}_1 \quad , \quad \mathfrak{A}_2 = \mathfrak{A}_3 + \mathfrak{C}_3 \quad , \quad \mathfrak{A} \sim \mathfrak{A}_2 \quad , \quad \mathfrak{A}_1 \sim \mathfrak{A}_3$$

that

$$\mathfrak{C}_1 \sim \mathfrak{C}_3$$

which is the first relation in 5).

Example. Let $\mathfrak{A} = (1, 2, 3, 4, \cdots)$.

$$\mathfrak{A}_1 = (2, 3, 4, 5 \cdots) \quad , \quad \mathfrak{A}_2 = (3, 4, 5, 6 \cdots)$$

$$\mathfrak{A}_3 = (5, 6, 7, 8 \cdots).$$

Then

$$\mathfrak{C}_1 = 1 \qquad \mathfrak{C}_3 = (3, 4).$$

Now $\mathfrak{A}$, $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ are all enumerable sets; hence

$$\mathfrak{A} \sim \mathfrak{A}_2 \quad , \quad \mathfrak{A}_1 \sim \mathfrak{A}_3.$$

But obviously $\mathfrak{C}_1$ is not equivalent to $\mathfrak{C}_3$, since a set containing only one element cannot be put in 1 to 1 correspondence with a set consisting of two elements.

240. 1. *If* $\mathfrak{A} > \mathfrak{B} > \mathfrak{C}$, *and* $\mathfrak{A} \sim \mathfrak{C}$, *then* $\mathfrak{A} \sim \mathfrak{B}$.

For by hypothesis a part of $\mathfrak{B}$, viz. $\mathfrak{C}$, is $\sim \mathfrak{A}$. But a part of $\mathfrak{A}$ is $\sim \mathfrak{B}$, viz. $\mathfrak{B}$ itself. We apply now 239.

2. *Let α be any cardinal number. If*

$$\alpha \leq \text{Card } \mathfrak{B} \leq \alpha,$$

then

$$\alpha = \text{Card } \mathfrak{B}.$$

For let Card $\mathfrak{A} = \alpha$. Then from

$$\alpha \leq \text{Card } \mathfrak{B}$$

it follows that $\mathfrak{A} \sim$ a part or the whole of $\mathfrak{B}$; while from

$$\text{Card } \mathfrak{B} \leq \alpha$$

it follows that $\mathfrak{B}$ is $\sim$ a part or the whole of $\mathfrak{A}$.

3. *Any part $\mathfrak{B}$ of an enumerable set $\mathfrak{A}$ is enumerable.*

For if $\mathfrak{B}$ is finite, it is enumerable. If infinite,

$$\text{Card } \mathfrak{B} \geq \aleph_0.$$

On the other hand

$$\text{Card } \mathfrak{B} \leq \text{Card } \mathfrak{A} = \aleph_0.$$

4. *Two infinite enumerable sets are equivalent.*

For both are equivalent to $\mathfrak{J}$, the set of positive integers.

241. 1. *Let* $\mathfrak{E}$ *be any enumerable set in* $\mathfrak{A}$; *set* $\mathfrak{A} = \mathfrak{E} + \mathfrak{B}$. *If* $\mathfrak{B}$ *is infinite,* $\mathfrak{A} \sim \mathfrak{B}$.

For $\mathfrak{B}$ being infinite, contains an infinite enumerable set $\mathfrak{F}$. Let $\mathfrak{B} = \mathfrak{F} + \mathfrak{G}$. Then

$$\mathfrak{A} = \mathfrak{E} + \mathfrak{F} + \mathfrak{G},$$

$$\mathfrak{B} = \mathfrak{F} + \mathfrak{G}.$$

But $\mathfrak{E} + \mathfrak{F} \sim \mathfrak{F}$. Hence $\mathfrak{A} \sim \mathfrak{B}$.

2. We may state 1 thus:

$$\text{Card}\,(\mathfrak{A} - \mathfrak{E}) = \text{Card}\,\mathfrak{A}$$

provided $\mathfrak{A} - \mathfrak{E}$ *is infinite.*

3. From 1 follows at once the theorem:

Let $\mathfrak{A}$ *be any infinite set and* $\mathfrak{E}$ *an enumerable set. Then*

$$\text{Card}\,(\mathfrak{A} + \mathfrak{E}) = \text{Card}\,\mathfrak{A}.$$

Some Space Transformations

242. 1. Let T be a transformation of space such that to each point x corresponds a single point x_T, and conversely.

Moreover, let x, y be *any* two points of space. After the transformation they go over into x_T, y_T. If

$$\text{Dist}\,(x, y) = \text{Dist}\,(x_T, y_T)$$

we call T a *displacement.*

If the displacement is defined by

$$x_1' = x_1 + a_1 \quad , \quad \cdots \quad x_m' = x_m + a_m$$

it is called a *translation.*

If the displacement is such that all the points of a line in space remain unchanged by T, it is called a *rotation* whose *axis* is the fixed line.

If $\mathfrak{R}$ denotes the original space, and $\mathfrak{R}_T$ the transformed space after displacement, we have, obviously,

$$\mathfrak{R} \sim \mathfrak{R}_T.$$

2. Let

$$y_1 = tx_1 \quad , \quad \cdots \quad y_m = tx_m \quad , \quad t > 0. \tag{1}$$

Then when x ranges over the m-way space $\mathfrak{X}$, y ranges over an m-way space $\mathfrak{Y}$. If we set $x \sim y$ as defined by 1),

$$\mathfrak{X} \sim \mathfrak{Y}.$$

Also

$$\text{Dist }(0, y) = t \text{ Dist }(0, x).$$

We call 1) a transformation of *similitude*. If $t > 1$, a figure in space is *dilated;* if $t < 1$, it is *contracted.*

3. Let Q be any point in space. About it as center, let us describe a sphere S of radius R. Let P be any other point. On the join of P, Q let us take a point P' such that

$$\text{Dist }(P', Q) = \frac{R^2}{\text{Dist }(P, Q)}.$$

Then P' is called the *inverse of P with respect to S.* This transformation of space is called *inversion.* Q is the *center of inversion.*

Obviously points without S go over into points within, and conversely. As $P \doteq \infty$, $P' \doteq Q$.

The correspondence between the old and new spaces is uniform, except there is no point corresponding to Q.

The Cardinal $\mathfrak{c}$

243. 1. *All or any part of space $\mathfrak{S}$ may be put in uniform correspondence with a point set lying in a given cube C.*

For let $\mathfrak{S}_\iota$ denote the points within and on a unit sphere S about the origin, while $\mathfrak{S}_e$ denotes the other points of space. By an inversion we can transform $\mathfrak{S}_e$ into a figure $\mathfrak{S}_j$ lying in S. By a transformation of similitude we can contract $\mathfrak{S}_\iota$, $\mathfrak{S}_j$ as much as we choose, getting $\mathfrak{S}'_\iota$, $\mathfrak{S}'_j$. We may now displace these figures so as to bring them within C in such a way as to have no points in common, the contraction being made sufficiently great. The

correspondence between $\mathfrak{S}$ and the resulting aggregate is obviously uniform since all the transformations employed are.

As a result of this and 240, 1 we see that the aggregate of all real numbers is $\sim$ to those lying in the interval $(0, 1)$; for example, the aggregate of all points of $\mathfrak{R}_m$ is $\sim$ to the points in a unit cube, or a unit sphere, etc.

244. 1. *The points lying in the unit interval* $\mathfrak{A} = (0^*, 1^*)$ *are not enumerable.*

For if they were, they could be arranged in a sequence

$$a_1, a_2, a_3 \cdots \tag{1}$$

Let us express the a's as decimals in the normal form. Then

$$a_n = \cdot\, a_{n1} a_{n2} a_{n3} \cdots$$

Consider the decimal

$$b = \cdot\, b_1 b_2 b_3 \cdots$$

also written in the normal form, where

$$b_1 \neq a_{1,1} \quad , \quad b_2 \neq a_{2,2} \quad , \quad b_3 \neq a_{3,3} \quad , \quad \cdots$$

Then b lies in $\mathfrak{A}$ and is yet different from any number in 1).

2. We have $\quad (0^*, 1^*) \sim (0, 1) \quad$, by 241, 3,

$$\sim (a, b) \quad , \text{ by 243,}$$

where a, b are finite or infinite.

Thus the cardinal number of any interval, finite or infinite, with or without its end points is the same.

We denote it by $\mathfrak{c}$ and call it *the cardinal number of the rectilinear continuum*, or *of the real number system* $\mathfrak{R}$.

Since $\mathfrak{R}$ contains the rational number system R, we have

$$\mathfrak{c} > \aleph_0.$$

3. *The cardinal number of the irrational or of the transcendental numbers in any interval* $\mathfrak{A}$ *is also* $\mathfrak{c}$.

For the non-irrational numbers in $\mathfrak{A}$ are the rational which are enumerable; and the non-transcendental numbers in $\mathfrak{A}$ are the algebraic which are also enumerable.

4. *The cardinal number of the Cantor set* $\mathfrak{C}$ *of* I, 272 *is* c.

For each point a of $\mathfrak{C}$ has the representation in the triadic system

$$a = \cdot\, a_1 a_2 a_3 \cdots \quad , \quad a = 0, 2.$$

But if we read these numbers in the dyadic system, replacing each $a_n = 2$ by the value 1, we get all the points in the interval $(0, 1)$. As there is a uniform correspondence between these two sets of points, the theorem is established.

245. *An enumerable set* $\mathfrak{A}$ *is not perfect, and conversely a perfect set is not enumerable.*

For suppose the enumerable set

$$\mathfrak{A} = a_1, a_2 \cdots \tag{1}$$

were perfect. In $D_{r_1}{}^*(a_1)$ lies an infinite partial set $\mathfrak{A}_1$ of $\mathfrak{A}$, since by hypothesis $\mathfrak{A}$ is perfect. Let a_{m_2} be the point of lowest index in $\mathfrak{A}_1$. Let us take $r_2 < r_1$ such that $D_{r_2}(a_{m_2})$ lies in $D_{r_1}^*(a_1)$. In $D_{r_2}^*(a_{m_2})$ lies an infinite partial set $\mathfrak{A}_2$ of $\mathfrak{A}_1$. Let a_{m_3} be the point of lowest index in $\mathfrak{A}_2$, etc.

Consider now the sequence

$$a_1 \quad , \quad a_{m^2} \quad , \quad a_{m_3} \quad \cdots$$

It converges to a point α by I, 127, 2. But α lies in $\mathfrak{A}$, since this is perfect. Thus α is some point of 1), say $\alpha = a_s$. But this leads to a contradiction. For a_s lies in every $D_{r_{m_n}}^*(a{m_n})$; on the other hand, no point in this domain has an index as low as m_n which $\doteq \infty$, as $n \doteq \infty$. Thus $\mathfrak{A}$ cannot be perfect.

Conversely, suppose the perfect set $\mathfrak{A}$ were enumerable. This is impossible, for we have just seen that when $\mathfrak{A}$ is enumerable it cannot be perfect.

246. *Let* $\mathfrak{A}$ *be the union of an enumerable set of aggregates* $\mathfrak{A}_n$ *each having the cardinal number* c. *Then* Card $\mathfrak{A} =$ c.

For let $\mathfrak{B}_n$ denote the elements of $\mathfrak{A}_n$ not in $\mathfrak{A}_1, \mathfrak{A}_2 \cdots \mathfrak{A}_{n-1}$. Then

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{B}_2 + \mathfrak{B}_3 + \cdots$$

Let $\mathfrak{C}_n$ denote the interval $(n-1, n^*)$. Then the cardinal number of $\mathfrak{C}_1 + \mathfrak{C}_2 + \cdots$ is c.

But
$$\text{Card } \mathfrak{B}_n \leq \text{Card } \mathfrak{C}_n.$$
Hence
$$\text{Card } \mathfrak{A} \leq \mathfrak{c} \quad , \quad \text{by 231, 5.} \tag{1}$$
On the other hand,
$$\text{Card } \mathfrak{A} \geq \text{Card } \mathfrak{A}_1 = \mathfrak{c}. \tag{2}$$
From 1), 2) we have the theorem, by 240, 2.

247. 1. As already stated, the complex $x = (x_1, x_2, \cdots x_n)$ denotes a point in n-way space. Let $x_1, x_2, \cdots$ denote an infinite enumerable set. We may also say that the complex

$$x = (x_1, x_2, \cdots \text{ in inf.})$$

denotes *a point in* ∞ *-way space* $\mathfrak{R}_\infty$.

2. *Let* $\mathfrak{A}$ *denote a point set in* $\mathfrak{R}_n$, *n finite or infinite. Then*

$$\text{Card } \mathfrak{A} \leq \mathfrak{c}. \tag{1}$$

For let us first consider the unit cube $\mathfrak{C}$ whose coördinates x_m range over $\mathfrak{B} = (0^*, 1^*)$. Let $\mathfrak{D}$ denote the diagonal of $\mathfrak{C}$. Then

$$\mathfrak{c} = \text{Card } \mathfrak{D} \leq \text{Card } \mathfrak{C}. \tag{2}$$

On the other hand we show Card $\mathfrak{C} \leq \mathfrak{c}$.

For let us express each coördinate x_m as a decimal in normal form. Then

$$x_1 = \cdot\, a_{11}a_{12}a_{13}a_{14} \cdots$$
$$x_2 = \cdot\, a_{21}a_{22}a_{23}a_{24} \cdots$$
$$x_3 = \cdot\, a_{31}a_{32}a_{33}a_{34} \cdots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

Let us now form the number

$$y = \cdot\, a_{11}a_{12}a_{21}a_{13}a_{22}a_{23} \cdots$$

obtained by reading the above table diagonally. Let $\mathfrak{Y}$ denote the set of y's so obtained as the x's range over their values. Then

$$\mathfrak{Y} < \mathfrak{B}.$$

For the point y, for example, in which $a_{1n} = 0$, $n = 1, 2, \cdots$ lies in $\mathfrak{B}$ but not in $\mathfrak{Y}$ as otherwise $x_1 = 0$. Let us now set $x \sim y$. Then $\mathfrak{C} \sim \mathfrak{Y}$ and hence

$$\text{Card } \mathfrak{C} \leq \mathfrak{c}. \tag{3}$$

From 2), 3) we have
$$\text{Card } \mathfrak{C} = \mathfrak{c}.$$

Let us now complete $\mathfrak{C}$ by adding its faces, obtaining the set C. By a transformation of similitude T we can bring C_T within $\mathfrak{C}$. Hence

$$\text{Card } \mathfrak{C} \geq \text{Card } C.$$

On the other hand, $\mathfrak{C}$ is a part of C, hence

$$\text{Card } \mathfrak{C} \leq \text{Card } C.$$

Thus Card $C = \mathfrak{c}$. The rest of the theorem follows now easily.

248. *Let $\mathfrak{F} = \{f\}$ denote the aggregate of one-valued continuous functions over a unit cube $\mathfrak{C}$ in $\mathfrak{R}_n$.*

Then

$$\text{Card } \mathfrak{F} = \mathfrak{c}.$$

Let C denote the rational points of $\mathfrak{C}$, *i.e.* the points all of whose coördinates are rational. Then any f is known when its values over C are known. For if α is an irrational point of $\mathfrak{C}$, we can approach it over a sequence of rational points $a_1, a_2 \cdots \doteq \alpha$. But f being continuous, $f(\alpha) = \lim f(a_n)$, and f is known at α. On the other hand, C being enumerable, we can arrange its points in a sequence

$$C = c_1, c_2, \cdots$$

Let now $\mathfrak{R}_\infty$ be a space of an infinite enumerable number of dimensions, and let $y = (y_1, y_2 \cdots)$ denote any one of its points. Let f have the value η_1 at c_1, the value η_2 at c_2 and so on for the points of C. Then the complex $\eta_1, \eta_2, \cdots$ completely determines f in $\mathfrak{C}$. But this complex also determines the point $\eta = (\eta_1, \eta_2 \cdots)$ in $\mathfrak{R}_\infty$. We now associate f with η. Thus

$$\text{Card } \mathfrak{F} \leq \text{Card } \mathfrak{R} \ \ = \mathfrak{c}.$$

But obviously Card $\mathfrak{F} \geq \mathfrak{c}$, for among the elements of $\mathfrak{F}$ there is an f which takes on any given value in the interval $(0, 1)$, at a given point of $\mathfrak{C}$.

249. *There exist aggregates whose cardinal number is greater than any given cardinal number.*

Let $\mathfrak{B} = \{b\}$ be an aggregate whose cardinal number $\mathfrak{b}$ is given. Let a be a symbol so related to $\mathfrak{B}$ that it has arbitrarily either the value 1 or 2 corresponding to each b of $\mathfrak{B}$. Let $\mathfrak{A}$ denote the

aggregate formed of all possible a's of this kind, and let $\mathfrak{a}$ be its cardinal number.

Let β be an arbitrary element of $\mathfrak{B}$. Let us associate with β that a which has the value 1 for $b = \beta$ and the value 2 for all other b's. This establishes a correspondence between $\mathfrak{B}$ and a part of $\mathfrak{A}$. Hence

$$\mathfrak{a} \geq \mathfrak{b}.$$

Suppose $\mathfrak{a} = \mathfrak{b}$. Then there exists a correspondence which associates with each b some one a and conversely. This is impossible.

For call a_b that element of $\mathfrak{A}$ which is associated with b. Then a_b has the value 1 or 2 for each β of $\mathfrak{B}$. There exists, however, in $\mathfrak{A}$ an element a' which for each β of $\mathfrak{B}$ has just the other determination than the one a_b has. But a' is by hypothesis associated with some element of $\mathfrak{B}$, say that

$$a' = a_{b'}.$$

Then for $b = b'$, a' must have that one of the two values 1, 2 which $a_{b'}$ has. But it has not, hence the contradiction.

250. *The aggregate of limited integrable functions* $\mathfrak{F}$ *defined over* $\mathfrak{A} = (0, 1)$ *has a cardinal number* $\mathfrak{f} > \mathfrak{c}$.

For let $f(x) = 0$ in $\mathfrak{A}$ except at the points $\mathfrak{C}$ of the discrete Cantor set of I, 272, and 229, Ex. 4. At each point of $\mathfrak{C}$ let f have the value 1 or 2 at pleasure. The aggregate $\mathfrak{G}$ formed of all possible such functions has a cardinal number $> \mathfrak{c}$, as the reasoning of 249 shows. But each f is continuous except in $\mathfrak{C}$, which is discrete. Hence f is integrable. But $\mathfrak{F} > \mathfrak{G}$. Hence

$$\mathfrak{f} > \mathfrak{c}.$$

Arithmetic Operations with Cardinals

251. *Addition of Cardinals.* Let $\mathfrak{A}$, $\mathfrak{B}$ be two aggregates without common element, whose cardinal numbers are $\mathfrak{a}$, $\mathfrak{b}$. We define the sum of $\mathfrak{a}$ and $\mathfrak{b}$ to be

$$\text{Card}(\mathfrak{A}, \mathfrak{B}) = \mathfrak{a} + \mathfrak{b}.$$

We have now the following obvious relations:

$$\aleph_0 + n = \aleph_0 \quad , \quad n \text{ a positive integer.} \tag{1}$$

$$\aleph_0 + \aleph_0 + \cdots + \aleph_0 = \aleph_0 \quad , \quad n \text{ terms.} \tag{2}$$

$$\aleph_0 + \aleph_0 + \cdots = \aleph_0 \quad , \quad \text{an infinite enumerable set of terms.} \tag{3}$$

If the cardinal numbers of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ *are* $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, *then* *

$$\mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} + \mathfrak{b}) + \mathfrak{c},$$

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}.$$

The first relation states that addition is *associative*, the second that it is *commutative.*

252. *Multiplication.*

1. Let $\mathfrak{A} = \{a\}$, $\mathfrak{B} = \{b\}$ have the cardinal numbers $\mathfrak{a}$, $\mathfrak{b}$. The union of all the pairs (a, b) forms a set called the *product of* $\mathfrak{A}$ *and* $\mathfrak{B}$. It is denoted by $\mathfrak{A} \cdot \mathfrak{B}$. We agree that (a, b) shall be the same as (b, a). Then

$$\mathfrak{A} \cdot \mathfrak{B} = \mathfrak{B} \cdot \mathfrak{A}.$$

We define the product of $\mathfrak{a}$ and $\mathfrak{b}$ to be

$$\text{Card } \mathfrak{A} \cdot \mathfrak{B} = \text{Card } \mathfrak{B} \cdot \mathfrak{A} = \mathfrak{a} \cdot \mathfrak{b} = \mathfrak{b} \cdot \mathfrak{a}.$$

2. *We have obviously the following formal relations as in finite cardinal numbers:*

$$\mathfrak{a}(\mathfrak{b} \cdot \mathfrak{c}) = (\mathfrak{a} \cdot \mathfrak{b})\mathfrak{c},$$

$$\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{b} \cdot \mathfrak{a},$$

$$\mathfrak{a}(\mathfrak{b} + \mathfrak{c}) = \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c},$$

which express respectively the associative, commutative, and distripulative properties of cardinal numbers.

Example 1. Let $\mathfrak{A} = \{a\}$, $\mathfrak{B} = \{b\}$ denote the points on two indefinite right lines. Then

$$\mathfrak{A} \cdot \mathfrak{B} = \{(a, b)\}.$$

If we take a, b to be the coördinates of a point in a plane $\mathfrak{R}_2$, then $\mathfrak{A} \cdot \mathfrak{B} = \mathfrak{R}_2$.

* The reader should note that here, as in the immediately following articles, $\mathfrak{c}$ is simply the cardinal number of $\mathfrak{C}$ which is any set, like $\mathfrak{A}$, $\mathfrak{B}$ ...

Example 2. Let $\mathfrak{A}=\{a\}$ denote the family of circles

$$x^2+y^2=a^2. \tag{1}$$

Let $\mathfrak{B}=\{b\}$ denote a set of segments of length b. We can interpret (a, b) to be the points on a cylinder whose base is 1) and whose height is b. Then $\mathfrak{A}\cdot\mathfrak{B}$ is the aggregate of these cylinders.

253. 1. $\aleph_0 = n\cdot\aleph_0$, *or* $n\mathfrak{e}=\mathfrak{e}$. (1

For let

$$\mathfrak{N}=(a_1, a_2, \cdots a_n),$$

$$\mathfrak{E}=(e_1, e_2 \cdots \text{in inf.})$$

Then

$$\begin{aligned}\mathfrak{N}\cdot\mathfrak{E} = &(a_1, e_1) \quad , \quad (a_1, e_2) \quad , \quad (a_1, e_3)\cdots\\ &(a_2, e_1) \quad , \quad (a_2, e_2) \quad , \quad (a_2, e_3)\cdots\\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ =&\mathfrak{E}_1+\mathfrak{E}_2+\cdots+\mathfrak{E}_n.\end{aligned}$$

The cardinal number of the set on the left is $n\aleph_0$, while the cardinal number of the set on the right is $\aleph_0$.

2. $$\mathfrak{e}\mathfrak{c}=\mathfrak{c}. \tag{2}$$

For let $\mathfrak{C}=\{c\}$ denote the points on a right line, and $\mathfrak{E}=(1, 2, 3, \cdots)$.

Then

$$\mathfrak{E}\mathfrak{C}=\{(n, c)\}$$

may be regarded as the points on a right line l_n. Obviously,

$$\text{Card }\{l_n\}=\mathfrak{c}.$$

Hence

$$\mathfrak{e}\mathfrak{c}=\text{Card }\mathfrak{E}\mathfrak{C}=\mathfrak{c}.$$

254. *Exponents.* Before defining this notion let us recall a problem in the theory of combinations, treated in elementary algebra.

Suppose that there are γ compartments

$$C_1, C_2, \cdots C_\gamma,$$

and that we have k classes of objects

$$K_1, K_2, \cdots K_k.$$

Let us place an object from any one of these classes in C_1, an object from any one of these classes in $C_2 \cdots$ and so on, for each compartment. The result is a certain *distribution* of the objects from these k classes K, among the γ compartments C.

The number of distributions of objects from k classes among γ compartments is k^γ.

For in C_1 we may put an object from any one of the k classes. Thus C_1 may be filled in k ways. Similarly C_2 may be filled in k ways. Thus the compartments C_1, C_2 may be filled in k^2 ways. Similarly C_1, C_2, C_3 may be filled in k^3 ways, etc.

255. 1. The totality of distributions of objects from k classes K among the γ compartments C form an aggregate which may be denoted by

$$K^C.$$

We call it the *distribution of K over C.* The number of distribution of this kind may be called the cardinal number of the set, and we have then

$$\text{Card } K^C = k^\gamma.$$

2. What we have here set forth for finite C and K may be extended to any aggregates, $\mathfrak{A} = \{a\}$, $\mathfrak{B} = \{b\}$ whose cardinal numbers we call $\mathfrak{a}$, $\mathfrak{b}$. Thus the totality of distributions of the a's among the b's, or the *distribution of $\mathfrak{A}$ over $\mathfrak{B}$*, is denoted by

$$\mathfrak{A}^{\mathfrak{B}},$$

and its cardinal number is taken to be the definition of the symbol $\mathfrak{a}^{\mathfrak{b}}$. Thus,

$$\text{Card} \cdot \mathfrak{A}^{\mathfrak{B}} = \mathfrak{a}^{\mathfrak{b}}.$$

256. *Example 1.* Let

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0 \qquad (1$$

have rational number coefficients. Each coefficient a_s can range over the enumerable set of elements in the rational number system $R = \{r\}$, whose cardinal number is $\aleph_0$. The n coefficients form a set $\mathfrak{A} = (a_1, \cdots a_n) = \{a\}$. To the totality of equations 1) corresponds a distribution of the r's among the a's, or the set

$$R^{\mathfrak{A}}$$

whose cardinal number is

$$\aleph_0^n = \mathfrak{e}^n.$$

As
$$\text{Card } R^{\mathfrak{A}} = \aleph_0 = \mathfrak{e}$$
we have the relation :
$$\aleph_0^n = \aleph_0 \quad , \quad \text{or } \mathfrak{e}^n = \mathfrak{e}$$
for any integer n.

On the other hand, the equations 1) may be associated with the complex
$$(a_1, \cdots a_n),$$
and the totality of equations 1) is associated with
$$\mathfrak{C} = \{(a_1, \cdots a_n)\}.$$
But
$$\{(a_1, a_2)\} = \{a_1\} \cdot \{a_2\},$$
$$\{(a_1, a_2, a_3)\} = \{(a_1, a_2)\} \cdot \{a_3\} \quad , \quad \text{etc.}$$
Hence
$$\mathfrak{C} = \{a_1\} \cdot \{a_2\} \cdots \{a_n\}.$$
Thus
$$\text{Card } \mathfrak{C} = \mathfrak{e} \cdot \mathfrak{e} \cdot \cdots \mathfrak{e} \quad , \quad n \textit{ times as factor.}$$
But
$$\text{Card } \mathfrak{C} = \text{Card } R^{\mathfrak{A}},$$
since each of these sets is associated uniformly with the equations 1). Thus
$$\mathfrak{e}^n = \mathfrak{e} \cdot \mathfrak{e} \cdot \cdots \mathfrak{e} \quad , \quad n \textit{ times as factor.}$$

257. *Example 2.* Any point x in m-way space $\mathfrak{R}_m$ depends on m coördinates $x_1, x_2, \cdots x_m$, each of which may range over the set of real numbers $\mathfrak{R}$, whose cardinal number is $\mathfrak{c}$. The m coördinates $x_1 \cdots x_m$ form a finite set
$$\mathfrak{X} = (x_1, \cdots x_m).$$
Thus to $\mathfrak{R}_m = \{x\}$ corresponds the distribution of the numbers in $\mathfrak{R}$, among the m elements of $\mathfrak{X}$, or the set
$$\mathfrak{R}^{\mathfrak{X}}$$
whose cardinal number is
$$\mathfrak{c}^m.$$
As
$$\text{Card } \mathfrak{R}^{\mathfrak{X}} = \mathfrak{c}$$
we have
$$\mathfrak{c}^m = \mathfrak{c} \qquad \textit{for any integer } m. \qquad (1$$
As in Example 1 we show
$$\mathfrak{c}^m = \mathfrak{c} \cdot \mathfrak{c} \cdot \cdots \mathfrak{c} \quad , \quad m \textit{ times as factor.}$$

258. $$\mathfrak{a}^{\mathfrak{b}+\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}} \cdot \mathfrak{a}^{\mathfrak{c}}. \tag{1}$$

To prove this we have only to show that

$$\mathfrak{A}^{\mathfrak{B}+\mathfrak{C}} \text{ and } \mathfrak{A}^{\mathfrak{B}} \cdot \mathfrak{A}^{\mathfrak{C}}$$

can be put in 1–1 correspondence. But this is obvious. For the set on the left is the totality of all the distributions of the elements of $\mathfrak{A}$ among the sets formed of $\mathfrak{B}$ and $\mathfrak{C}$. On the other hand, the set on the right is formed of a combination of a distribution of the elements of $\mathfrak{A}$ among the $\mathfrak{B}$, and among the $\mathfrak{C}$. But such a distribution may be regarded as the distribution first considered.

259. $$(\mathfrak{a}^{\mathfrak{b}})^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}\cdot\mathfrak{c}}. \tag{1}$$

We have only to show that we can put in 1–1 correspondence the elements of

$$(\mathfrak{A}^{\mathfrak{B}})^{\mathfrak{C}} \text{ and } \mathfrak{A}^{\mathfrak{B}\cdot\mathfrak{C}}. \tag{2}$$

Let $\mathfrak{A} = \{a\}$, $\mathfrak{B} = \{b\}$, $\mathfrak{C} = \{c\}$. We note that $\mathfrak{A}^{\mathfrak{B}}$ is a union of distributions of the a's among the b's, and that the left side of 2) is formed of the distributions of these sets among the c's. These are obviously associated uniformly with the distributions of the a's among the elements of $\mathfrak{B} \cdot \mathfrak{C}$.

260. 1. $$\mathfrak{c}^n = (m^{\mathfrak{e}})^n = m^{n\mathfrak{e}} = m^{\mathfrak{e}} = \mathfrak{c} \tag{1}$$

where m, n are positive integers.

For each number in the interval $\mathfrak{C} = (0, 1^*)$ can be represented in normal form once and once only by

$$\cdot\, a_1 a_2 a_3 \cdots \text{ in the } m\text{-adic system,} \tag{2}$$

where the $\qquad 0 \leq a_s < m. \qquad$ [I, 145].

Now the set of numbers 2) is the distribution of $\mathfrak{M} = (0, 1, 2, \cdots m-1)$ over $\mathfrak{E} = (a_1, a_2, a_3 \cdots)$, or

$$\mathfrak{M}^{\mathfrak{E}}$$

whose cardinal number is

$$m^{\mathfrak{e}}.$$

On the other hand, the cardinal number $\mathfrak{C}$ is $\mathfrak{c}$.

Hence, $$m^{\mathfrak{e}} = \mathfrak{c}.$$

As $n^{\mathfrak{e}} = \mathfrak{e}$, we have 1), using 1) in 257.

2. *The result obtained in* 247, 2 *may be stated:*

$$\mathfrak{c}^{\mathfrak{e}} = \mathfrak{c}. \tag{3}$$

3. $$\mathfrak{e}^{\mathfrak{e}} = \mathfrak{c}. \tag{4}$$

For obviously $$n^{\mathfrak{e}} \leq \mathfrak{e}^{\mathfrak{e}} \leq \mathfrak{c}^{\mathfrak{e}}.$$

But by 3), $\mathfrak{c}^{\mathfrak{e}} = \mathfrak{c}$ and by 1) $n^{\mathfrak{e}} = \mathfrak{c}$.

261. 1. *The cardinal number* $\mathfrak{t}$ *of all functions* $f(x_1 \cdots x_m)$ *which take on but two values in the domain of definition* $\mathfrak{A}$, *of cardinal number* $\mathfrak{a}$, *is* $2^{\mathfrak{A}}$.

Moreover, $$2^{\mathfrak{A}} > \mathfrak{a}.$$

This follows at once from the reasoning of 249.

2. *Let* $\mathfrak{f}$ *be the cardinal number of the class of all functions defined over a domain* $\mathfrak{A}$ *whose cardinal number is* $\mathfrak{c}$. *Then*

$$\mathfrak{f} = \mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}} > \mathfrak{c}. \tag{1}$$

For the class of functions which have but two values in $\mathfrak{A}$ is by 1, $2^{\mathfrak{c}}$.

On the other hand, obviously

$$\mathfrak{f} = \mathfrak{c}^{\mathfrak{c}}.$$

But

$$\begin{aligned} \mathfrak{c}^{\mathfrak{c}} &= (2^{\mathfrak{e}})^{\mathfrak{c}}, && \text{by 260, 1)} \\ &= 2^{\mathfrak{e}\mathfrak{c}}, && \text{by 259, 1)} \\ &= 2^{\mathfrak{c}}, && \text{by 253, 2).} \end{aligned}$$

Thus, $$\mathfrak{c}^{\mathfrak{c}} = 2^{\mathfrak{c}}.$$

That $$\mathfrak{f} > \mathfrak{c}$$

follows from 250, since the class of functions there considered lies in the class here considered.

3. *The cardinal number* $\mathfrak{i}$ *of the class of limited integrable functions in the interval* $\mathfrak{A}$ *is* $= \mathfrak{f}$, *the cardinal number of all limited functions defined over* $\mathfrak{A}$.

For let $\mathfrak{D}$ be a Cantor set in $\mathfrak{A}$ [I, 272]. Being discrete, any limited function defined over $\mathfrak{D}$ is integrable. But by 229, Ex. 4, the points of $\mathfrak{A}$ may be set in uniform correspondence with the points of $\mathfrak{D}$.

4. *The set of all functions*

$$f(x) = f_1(x) + f_2(x) + \cdots \tag{2}$$

which are the sum of convergent series, and whose terms are continuous in $\mathfrak{A}$, *has the cardinal number* $\mathfrak{c}$.

For the set $\mathfrak{F}$ of continuous functions in $\mathfrak{A}$ has the cardinal number $\mathfrak{c}$ by 248. These functions are to be distributed among the enumerable set $\mathfrak{E}$ of terms in 2). Hence the set of these functions is

$$\mathfrak{F}^{\mathfrak{E}},$$

whose cardinal number is

$$\mathfrak{c}^{\mathfrak{e}} = \mathfrak{c}.$$

Remark. Not every integrable function can be represented by the series 2).

For the class of integrable functions has a cardinal number $> \mathfrak{c}$, by 250.

5. *The cardinal number of all enumerable sets in an m-way space* $\mathfrak{R}_m$ *is* $\mathfrak{c}$.

For it is obviously the cardinal number of the distribution of $\mathfrak{R}_m$ over an enumerable set $\mathfrak{E}$, or

$$\text{Card } \mathfrak{R}_m^{\mathfrak{E}} = \mathfrak{c}^{\mathfrak{e}} = \mathfrak{c}.$$

Numbers of Liouville

262. In I, 200 we have defined algebraic numbers as roots of equations of the type

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0 \tag{1}$$

where the coefficients a are integers. All other numbers in $\mathfrak{R}$ we said were transcendental. We did not take up the question whether there are any transcendental numbers, whether in fact, not all numbers in $\mathfrak{R}$ are roots of equations of the type 1).

The first to actually show the existence of transcendental numbers was *Liouville*. He showed how to form an infinity of such numbers. At present we have practical means of deciding whether a given number is algebraic or not. It was one of the signal achievements of *Hermite* to have shown that $e = 2.71818 \cdots$ is transcendental.

Shortly after *Lindemann*, adapting Hermite's methods, proved that $\pi = 3.14159 \cdots$ is also transcendental. Thereby that famous problem the *Quadrature of the Circle* was answered in the negative. The researches of Hermite and Lindemann enable us also to form an infinity of transcendental numbers. It is, however, not our purpose to give an account of these famous results. We shall limit our considerations to certain numbers which we call the numbers of Liouville.

In passing let us note that the existence of transcendental numbers follows at once from 235, 2 and 244, 2. b 288

For the cardinal number of the set of real algebraic number is $\mathfrak{e}$, and that of the set of all real numbers is $\mathfrak{c}$, and $\mathfrak{c} > \mathfrak{e}$.

263. In algebra it is shown that any algebraic number α is a root of an *irreducible* equation,

$$f(x) = a_0 x^m + a_1 x^{m-1} + \cdots + a_m = 0 \tag{1}$$

whose coefficients are integers without common divisor. We say the *order of α is m*.

We prove now the theorem

Let

$$r_n = \frac{p_n}{q_n} \quad , \quad p_n, q_n \textit{ relatively prime,}$$

$\doteq \alpha$, *an algebraic number of order m, as $n \doteq \infty$. Then*

$$|\alpha - r_n| > \frac{1}{q_n^{m-1}} \quad , \quad n > \nu. \tag{2}$$

For let α be a root of 1). We may take $\delta > 0$ so small that $f(x) \neq 0$ in $D_\delta^*(\alpha)$, and s so large that r_n lies in $D_\delta(\alpha)$, for $n > s$. Thus

$$|f(r_n)| = \left| \frac{a_0 p_n^m + a_1 p_n^{m-1} q_n + \cdots + a_m q_n^m}{q_n^m} \right| \geq \frac{1}{q_n^m}, \tag{3}$$

for $n > s$, since the numerator of the middle member is an integer, and hence ≥ 1.

On the other hand, by the Law of the Mean [I, 397],

$$f(r_n) - f(\alpha) = (r_n - \alpha) f'(\beta)$$

where β lies in $D_\delta(\alpha)$. Now $f(\alpha) = 0$ and $f'(\beta) <$ some M. Hence

$$|r_n - \alpha| > \frac{f(r_n)}{M} \geq \frac{1}{M q_n^m}, \tag{4}$$

on using 3). But however large M is, there exists a ν, such that $q_n > M$, for any $n > \nu$. This in 4) gives 2).

264. 1. *The numbers*

$$L = \frac{a_1}{10^{1!}} + \frac{a_2}{10^{2!}} + \frac{a_3}{10^{3!}} + \cdots \tag{1}$$

where $a_n < 10^n$, and not all of them vanish after a certain index, are transcendental.

For if L is algebraic, let its order be m. Then if L_n denotes the sum of the first n terms of 1), there exists a ν such that

$$\eta = |L - L_n| > \frac{1}{10^{(m+1)n!}} \quad , \quad \text{for } n > \nu. \tag{2}$$

But

$$\eta = \frac{a_{n+1}}{10^{(n+1)!}} + \cdots < \frac{1}{10^{(m+1)n!}} \quad , \quad n > \nu', \tag{3}$$

ν' being taken sufficiently large. But 3) contradicts 2).

The numbers 1) we call the *numbers of Liouville.*

2. *The set of Liouville numbers has the cardinal number* $\mathfrak{c}$.

For all real numbers in the interval $(0^*, 1)$ can be represented by

$$\beta = \frac{b_1}{10^1} + \frac{b_2}{10^2} + \frac{b_3}{10^3} + \cdots \quad , \quad 0 \leq b_n \leq 9,$$

where not all the b's vanish after a certain index. The numbers

$$\lambda = \frac{b_1}{10^{1!}} + \frac{b_2}{10^{2!}} + \frac{b_3}{10^{3!}} + \cdots$$

can obviously be put in uniform correspondence with the set $\{\beta\}$. Thus Card $\{\lambda\} = \mathfrak{c}$. But $\{L\} > \{\lambda\}$, hence Card $\{L\} \geq \mathfrak{c}$. On the other hand, the numbers $\{L\}$ form only a part of the numbers in $(0^*, 1)$. Hence Card $\{L\} \leq \mathfrak{c}$.

CHAPTER IX

ORDINAL NUMBERS

Ordered Sets

265. An aggregate $\mathfrak{A}$ is *ordered*, when a, b being any two of its elements, either a precedes b, or a succeeds b, according to some law; such that if a precedes b, and b precedes c, then a shall precede c. The fact that a precedes b may be indicated by

$$a < b.$$

Then

$$a > b$$

states that a succeeds b.

Example 1. The aggregates

$$1, 2, 3, \cdots$$

$$2, 4, 6, \cdots$$

$$a_1, a_2, a_3, \cdots$$

$$\cdots -3, -2, -1, 0, 1, 2, 3, \cdots$$

$$\cdots a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, \cdots$$

are ordered.

Example 2. The rational number system R can be ordered in an infinite variety of ways. For, being enumerable, they can be arranged in a sequence

$$r_1, r_2, r_3, \cdots r_n \cdots$$

Now interchange r_1 with r_n. In this way we obtain an infinity of sequences.

Example 3. The points of the circumference of a circle may be ordered in an infinite variety of ways.

For example, let two of its points P_1, P_2 make the angles $\alpha + \theta_1$, $\alpha + \theta_2$ with a given radius, the angle θ varying from 0 to 2π. Let P_1 precede P_2 when $\theta_1 < \theta_2$.

Example 4. The positive integers $\mathfrak{J}$ may be ordered in an infinite variety of ways besides their natural order. For instance, we may write them in the order

$$1,\ 3,\ 5,\ \cdots\ 2,\ 4,\ 6,\ \cdots$$

so that the odd numbers precede the even. Or in the order

$$1,\ 4,\ 7,\ 10,\ \cdots\ 2,\ 5,\ 8,\ 11,\ \cdots\ 3,\ 6,\ 9,\ 12,\ \cdots$$

and so on. We may go farther and arrange them in an infinity of sets. Thus in the first set put all primes; in the second set the products of two primes; in the third set the products of three primes; etc., allowing repetitions of the factors. Let any number in set m precede all the numbers in set $n > m$. The numbers in each set may be arranged in order of magnitude.

Example 5. The points of the plane $\mathfrak{R}_2$ may be ordered in an infinite variety of ways. Let L_y denote the right line parallel to the x-axis at a distance y from this axis, taking account of the sign of y. We order now the points of $\mathfrak{R}_2$ by stipulating that any point on $L_{y'}$ precedes the points on any $L_{y''}$ when $y' < y''$, while points on any L_y shall have the order they already possess on that line due to their position.

266. *Similar Sets.* Let $\mathfrak{A}$, $\mathfrak{B}$ be ordered and equivalent. Let $a \sim b$, $\alpha \sim \beta$. If when $a < \alpha$ in $\mathfrak{A}$, $b < \beta$ in $\mathfrak{B}$, we say $\mathfrak{A}$ is *similar* to $\mathfrak{B}$, and write

$$\mathfrak{A} \simeq \mathfrak{B}.$$

Thus the two ordered and equivalent aggregates are similar when corresponding elements in the two sets occur in the same relative order.

Example 1. Let

$$\mathfrak{A} = 1,\ 2,\ 3,\ \cdots$$

$$\mathfrak{B} = a_1,\ a_2,\ a_3,\ \cdots$$

In the correspondence $\mathfrak{A} \sim \mathfrak{B}$, let n be associated with a_n. Then $\mathfrak{A} \simeq \mathfrak{B}$.

Example 2. Let

$$\mathfrak{A} = 1,\ 2,\ 3,\ \cdots$$

$$\mathfrak{B} = a_1\ a_2 \cdots a_m,\ b_1,\ b_2,\ b_3 \cdots$$

In the correspondence $\mathfrak{A} \sim \mathfrak{B}$, let $a_r \sim r$ for $r = 1, 2, \cdots m$; also let $b_n \sim m + n$, $n = 1, 2 \cdots$ Then $\mathfrak{A} \simeq \mathfrak{B}$.

Example 3. Let

$$\mathfrak{A} = 1, 2, 3, \cdots$$

$$\mathfrak{B} = b_1, b_2 \cdots a_1, a_2 \cdots a_m.$$

Let the correspondence between $\mathfrak{A}$ and $\mathfrak{B}$ be the same as in Ex. 2. Then $\mathfrak{A}$ is not similar to $\mathfrak{B}$. For 1 is the first element in $\mathfrak{A}$ while its associated element a_1 is not first in $\mathfrak{B}$.

Example 4. Let

$$\mathfrak{A} = 1, 2, 3, \cdots$$

$$\mathfrak{B} = a_1, a_2 \cdots b_1, b_2 \cdots$$

Let $a_n \sim 2n$, $b_n \sim 2n - 1$. Then $\mathfrak{A} \sim \mathfrak{B}$ but $\mathfrak{A}$ is not $\simeq \mathfrak{B}$.

267. *Let* $\mathfrak{A} \simeq \mathfrak{B}$, $\mathfrak{B} \simeq \mathfrak{C}$. *Then* $\mathfrak{A} \simeq \mathfrak{C}$.

For let $a \sim b$, $a' \sim b'$ in $\mathfrak{A} \sim \mathfrak{B}$. Let $b \sim c$, $b' \sim c'$ in $\mathfrak{B} \sim \mathfrak{C}$. Let us establish a correspondence $\mathfrak{A} \sim \mathfrak{C}$ by setting $a \sim c$, $a' \sim c'$. Then if $a < a'$ in $\mathfrak{A}$, $c < c'$ in $\mathfrak{C}$. Hence $\mathfrak{A} \simeq \mathfrak{C}$.

Eutactic Sets

268. Let $\mathfrak{A}$ be any ordered aggregate, and $\mathfrak{B}$ a part of $\mathfrak{A}$, the elements of $\mathfrak{B}$ being kept in the same relative order as in $\mathfrak{A}$. If $\mathfrak{A}$ and each $\mathfrak{B}$ both have a first element, we say that $\mathfrak{A}$ is *well ordered*, or *eutactic*.

Example 1. $\mathfrak{A} = 2, 3, \cdots 500$ is well ordered. For it has a first element 2. Moreover any part of $\mathfrak{A}$ as 6, 15, 25, 496 also has a first element.

Example 2. $\mathfrak{A} = 12, 13, 14, \cdots$ in inf. is well ordered. For it has a first element 12, and any part $\mathfrak{B}$ of $\mathfrak{A}$ whose elements preserve the same relative order as in $\mathfrak{A}$, has a first element, viz. the least number in $\mathfrak{B}$.

The condition that the elements of $\mathfrak{B}$ should keep the same relative order as in $\mathfrak{A}$ is necessary. For $B = \cdots 28, 26, 24, 22, 20, 21, 23, 25, 27, \ldots$ is a partial aggregate having no first element. But the elements of B do not have the order they have in $\mathfrak{A}$.

Example 3. Let $\mathfrak{A}$ = rational numbers in the interval (0, 1) arranged in their order of magnitude. Then $\mathfrak{A}$ is ordered. It also has a first element, viz. 0. It is not well ordered however. For the partial set $\mathfrak{B}$ consisting of the *positive* rational numbers of $\mathfrak{A}$ has no first element.

Example 4. An ordered set which is not well ordered may sometimes be made so by ordering its elements according to another law.

Thus in Ex. 3, let us arrange $\mathfrak{A}$ in a manner similar to 233. Obviously $\mathfrak{A}$ is now well ordered.

Example 5. $\mathfrak{A} = a_1, a_2 \cdots b_1, b_2 \cdots$ is well ordered. For a_1 is the first element of $\mathfrak{A}$; and any part of $\mathfrak{A}$ as

$$a_1, \ a_2 \cdots$$

$$b_1, \ b_2 \cdots$$

$$a_{\iota_1}, \ a_{\iota_2} \cdots b_{\kappa_1}, \ b_{\kappa_2} \cdots$$

has a first element.

269. 1. *Every partial set $\mathfrak{B}$ of a well-ordered aggregate $\mathfrak{A}$ is well ordered.*

For $\mathfrak{B}$ has a first element, since it is a part of $\mathfrak{A}$ which is well ordered. If $\mathfrak{C}$ is a part of $\mathfrak{B}$, it is also a part of $\mathfrak{A}$, and hence has a first element.

2. *If a is not the last element of a well-ordered aggregate $\mathfrak{A}$, there is an element of $\mathfrak{A}$ immediately following a.*

For let $\mathfrak{B}$ be the part of $\mathfrak{A}$ formed of the elements after a. It has a first element b since $\mathfrak{A}$ is well ordered. Suppose now

$$a < c < b.$$

Then b is not the first element of $\mathfrak{B}$, as $c < b$ is in $\mathfrak{B}$.

3. When convenient the element immediately succeeding a may be denoted by

$$a + 1.$$

Similarly we may denote the element immediately preceding a, when it exists, by

$$a - 1.$$

For example, let

$$\mathfrak{A} = a_1 a_2 \cdots b_1 b_2 \cdots$$

Then
$$a_n + 1 = a_{n+1} \quad , \quad b_m + 1 = b_{m+1}$$
$$a_n - 1 = a_{n-1} \quad , \quad b_m - 1 = b_{m-1}.$$

There is, however, no $b_1 - 1$.

270. 1. *If $\mathfrak{A}$ is well ordered, it is impossible to pick out an infinite sequence of the type*

$$a_1 > a_2 > a_3 > \cdots \tag{1}$$

For
$$\mathfrak{B} = \cdots a_3, a_2, a_1$$

is a part of $\mathfrak{A}$ whose elements occur in the same relative order as in $\mathfrak{A}$, and $\mathfrak{B}$ has no first element.

2. A sequence as 1) may be called a *decreasing sequence*, while

$$a_1 < a_2 < a_3 \cdots$$

may be called *increasing.*

In every infinite well ordered aggregate there exist increasing sequences.

3. *Let $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \cdots$ be a well ordered set. Let $\mathfrak{A} = \{a\}$ be well ordered in the a's, $\mathfrak{B} = \{b\}$ be well ordered in the b's, etc. The set*

$$\mathfrak{U} = \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \cdots$$

is well ordered with regard to the little letters a, b ⋯

For $\mathfrak{U}$ has a first element in the little letters, viz. the first element of $\mathfrak{A}$. Moreover, any part of $\mathfrak{U}$, as $\mathfrak{B}$, has a first element in the little letters. For if it has not, there exists in $\mathfrak{B}$ an infinite decreasing sequence

$$t > s > r > \cdots$$

This, however, is impossible, as such a sequence would determine a similar sequence in $\mathfrak{U}$ as

$$\mathfrak{T} > \mathfrak{S} > \mathfrak{R} > \cdots$$

which is impossible as $\mathfrak{U}$ is well ordered with regard to $\mathfrak{A}, \mathfrak{B} \cdots$

4. *Let*
$$\mathfrak{A} < \mathfrak{B} < \mathfrak{C} < \cdots \tag{1}$$

Let each element of $\mathfrak{A}$ precede each element of $\mathfrak{B}$, etc.

Let each $\mathfrak{A}$, $\mathfrak{B}$, $\cdots$ *be well ordered.*

Let
$$\mathfrak{B} = \mathfrak{A} + B, \qquad \mathfrak{C} = \mathfrak{B} + C \cdots$$

Then
$$\mathfrak{S} = \mathfrak{A} + B + C + \cdots$$

is a well ordered set, $\mathfrak{S}$ *preserving the relative order of elements intact.*

For $\mathfrak{S}$ has a first element, viz. the first element of $\mathfrak{A}$. Any part S of $\mathfrak{S}$ has a first element. For, if not, there exists in $\mathfrak{S}$ an infinite decreasing sequence

$$r > q > p > \cdots \tag{2}$$

Now r lies in some set of 1) as $\mathfrak{R}$. Hence $q, p, \cdots$ also lie in $\mathfrak{R}$. But in $\mathfrak{R}$ there is no sequence as 2).

5. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\cdots$ be an ordered set of well ordered aggregates, no two of which have an element in common. The reader must guard against assuming that $\mathfrak{A} + \mathfrak{B} + \mathfrak{C} + \cdots$, keeping the relative order intact, is necessarily well ordered.

For let us modify Ex. 5 in 265 by taking instead of all the points on each L_y only a well ordered set which we denote by $\mathfrak{A}_y$.

Then the sum
$$\mathfrak{A} = \Sigma\mathfrak{A}_y$$

has a definite meaning. The elements of $\mathfrak{A}$ we supposed arranged as in Ex. 5 of 265.

Obviously $\mathfrak{A}$ is not well ordered.

Sections

271. We now introduce a notion which in the theory of well-ordered sets plays a part analogous to Dedekind's partitions in the theory of the real number system $\mathfrak{R}$. Cf. I, 128.

Let $\mathfrak{A}$ be a well ordered set. The elements preceding a given element a of $\mathfrak{A}$ form a partial set called the *section of* $\mathfrak{A}$ *generated by* a. We may denote it by

$$Sa,$$

or by the corresponding small letter $\mathfrak{a}$.

Example 1. Let
$$\mathfrak{A} = 1, 2, 3, \cdots$$
Then
$$S100 = 1, 2, \cdots 99$$
is the section of $\mathfrak{A}$ generated by the element 100.

Example 2. Let
$$\mathfrak{A} = a_1, a_2 \cdots b_1, b_2 \cdots$$
Then
$$Sb_5 = a_1 a_2 \cdots b_1 b_2 b_3 b_4$$
is the section generated by b_5.
$$Sb_1 = a_1 a_2 \cdots$$
that generated by b_1, etc.

272. 1. *Every section of a well ordered aggregate is well ordered.*

For each section of $\mathfrak{A}$ is a partial aggregate of $\mathfrak{A}$, and hence well ordered by 269, 1.

2. *In the well ordered set $\mathfrak{A}$, let $a < b$. Then Sa is a section of Sb.*

3. *Let $\mathfrak{S}$ denote the aggregate of sections of an infinite well ordered set $\mathfrak{A}$. If we order $\mathfrak{S}$ such that $Sa < Sb$ in $\mathfrak{S}$ when $a < b$ in $\mathfrak{A}$, $\mathfrak{S}$ is well ordered.*

For the correspondence between $\mathfrak{A}$ and $\mathfrak{S}$ is uniform and similar.

273. *Let $\mathfrak{A}$, $\mathfrak{B}$ be well ordered and $\mathfrak{A} \simeq \mathfrak{B}$. If $a \sim b$, then $Sa \simeq Sb$.*

For in $\mathfrak{A}$ let $a'' < a' > a$. Let $b' \sim a'$ and $b'' \sim a''$. Since $\mathfrak{A} \simeq \mathfrak{B}$, we have
$$b'' < b' < b;$$
hence the theorem.

274. If $\mathfrak{A}$ is well ordered, $\mathfrak{A}$ is not similar to any one of its sections.

For if $\mathfrak{A} \simeq Sa$, to a in $\mathfrak{A}$ corresponds an element $a_1 < a$ in Sa. To a_1 in $\mathfrak{A}$ corresponds an element a_2 in Sa, etc. In this way we obtain an infinite decreasing sequence
$$a > a_1 > a_2 > \cdots,$$
which is impossible by 270, 1.

275. *Let $\mathfrak{A}$, $\mathfrak{B}$ be well ordered and $\mathfrak{A} \simeq \mathfrak{B}$. Then to Sa in $\mathfrak{A}$ cannot correspond two sections Sb, $S\beta$ each $\simeq Sa$.*

For let $b < \beta$, and $Sa \simeq Sb$, $Sa \simeq S\beta$. Then

$$Sb \simeq S\beta, \qquad \text{by 267.} \tag{1}$$

But 1) contradicts 274.

276. *Let $\mathfrak{A}$, $\mathfrak{B}$ be two well ordered aggregates. It is impossible to establish a uniform and similar correspondence between $\mathfrak{A}$ and $\mathfrak{B}$ in more than one way.*

For say $Sa \simeq Sb$ in one correspondence, and $Sa \simeq S\beta$ in another, b, β being different elements of $\mathfrak{B}$. Then

$$Sb \simeq S\beta, \qquad \text{by 267.}$$

This contradicts 275.

277. 1. We can now prove the following theorem, which is the converse of 273.

Let $\mathfrak{A}$, $\mathfrak{B}$ be well ordered. If to each section of $\mathfrak{A}$ corresponds one similar section of $\mathfrak{B}$, and conversely, then $\mathfrak{B} \simeq \mathfrak{A}$.

Let us first show that $\mathfrak{A} \sim \mathfrak{B}$. Since to any Sa of $\mathfrak{A}$ corresponds a similar section Sb in $\mathfrak{B}$, let us set $a \sim b$. No other $a' \sim b$, and no other $b' \sim a$, as then $Sa' \simeq Sb$ or $Sb' \simeq Sa$, which contradicts 274. Let the first element of $\mathfrak{A}$ correspond to the first of $\mathfrak{B}$. Thus the correspondence we have set up between $\mathfrak{A}$ and $\mathfrak{B}$ is uniform and $\mathfrak{A} \sim \mathfrak{B}$.

We show now that this correspondence is *similar*. For let

$$a \sim b \text{ and } a' \sim b', \qquad a' < a.$$

Then $b' < b$. For a' lies in $Sa \simeq Sb$ and $b' \sim a'$ lies in Sb.

2. From 1 and 273 we have now the fundamental theorem :

In order that two well-ordered sets $\mathfrak{A}$, $\mathfrak{B}$ be similar, it is necessary and sufficient that to each section of $\mathfrak{A}$ corresponds a similar section of $\mathfrak{B}$, and conversely.

278. *Let $\mathfrak{A}$, $\mathfrak{B}$ be well ordered. If to each section of $\mathfrak{A}$ corresponds a similar section of $\mathfrak{B}$, but not conversely, then $\mathfrak{A}$ is similar to a section of $\mathfrak{B}$.*

Let us begin by ordering the sections of $\mathfrak{A}$ and $\mathfrak{B}$ as in 272, 3. Let B denote the aggregate of sections of $\mathfrak{B}$ to which similar sections of $\mathfrak{A}$ do not correspond. Then B is well ordered and has a first section, say Sb. Let $\beta < b$. Then to $S\beta$ in $\mathfrak{B}$ corresponds by hypothesis a similar section $S\alpha$ in $\mathfrak{A}$. On the other hand, to any section Sa' of $\mathfrak{A}$ corresponds a similar section Sb' of $\mathfrak{B}$. Obviously $b' < b$. Thus to any section of $\mathfrak{A}$ corresponds a similar section of Sb and conversely. Hence $\mathfrak{A} \simeq Sb$ by 277, 1.

279. *Let $\mathfrak{A}$, $\mathfrak{B}$ be well ordered. Either $\mathfrak{A}$ is similar to $\mathfrak{B}$ or one is similar to a section of the other.*

For either:

1° To each section of $\mathfrak{A}$ corresponds a similar section of $\mathfrak{B}$ and conversely;

or 2° To each section of one corresponds a similar section of the other but not conversely;

or 3° There is at least one section in both $\mathfrak{A}$ and $\mathfrak{B}$ to which no similar section corresponds in the other.

If 1° holds, $\mathfrak{A} \simeq \mathfrak{B}$ by 277, 1. If 2° holds, either $\mathfrak{A}$ or $\mathfrak{B}$ is similar to a section of the other.

We conclude by showing 3° is impossible.

For let A be the set of sections of $\mathfrak{A}$ to which no similar section in $\mathfrak{B}$ corresponds. Let B have the same meaning for $\mathfrak{B}$. If we suppose $\mathfrak{A}$, $\mathfrak{B}$ ordered as in 272, 3, A will have a first section say $S\alpha$, and B a first section $S\beta$.

Let $a < \alpha$. Then to Sa in $\mathfrak{A}$ corresponds by hypothesis a section Sb of $S\beta$ as in 278. Similarly if $b' < \beta$, to Sb' of $\mathfrak{B}$ corresponds a section Sa' of $S\alpha$. But then $S\alpha \simeq S\beta$ by 277, 1, and this contradicts the hypothesis.

Ordinal Numbers

280. 1. With each well ordered aggregate $\mathfrak{A}$ we associate an attribute called its *ordinal number*, which we define as follows:

1° If $\mathfrak{A} \simeq \mathfrak{B}$, they have the *same* ordinal number.

2° If $\mathfrak{A} \simeq$ a section of $\mathfrak{B}$, the ordinal number of $\mathfrak{A}$ is *less* than that of $\mathfrak{B}$.

3° If a section of $\mathfrak{A}$ is $\simeq \mathfrak{B}$, the ordinal number of $\mathfrak{A}$ is *greater* than that of $\mathfrak{B}$.

The ordinal number of $\mathfrak{A}$ may be denoted by

$$\text{Ord}\ \mathfrak{A},$$

or when no ambiguity can arise, by the corresponding small letter $\mathfrak{a}$. As any two well ordered aggregates $\mathfrak{A}$, $\mathfrak{B}$ fall under one and only one of the three preceding cases, any two ordinal numbers $\mathfrak{a}$, $\mathfrak{b}$ satisfy one of the three following relations, and only one, viz.:

$$\mathfrak{a} = \mathfrak{b} \quad , \quad \mathfrak{a} < \mathfrak{b} \quad , \quad \mathfrak{a} > \mathfrak{b},$$

and if $\mathfrak{a} < \mathfrak{b}$, it follows that $\mathfrak{b} > \mathfrak{a}$.

Obviously they enjoy also the following properties.

2. *If*

$$\mathfrak{a} = \mathfrak{b} \quad , \quad \mathfrak{b} = \mathfrak{c} \quad , \quad \textit{then}\ \mathfrak{a} = \mathfrak{c}.$$

For if $\mathfrak{c} = \text{Ord}\ \mathfrak{C}$, the first two relations state that

$$\mathfrak{A} \simeq \mathfrak{B} \quad , \quad \mathfrak{B} \simeq \mathfrak{C}.$$

But then

$$\mathfrak{A} \simeq \mathfrak{C} \quad , \quad \text{by } 267.$$

Hence

$$\mathfrak{a} = \mathfrak{c}.$$

3. *If*

$$\mathfrak{a} > \mathfrak{b} \quad , \quad \mathfrak{b} > \mathfrak{c} \quad , \quad \textit{then}\ \mathfrak{a} > \mathfrak{c}.$$

281. 1. Let $\mathfrak{A}$ be a finite aggregate, embracing say n elements. Then we set

$$\text{Ord}\ \mathfrak{A} = n.$$

Thus the ordinal number of a finite aggregate has exactly similar properties to those of finite cardinal numbers. The ordinal number of a finite aggregate is called *finite*, otherwise *transfinite*.

The ordinal number belonging to the well ordered set formed of the positive integers

$$\mathfrak{J} = 1, 2, 3, \cdots$$

we call ω.

2. *The least transfinite ordinal number is* ω.

For suppose $\alpha = \text{Ord}\ \mathfrak{A} < \omega$, is transfinite. Then $\mathfrak{A}$ is $\simeq$ a section of $\mathfrak{J}$. But every section of $\mathfrak{J}$ is finite, hence the contradiction.

3. The cardinal number of a set $\mathfrak{A}$ is independent of the order in which the elements of $\mathfrak{A}$ occur. This is not so in general for ordinal numbers.

For example, let
$$\mathfrak{A} = 1, 2, 3, \cdots$$
$$\mathfrak{B} = 1, 3, 5, \cdots 2, 4, 6, \cdots$$

Here
$$\text{Card } \mathfrak{A} = \text{Card } \mathfrak{B} = \aleph_0.$$

But
$$\text{Ord } \mathfrak{A} < \text{Ord } \mathfrak{B},$$

since $\mathfrak{A}$ is similar to a section of $\mathfrak{B}$, viz. the set of odd numbers, $1, 3, 5, \cdots$

282. 1. *Addition of Ordinals.* Let $\mathfrak{A}$, $\mathfrak{B}$ be well ordered sets without common elements. Let $\mathfrak{C}$ be the aggregate formed by placing the elements of $\mathfrak{B}$ after those of $\mathfrak{A}$, leaving the order in $\mathfrak{B}$ otherwise unchanged. Then the ordinal number of $\mathfrak{C}$ is called the *sum* of the ordinal numbers of $\mathfrak{A}$ and $\mathfrak{B}$, or

$$\text{Ord } \mathfrak{C} = \text{Ord } \mathfrak{A} + \text{Ord } \mathfrak{B},$$

or
$$\mathfrak{c} = \mathfrak{a} + \mathfrak{b}.$$

The extension of this definition to any set of well-ordered aggregates such that the result is well ordered is obvious.

2. *We note that*
$$\mathfrak{a} + \mathfrak{b} > \mathfrak{a}, \qquad \mathfrak{a} + \mathfrak{b} \geq \mathfrak{b}.$$

For $\mathfrak{A}$ is similar to a section of $\mathfrak{C}$, and $\mathfrak{B}$ is equivalent to a part of $\mathfrak{C}$.

3. *The addition of ordinal numbers is associative.*

This is an immediate consequence of the definition of addition.

4. The addition of ordinal numbers is not always commutative.

Thus if
$$\mathfrak{A} = (a_1 a_2 \cdots \text{ in inf.}), \qquad \text{Ord } \mathfrak{A} = \omega,$$
$$\mathfrak{B} = (b_1 b_2 \cdots b_n), \qquad \text{Ord } \mathfrak{B} = n;$$

let
$$\mathfrak{C} = (a_1 a_2 \cdots b_1 b_2 \cdots b_n), \qquad \text{Ord } \mathfrak{C} = \mathfrak{c},$$
$$\mathfrak{D} = (b_1 \cdots b_n a_1 a_2 \cdots), \qquad \text{Ord } \mathfrak{D} = \mathfrak{d}.$$

Then
$$\mathfrak{c} = \omega + n \quad , \quad \mathfrak{d} = n + \omega.$$

But $\mathfrak{A} \simeq$ a section of $\mathfrak{C}$, viz.: $\simeq Sb_1$, while $\mathfrak{D} \simeq \mathfrak{A}$. Hence

$$\omega < \mathfrak{c} \quad , \quad \omega = \mathfrak{d},$$

or

$$\omega + n > \omega \quad , \quad n + \omega = \omega.$$

5. *If* $\mathfrak{a} > \mathfrak{b}$, *then* $\mathfrak{c} + \mathfrak{a} > \mathfrak{c} + \mathfrak{b}$, *and* $\mathfrak{a} + \mathfrak{c} \geq \mathfrak{b} + \mathfrak{c}$.

For let

$$\mathfrak{a} = \text{Ord}\ \mathfrak{A}, \qquad \mathfrak{b} = \text{Ord}\ \mathfrak{B}, \qquad \mathfrak{c} = \text{Ord}\ \mathfrak{C}.$$

Since $\mathfrak{a} > \mathfrak{b}$, we can take for $\mathfrak{B}$ a section Sb of $\mathfrak{A}$. Then $\mathfrak{c} + \mathfrak{a}$ is the ordinal number of

$$\mathfrak{C} + \mathfrak{A}, \tag{1}$$

and $\mathfrak{c} + \mathfrak{b}$ is the ordinal number of

$$\mathfrak{C} + Sb, \tag{2}$$

preserving the relative order of the elements.

But 2) is a section of 1), and hence $\mathfrak{c} + \mathfrak{a} > \mathfrak{c} + \mathfrak{b}$.

The proof of the rest of the theorem is obvious.

283. 1. *The ordinal number immediately following* $\mathfrak{a}$ *is* $\mathfrak{a} + 1$.

For let $\mathfrak{a} = \text{Ord}\ \mathfrak{A}$. Let $\mathfrak{B}$ be a set formed by adding after all the elements of $\mathfrak{A}$ another element b. Then

$$\mathfrak{a} + 1 = \text{Ord}\ \mathfrak{B} = \mathfrak{b}.$$

Suppose now

$$\mathfrak{a} < \mathfrak{c} < \mathfrak{b} \quad , \quad \mathfrak{c} = \text{Ord}\ \mathfrak{C}. \tag{1}$$

Then $\mathfrak{C}$ is similar to a section of $\mathfrak{B}$. But the greatest section of $\mathfrak{B}$ is $Sb = \mathfrak{A}$. Hence

$$\mathfrak{c} \leq \mathfrak{a},$$

which contradicts 1).

2. *Let* $\mathfrak{a} > \mathfrak{b}$. *Then there is one and only one ordinal number* $\mathfrak{d}$ *such that*

$$\mathfrak{a} = \mathfrak{b} + \mathfrak{d}.$$

For let

$$\mathfrak{a} = \text{Ord}\ \mathfrak{A} \quad , \quad \mathfrak{b} = \text{Ord}\ \mathfrak{B}.$$

We may take $\mathfrak{B}$ to be a section Sb of $\mathfrak{A}$. Let $\mathfrak{D}$ denote the set of elements of $\mathfrak{A}$, coming after Sb. It is well ordered and has an ordinal number $\mathfrak{d}$. Then

$$\mathfrak{A} = \mathfrak{B} + \mathfrak{D},$$

preserving the relative order, and hence

$$\mathfrak{a} = \mathfrak{b} + \mathfrak{d}.$$

There is no other number, as 282, 5 shows.

284. 1. *Multiplication of Ordinals.* Let $\mathfrak{A}$, $\mathfrak{B}$ be well-ordered aggregates having $\mathfrak{a}$, $\mathfrak{b}$ as ordinal numbers. Let us replace each element of $\mathfrak{A}$ by an aggregate $\simeq \mathfrak{B}$. The resulting aggregate $\mathfrak{C}$ we denote by

$$\mathfrak{B} \cdot \mathfrak{A}.$$

As $\mathfrak{C}$ is a well-ordered set by 270, 3 it has an ordinal number $\mathfrak{c}$. We define now the *product* $\mathfrak{b} \cdot \mathfrak{a}$ to be $\mathfrak{c}$, and write

$$\mathfrak{b} \cdot \mathfrak{a} = \mathfrak{c}.$$

We say $\mathfrak{c}$ is the result of *multiplying* $\mathfrak{a}$ *by* $\mathfrak{b}$, and call $\mathfrak{a}$, $\mathfrak{b}$ *factors.* We write

$$\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}^2 \quad , \quad \mathfrak{a} \cdot \mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}^3 \quad , \quad \text{etc.}$$

2. *Multiplication is associative.*

This is an immediate consequence of the definition.

3. *Multiplication is not always commutative.*

For example, let

$$\mathfrak{A} = (a_1 a_2),$$

$$\mathfrak{B} = (1, 2, 3 \cdots \text{in inf.}).$$

Then

$$\mathfrak{B} \cdot \mathfrak{A} = (b_1 b_2 b_3 \cdots, \quad c_1 c_2 c_3 \cdots).$$

$$\mathfrak{A} \cdot \mathfrak{B} = (b_1, c_1, \quad b_2, c_2, \quad b_3, c_3, \cdots).$$

Hence

$$\text{Ord}\,(\mathfrak{B} \cdot \mathfrak{A}) = \omega \cdot 2 > \omega,$$

$$\text{Ord}\,(\mathfrak{A} \cdot \mathfrak{B}) = 2\,\omega = \omega.$$

4. *If* $\mathfrak{a} < \mathfrak{b}$, *then* $\mathfrak{c}\mathfrak{a} < \mathfrak{c}\mathfrak{b}$.

For $\mathfrak{C} \cdot \mathfrak{A}$ is a section of $\mathfrak{C} \cdot \mathfrak{B}$.

Limitary Numbers

285. 1. *Let*

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdots \tag{1}$$

be an infinite increasing enumerable sequence of ordinal numbers. There exists a first ordinal number α greater than every α_n.

Let

$$\alpha_n = \text{Ord}\,\mathfrak{A}_n.$$

Since $\alpha_{n-1} < \alpha_n$, $\mathfrak{A}_{n-1}$ is similar to a section of $\mathfrak{A}_n$. For simplicity we may take $\mathfrak{A}_{n-1}$ to be a section of $\mathfrak{A}_n$. Let, therefore,

$$\mathfrak{A}_n = \mathfrak{A}_{n-1} + \mathfrak{B}_n.$$

Consider now

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{B}_2 + \mathfrak{B}_3 + \cdots$$

keeping the relative order of the elements intact. Then $\mathfrak{A}$ is well ordered and has an ordinal number α.

As any $\mathfrak{A}_n$ is a section of $\mathfrak{A}$,

$$\alpha_n < \alpha.$$

Moreover any number $\beta < \alpha$ is also $<$ some α_m. For if $\mathfrak{B}$ has the ordinal number β, $\mathfrak{B}$ must be similar to a section of $\mathfrak{A}$. But there is no last section of $\mathfrak{A}$.

2. The number α we have just determined is called the *limit of the sequence* 1). We write

$$\alpha = \lim \alpha_n \quad , \quad \text{or } \alpha_n \doteq \alpha.$$

We also say that α *corresponds to the sequence* 1).

All numbers corresponding to infinite enumerable increasing sequences of ordinal numbers are called *limitary*.

3. *If every* α_n *in* 1) *is* $< \beta$, *then* $\alpha \leq \beta$.

For if $\beta < \alpha$, α is not the least ordinal number greater than every α_n.

4. *If* $\beta < \alpha$, β *is* $<$ *some* α_n.

286. *In order that*

$$\alpha_1 < \alpha_2 < \cdots \qquad (1$$

$$\beta_1 < \beta_2 < \cdots \qquad (2$$

define the same number λ *it is necessary and sufficient that each number in either sequence is surpassed by a number in the other.*

For let

$$\alpha_n \doteq \alpha \quad , \quad \beta_n \doteq \beta.$$

If no β_n is greater than α_m, $\beta < \alpha_m < \alpha$, by 285, 3, and $\alpha \neq \beta$.

On the other hand, if each $\alpha_m <$ some β_n, $\alpha \leq \beta$. Similarly $\beta \leq \alpha$.

287. *Cantor's Principles of Generating Ordinals.* We have now two methods of generating ordinal numbers. First, by adding 1 to any ordinal number α. In this way we get

$$\alpha, \alpha + 1, \alpha + 2, \cdots$$

Secondly, by taking the limit of an infinite enumerable increasing sequence of ordinal numbers, as

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdots$$

Cantor calls these two methods the *first* and *second principles* of generating ordinal numbers.

Starting with the ordinal number 1, we get by successive applications of the first principle the numbers

$$1, 2, 3, 4, \cdots$$

The limit of this sequence is ω by 285, 1. Using the first principle alone, this number would not be attained; to get it requires the application of the second principle. Making use of the first principle again, we obtain

$$\omega + 1, \qquad \omega + 2, \qquad \omega + 3, \cdots$$

The second principle gives now the limitary number $\omega + \omega = \omega 2$ by 285, 1. From this we get, using the first principle, as before,

$$\omega 2 + 1, \qquad \omega 2 + 2, \qquad \omega 2 + 3, \cdots$$

whose limit is $\omega 3$. In this way we may obtain the numbers

$$\omega m + n \quad , \quad m, n \text{ finite.}$$

The limit of any increasing sequence of these numbers as

$$\omega \quad , \quad \omega 2 \quad , \quad \omega 3 \quad , \quad \omega 4, \cdots$$

is $\omega \cdot \omega = \omega^2$, by 285, 1.

From ω^2 we can get numbers of the type

$$\omega^2 l + \omega m + n \qquad l, m, n \text{ finite.}$$

Obviously we may proceed in this way indefinitely and obtain all numbers of the type

$$\omega^n a_0 + \omega^{n-1} a_1 + \omega^{n-2} a_2 + \cdots + a_n \tag{1}$$

where $a_0, a_1 \cdots a_n$ are finite ordinals.

But here the process does not end. For the sequence

$$\omega < \omega^2 < \omega^3 < \cdots$$

has a limit which we denote by ω^ω.

Continuing we obtain

$$\omega^{\omega^\omega}, \quad \omega^{\omega^{\omega^\omega}}, \quad \text{etc.}$$

288. It is interesting to see how we may obtain well ordered sets of points whose ordinal numbers are the numbers just considered.

In the unit interval $\mathfrak{A} = (0, 1)$, let us take the points

$$\tfrac{1}{2}\ , \ \tfrac{3}{4}\ , \ \tfrac{7}{8}\ , \ \tfrac{15}{16} \cdots \tag{1}$$

These form a well ordered set whose ordinal number is ω.

The points 1) divided $\mathfrak{A}$ into a set of intervals,

$$\mathfrak{A}_1\ , \ \mathfrak{A}_2\ , \ \mathfrak{A}_3 \cdots \tag{2}$$

In m of these intervals, let us take a set similar to 1). This gives us a set whose ordinal number is ωm.

In each interval 2), let us take a set similar to 1). This gives us a set whose ordinal number is ω^2. The points of this set divide $\mathfrak{A}$ into a set of ω^2 intervals. In each of these intervals, let us take a set of points similar to 1). This gives a set of points whose ordinal number is ω^3, etc.

Let us now put in $\mathfrak{A}_1$ a set of points $\mathfrak{B}_1$ whose ordinal number is ω. In $\mathfrak{A}_2$ let us put a set $\mathfrak{B}_2$ whose ordinal number is ω^2, and so on, for the other intervals of 2).

We thus get in $\mathfrak{A}$ the well ordered set

$$\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2 + \mathfrak{B}_3 + \cdots$$

whose ordinal number is the limit of

$$\omega\ , \ \omega + \omega^2\ , \ \omega + \omega^2 + \omega^3\ , \ \cdots$$

This by 286 has the same limit as

$$\omega\ , \ \omega^2\ , \ \omega^3\ , \ \cdots \ \text{or} \ \omega^\omega.$$

With this set we may now form a set whose ordinal number is ω^{ω^ω}, etc.

Classes of Ordinals

289. Cantor has divided the ordinal numbers into classes.

Class 1, denoted by Z_1, embraces all finite ordinal numbers.

Class 2, denoted by Z_2, embraces all transfinite ordinal numbers corresponding to well ordered enumerable sets; that is, to sets whose cardinal number is $\aleph_0$. For this reason we also write

$$Z_2 = Z(\aleph_0).$$

It will be shown in 293, 1 that Z_2 is not enumerable. Moreover if we set

$$\aleph_1 = \text{Card } Z_2,$$

there is no cardinal number between $\aleph_0$ and $\aleph_1$ as will be shown in 294. We are thus justified in saying that Class 3, denoted by Z_3 or $Z(\aleph_1)$, embraces all ordinal numbers corresponding to well ordered sets whose cardinal number is $\aleph_1$, etc.

Let $\beta = \text{Ord } \mathfrak{B}$ be any ordinal number. Then all the numbers $\alpha < \beta$ correspond to sections of $\mathfrak{B}$. These sections form a well ordered set by 272, 3. Therefore if we arrange the numbers $\alpha < \beta$ in an order such that α' precedes α when $S\alpha' < S\alpha$, they are *well ordered*. We shall call this the natural order. Then the first number in Z_1 is 1, the first number of Z_2 is ω. The first number in Z_3 is denoted by Ω.

290. As the numbers in Class 1 are the positive integers, they need no comment here. Let us therefore turn to Class 2.

If α *is in* Z_2, *so is* $\alpha + 1$.

For let $\alpha = \text{Ord } \mathfrak{A}$. Let $\mathfrak{B}$ be the well ordered set obtained by placing an element b after all the elements of $\mathfrak{A}$. Then

$$\alpha + 1 = \text{Ord } \mathfrak{B}.$$

But $\mathfrak{B}$ is enumerable since $\mathfrak{A}$ is.

Hence $\alpha + 1$ lies in Z_2.

291. *Let*

$$\alpha_1 < \alpha_2 < \alpha_3 < \cdots$$

be an enumerable infinite set of numbers in Z_2. *Then* $\alpha = \lim \alpha_n$ *lies in* Z_2.

For using the notation employed in the proof of 285, 1, α is the ordinal number of

$$\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{B}_1 + \mathfrak{B}_2 + \cdots$$

But $\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{B}_2 \cdots$ are each enumerable.

Hence $\mathfrak{A}$ is enumerable by 235, 1, and α lies in Z_2.

292. We prove now the converse of 290 and 291.

Each number α in Z_2, except ω, is obtained by adding 1 *to some number in Z_2; or it is the limit of an infinite enumerable increasing set of numbers in Z_2.*

For, let $\alpha = \text{Ord}\,\mathfrak{A}$. Suppose *first*, that $\mathfrak{A}$ has a last element, say a. Since $\mathfrak{A}$ is enumerable, so is Sa. Hence

$$\beta = \text{Ord} \cdot Sa$$

is in Z_2. Then

$$\alpha = \beta + 1.$$

Suppose *secondly*, that $\mathfrak{A}$ has no last element. All the numbers $\beta < \alpha$ in Z_2 belong to sections of $\mathfrak{A}$. Since $\mathfrak{A}$ is enumerable, the numbers β are enumerable. Let them be arranged in a sequence

$$\beta_1, \beta_2, \beta_3 \cdots \tag{1}$$

Since they have no greatest, let β_1' be the first number in it $> \beta_1$, let β_2' be the first number in it $> \beta_1'$, etc. We get thus the sequence

$$\beta_1 < \beta_1' < \beta_2' < \cdots \tag{2}$$

whose limit is λ, say.

Then $\lambda = \alpha$. For λ is $>$ any number in 1), which embraces all the numbers of $Z_2 < \alpha$. Moreover it is the least number which enjoys this property.

293. 1. *The numbers of Z_2 are not enumerable.*

For suppose they were. Let us arrange them in the sequence

$$\alpha_1, \alpha_2, \alpha_3 \cdots \tag{1}$$

Then, as in 292, there exists in this sequence the infinite enumerable sequence

$$\alpha_1 < \alpha_1' < \alpha_2' < \cdots \tag{2}$$

such that there are numbers in 2) greater than any given number in 1).

Let $\alpha'_n \doteq \alpha'$. Then α' lies in Z_2 by 291. On the other hand, by 285, α' is $>$ any number in 2), and therefore $>$ any number in 1). But 1) embraces all the numbers of Z_2, by hypothesis. We are thus led to a contradiction.

2. We set

$$\aleph_1 = \text{Card } Z_2.$$

294. *There is no cardinal number between* $\aleph_0$ *and* $\aleph_1$.

For let $\alpha = \text{Card } \mathfrak{A}$ be such a number. Then $\mathfrak{A}$ is $\sim$ an infinite partial aggregate of Z_2, which without loss of generality may be taken to be a section of Z_2. But every such section is enumerable. Hence $\mathfrak{A}$ is enumerable and $\alpha = \aleph_0$, which is a contradiction.

295. We have just seen that the numbers in Z_2 are not enumerable. Let us order them so that each number is less than any succeeding number. We shall call this the *natural order.*

1. *The numbers of* Z_2 *when arranged in their natural order form a well ordered set.*

For Z_2 has a first element ω. Moreover any partial set Z, the relative order being preserved, has a first element. For if it has not, there exists an infinite enumerable decreasing sequence

$$\alpha > \beta > \gamma > \cdots$$

This, however, is not possible. For $\beta, \gamma, \cdots$ form a part of $S\alpha$ which is well ordered.

There is thus one well ordered set having $\aleph_1$ as cardinal number. Let

$$\Omega = \text{Ord } Z_2.$$

Let now $\mathfrak{A}$ be an enumerable well ordered set whose ordinal number is α. The set

$$Z_2 + \mathfrak{A},$$

the elements of $\mathfrak{A}$ coming after Z_2, has the cardinal number $\aleph_1$ by 241, 3. It is well ordered by 270, 3. It has therefore an ordinal number which lies in Z_3, viz. $\Omega + \alpha$ by 282, 1. Thus Z_3 embraces an infinity of numbers.

2. *The least number in* Z_3 *is* Ω.

For to any number $\alpha < \Omega$ corresponds a section $\mathfrak{A}$ of Z_2. Hence α lies in Z_2.

296. 1. *An aggregate formed of an $\aleph_1$ set of $\aleph_1$ sets is an $\aleph_1$ set.* Consider the set

$$A = \begin{array}{llllllll} a_{11}, & a_{12}, & a_{13} & \cdots & a_{1\omega} & \cdots & a_{1\alpha} & \cdots \\ a_{21}, & a_{22}, & a_{23} & \cdots & a_{2\omega} & \cdots & a_{2\alpha} & \cdots \\ a_{\omega 1}, & a_{\omega 2}, & a_{\omega 3} & \cdots & a_{\omega\omega} & \cdots & a_{\omega\alpha} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{\alpha 1}, & a_{\alpha 2}, & a_{\alpha 3} & \cdots & a_{\alpha\omega} & \cdots & a_{\alpha\alpha} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array} \qquad \alpha < \Omega$$

Here each row is an $\aleph_1$ set. As there are an $\aleph_1$ set of rows, A is an $\aleph_1$ set of $\aleph_1$ sets. To show that A is an $\aleph_1$ set, we associate each $a_{\iota\kappa}$ with some number in the first two number classes.

In the first place the elements $a_{\iota\kappa}$ where ι $\kappa < \omega$ may be associated with the numbers $1, 2, 3, \cdots < \omega$. The elements $a_{\iota\omega}$, $a_{\omega\kappa}$ lying just inside the ω^{th} square and which are characterized by the condition that $\iota = 1, 2, \cdots \omega$; $\kappa = 1, 2 \cdots < \omega$ form an enumerable set and may therefore be associated with the ordinals $\omega, \omega + 1, \cdots < \omega 2$. For the same reason the elements just inside the $\omega + 1^{\text{st}}$ square may be associated with the ordinals $\omega 2, \omega 2 + 1, \cdots < \omega 3$. In this way we may continue. For when we have arrived at the α^{th} row and column (edge of the α^{th} square) we have only used up an enumerable set of numbers in the sequence

$$1, 2, \cdots \omega \cdots < \Omega \tag{1}$$

in our process of association. There are thus still an $\aleph_1$ set left in 1) to continue the process of association.

2. As a corollary of 1 we have:

The ordinal numbers

$$\Omega^2, \qquad \Omega^3, \qquad \Omega^4, \cdots$$

lie in Z_3.

297. 1. *Let*

$$\alpha < \beta < \gamma < \cdots \tag{1}$$

be an increasing sequence of numbers in Z_3 having $\aleph_1$ as cardinal number and such that any section of 1) *has $\aleph_0$ as its cardinal. There exists a first ordinal number λ in Z_3 greater than any number in* 1).

For let

$$\alpha = \text{Ord}\,\mathfrak{A}, \qquad \beta = \text{Ord}\,\mathfrak{B}, \qquad \gamma = \text{Ord}\,\mathfrak{C} \cdots$$

Since $\alpha < \beta$ we may take $\mathfrak{A}$ to be a section of $\mathfrak{B}$. Similarly we may suppose $\mathfrak{B}$ is a section of $\mathfrak{C}$, etc.

Let now
$$\mathfrak{B} = \mathfrak{A} + B, \qquad \mathfrak{C} = \mathfrak{B} + C, \cdots$$
Consider now
$$\mathfrak{L} = \mathfrak{A} + B + C + \cdots$$
keeping the relative order intact. Then $\mathfrak{L}$ is well ordered by 270, 4. Let
$$\lambda = \text{Ord } \mathfrak{L}.$$

Since Card $\mathfrak{L} = \aleph_1$, by 296, 1, λ lies in Z_3.

As any $\mathfrak{A}$, $\mathfrak{B}$, $\cdots$ is a section of $\mathfrak{L}$,
$$\alpha < \beta < \cdots < \lambda.$$

Moreover, any number $\mu < \lambda$ is also $<$ some $\alpha, \beta, \gamma \cdots$ For if $\mathfrak{M}$ has ordinal number μ, $\mathfrak{M}$ must be similar to a section of $\mathfrak{L}$. But there is no last section in $\mathfrak{L}$.

2. We shall call sequences of the type 1), *an* $\aleph_1$ *sequence.* The number λ whose existence we have just established, we shall call the *limit of* 1). We shall also write
$$\alpha < \beta < \gamma \cdots \doteq \lambda$$
to indicate that $\alpha, \beta, \cdots$ is an $\aleph_1$ sequence whose limit is λ.

298. 1. The preceding theorem gives us a third method of generating ordinal numbers. We call it the *third principle.*

We have seen that the first and second principles suffice to generate the numbers of the first two classes of ordinal numbers but do not suffice to generate even the first number, viz. Ω in Z_3. We prove now the following fundamental theorem :

2. *The three principles already described are necessary and sufficient to generate the numbers in* Z_3.

For let $\alpha =$ Ord be any number of Z_3. If $\mathfrak{A}$ has a last element, reasoning similar to 292, 1 shows that
$$\alpha = \beta + 1.$$

If $\mathfrak{A}$ has no last element, all the numbers of $Z_3 < \alpha$ form an $\aleph_0$ or $\aleph_1$ set. In the former case
$$\alpha = \Omega + \beta,$$

where β lies in Z_2. In the latter case, reasoning similar to 292, 1 shows that we can pick out an $\aleph_1$ increasing sequence

$$\beta_1 < \beta_2' < \beta_3' \cdots \doteq \alpha.$$

299. 1. *The numbers of Z_3 form a set whose cardinal number α is $> \aleph_1$.*

The proof is entirely similar to 293, 1. Suppose, in fact, that $\alpha = \aleph_1$. Let us arrange the elements of Z_3 in the $\aleph_1$ sequence

$$\alpha_1 \ , \ \alpha_2 \cdots \alpha_\omega \cdots \alpha_\Omega \cdots \tag{1}$$

As in 292, there exists in this sequence an $\aleph_1$ increasing sequence

$$\alpha_1' < \alpha_2' < \cdots \doteq \alpha'. \tag{2}$$

Then α' lies in Z_3 by 297, 1. On the other hand α' is greater than any number in 2) and hence greater than any number in 1). But 1) embraces all the numbers in Z_3 by hypothesis. We are thus led to a contradiction.

2. We set

$$\aleph_2 = \text{Card}\, Z_3 .$$

3. *There is no cardinal number between $\aleph_1$ and $\aleph_2$.*

For let $\alpha = \text{Card}\, \mathfrak{A}$ be such a number. Then $\mathfrak{A}$ is equivalent to a section of Z_3. But every such section has the cardinal number $\aleph_1$.

300. The reasoning of the preceding paragraphs may be at once generalized. The ordinal numbers of Z_n corresponding to well ordered sets of cardinal number $\aleph_{n-2}$ form a well ordered set having a greater cardinal number α than $\aleph_{n-2}$. Moreover there is no cardinal lying between $\aleph_{n-2}$ and α. We may therefore appropriately denote α by $\aleph_{n-1}$. The $\aleph_{n-2}$ sequence of ordinal numbers

$$\alpha < \beta < \gamma \cdots$$

lying in Z_n has a limit lying in Z_n, and this fact embodies the n^{th} *principle* for generating ordinal numbers. The first n principles are just adequate to generate the numbers of Z_n. They do not suffice to generate even the first number in Z_{n+1}.

Finally we note that an $\aleph_n$ set of $\aleph_n$ sets forms an $\aleph_n$ set.

Sequence 2° There is a first, a second etc no.
e.g. $\{\frac{1}{n}\}$ is a sequence
? Rational nos in (0, 1) is not a sequence because 2° does not hold. (e.g) no 2nd number

CHAPTER X

POINT SETS

Pantaxis

301. 1. (*Borel.*) *Let each point of the limited or unlimited set $\mathfrak{A}$ lie at the center of a cube $\mathfrak{C}$. Then there exists an enumerable set of non-overlapping cubes $\{\mathfrak{c}\}$ such that each $\mathfrak{c}$ lies within some $\mathfrak{C}$, and each point of $\mathfrak{A}$ lies in some $\mathfrak{c}$. If $\mathfrak{A}$ is limited and complete, there is a finite set $\{\mathfrak{c}\}$ having this property.*

Find all lim. pts 167.

For let $D_1, D_2 \cdots$ be a sequence of superposed cubical divisions of norms $\doteq 0$. Any cell of D_1 which lies within some $\mathfrak{C}$ and which contains a point of $\mathfrak{A}$ we call a black cell; the other cells of D we call white. The black cells are not further subdivided. The division D_2 divides each white cell. Any of these subdivided cells which lies within some $\mathfrak{C}$ and contains a point of $\mathfrak{A}$ we call a black cell, the others are white. Continuing we get an enumerable set of non-overlapping cubical cells $\{\mathfrak{c}\}$.

Each point a of $\mathfrak{A}$ lies within some $\mathfrak{c}$. For a is the center of some $\mathfrak{C}$. But when n is taken sufficiently large, a lies in a cell of D_n, which cell lies within $\mathfrak{C}$.

Let now $\mathfrak{A}$ be limited and complete. Each a lies within a cube $\mathfrak{c}$, or on the faces of a finite number of these $\mathfrak{c}$. With a we associate the diagonal δ of the smallest of these cubes. Suppose $\text{Min}\,\delta = 0$ in $\mathfrak{A}$. As $\mathfrak{A}$ is complete, there is a point α in $\mathfrak{A}$ such that $\text{Min}\,\delta = 0$, in any $V_\eta(\alpha)$. This is not possible, since if η is taken sufficiently small, all the points of V_η lie in a finite number of the cubes $\mathfrak{c}$.

Thus $\text{Min}\,\delta > 0$. As the $\mathfrak{c}$'s do not overlap, there are but a finite number.

2. In the foregoing theorem the points of $\mathfrak{A}$ are not necessarily inner points of the cubes $\mathfrak{c}$. Let a be a point of $\mathfrak{A}$ on the face of one of these $\mathfrak{c}$. Since a lies within some $\mathfrak{C}$, it is obvious that the

cells of some D_n, n sufficiently large, which surround a form a cube c, lying within $\mathfrak{C}$. Thus the points of $\mathfrak{A}$ lie within an enumerable set of cells $\{c\}$, each c lying within some $\mathfrak{C}$. The cells c of course will in general overlap. Obviously also, if $\mathfrak{A}$ is complete, the points of $\mathfrak{A}$ will lie within a finite number of these c's.

302. *If $\mathfrak{A}$ is dense, $\mathfrak{A}'$ is perfect.*

For, in the first place, $\mathfrak{A}'$ is dense. In fact, let α be a point of $\mathfrak{A}'$. Then in any $D^*(\alpha)$ there are points of $\mathfrak{A}$. Let a be such a point. Since $\mathfrak{A}$ is dense, it is a limiting point of $\mathfrak{A}$ and hence is a point of $\mathfrak{A}'$. Thus in any $D^*(\alpha)$ there are points of $\mathfrak{A}'$.

Secondly, $\mathfrak{A}'$ is complete, by I, 266.

303. *Let $\mathfrak{B}$ be a complete partial set of the perfect aggregate $\mathfrak{A}$. Then $\mathfrak{C} = \mathfrak{A} - \mathfrak{B}$ is dense.*

For if $\mathfrak{C}$ contains the isolated point c, all the points of $\mathfrak{A}$ in $D_r^*(c)$ lie in $\mathfrak{B}$, if r is taken sufficiently small. But $\mathfrak{B}$ being complete, c must then lie in $\mathfrak{B}$.

Remark. We take this occasion to note that a finite set is to be regarded as complete.

304. 1. *If $\mathfrak{A}$ does not embrace all $\mathfrak{R}_n$, it has at least one frontier point in $\mathfrak{R}_n$.*

For let a be a point of $\mathfrak{A}$, and b a point of $\mathfrak{R}_n$ not in $\mathfrak{A}$. The points on the join of a, b have coördinates

$$x_\iota = a_\iota + \theta(b_\iota - a_\iota) = x_\iota(\theta), \quad 0 \leq \theta \leq 1, \quad \iota = 1, 2, \cdots n.$$

Let θ' be the maximum of those θ's such that $x(\theta)$ belongs to $\mathfrak{A}$ if $\theta < \theta'$. Then $x(\theta')$ is a frontier point of $\mathfrak{A}$.

2. Let $\mathfrak{A}$, $\mathfrak{B}$ have no point in common. If Dist $(\mathfrak{A}, \mathfrak{B}) > 0$, we say $\mathfrak{A}$, $\mathfrak{B}$ are *exterior* to each other.

305. 1. Let $\mathfrak{A} = \{a\}$ be a limited or unlimited point set in $\mathfrak{R}_m$. We say $\mathfrak{B} < \mathfrak{A}$ is *pantactic* in $\mathfrak{A}$, when in each $D_\delta(a)$ there is a point $\mathfrak{B}$.

We say $\mathfrak{B}$ is *apantactic* in $\mathfrak{A}$ when in each $D_\delta(a)$ there is a point α of $\mathfrak{A}$ such that some $D_\eta(\alpha)$ contains no point of $\mathfrak{B}$.

Example 1. Let $\mathfrak{A}$ be the unit interval $(0, 1)$, and $\mathfrak{B}$ the rational points in $\mathfrak{A}$. Then $\mathfrak{B}$ is pantactic in $\mathfrak{A}$.

Example 2. Let $\mathfrak{A}$ be the interval $(0, 1)$, and $\mathfrak{B}$ the Cantor set of I, 272. Then $\mathfrak{B}$ is apantactic in $\mathfrak{A}$.

2. *If* $\mathfrak{B} < \mathfrak{A}$ is *pantactic in* $\mathfrak{A}$, $\mathfrak{A}$ *contains no isolated points not in* $\mathfrak{B}$.

For let a be a point of $\mathfrak{A}$ not in $\mathfrak{B}$. Then by definition, in any $D_\delta(a)$ there is a point of $\mathfrak{B}$. Hence there are an infinity of points of $\mathfrak{B}$ in this domain. Hence a is a limiting point of $\mathfrak{A}$.

306. Let $\mathfrak{A}$ be complete. We say $\mathfrak{B} \leq \mathfrak{A}$ is of the 1° *category in* $\mathfrak{A}$, if $\mathfrak{B}$ is the union of an enumerable set of apantactic sets in $\mathfrak{A}$.

If $\mathfrak{B}$ is not of the 1° category, we say it is of the 2° *category.*

Sets of the 1° category may be called *Baire* sets.

Example. Let $\mathfrak{A}$ be the unit interval, and $\mathfrak{B}$ the rational points in it. Then $\mathfrak{B}$ is of the 1° category.

For $\mathfrak{B}$ being enumerable, let $\mathfrak{B} = \{b_n\}$. But each b_n is a single point and is thus apantactic in $\mathfrak{A}$.

The same reasoning shows that if $\mathfrak{B}$ is any enumerable set in $\mathfrak{A}$, then $\mathfrak{B}$ is of the 1° category.

307. 1. *If* $\mathfrak{B}$ *is of the* 1° *category in* $\mathfrak{A}$, $\mathfrak{A} - \mathfrak{B} = B$ *is* > 0.

For since $\mathfrak{B}$ is of the 1° category in $\mathfrak{A}$, it is the union of an enumerable set of apantactic sets $\{\mathfrak{B}_n\}$. Then by definition there exist points $a_1, a_2, \cdots$ in $\mathfrak{A}$ such that

$$D_{\delta_1}(a_1) > D_{\delta_2}(a_2) > \cdots \quad , \quad \delta_n \doteq 0 \tag{1}$$

where $D(a_1)$ contains no point of $\mathfrak{B}_1$, $D(a_2)$ no point of $\mathfrak{B}_2$, etc. Let b be the point determined by 1). Since $\mathfrak{A}$ is complete by definition, b is a point of $\mathfrak{A}$. As it is not in any $\mathfrak{B}_n$, it is not in $\mathfrak{B}$. Hence B contains at least one point.

2. *Let* $\mathfrak{A}$ *be the union of an enumerable set of sets* $\{\mathfrak{A}_n\}$, *each* $\mathfrak{A}_n$ *being of the* 1° *category in* $\mathfrak{B}$. *Then* $\mathfrak{A}$ *is of the* 1° *category in* $\mathfrak{B}$.

This is obvious, since the union of an enumerable set of enumerable sets is enumerable.

3. *Let $\mathfrak{B}$ be of the $1°$ category in $\mathfrak{A}$. Then $B = \mathfrak{A} - \mathfrak{B}$ is of the $2°$ category in $\mathfrak{A}$.*

For otherwise $\mathfrak{B} + B$ would be of the $1°$ category in $\mathfrak{A}$. But

$$\mathfrak{A} - (\mathfrak{B} + B) = 0,$$

and this violates 1.

4. It is now easy to give examples of sets of the $2°$ category. For instance, the irrational points in the interval $(0, 1)$ form a set of the $2°$ category.

308. *Let $\mathfrak{A}$ be a set of the $1°$ category in the cube $\mathfrak{Q}$. Then $A = \mathfrak{Q} - \mathfrak{A}$ has the cardinal number $\mathfrak{c}$.*

If A has an inner point, $D_\delta(a)$, for sufficiently small δ, lies in A. As Card $D_\delta = \mathfrak{c}$, the theorem is proved.

Suppose that A has no inner point. Let $\mathfrak{A}$ be the union of the apantactic sets $\mathfrak{A}_1 < \mathfrak{A}_2 < \cdots$ in $\mathfrak{Q}$. Let $A_n = \mathfrak{Q} - \mathfrak{A}_n$. Let q_n be the maximum of the sides of the cubes lying wholly in A_n. Obviously $q_n \doteq 0$, since by hypothesis A has no inner points. Let Q be a cube lying in A_1. As $q_n \doteq 0$, there exists an n_1 such that Q has at least two cubes lying in A_{n_1}; call them Q_0, Q_1. There exists an $n_2 > n_1$ such that Q_0, Q_1 each have two cubes in A_{n_2}; call them

$$Q_{0,0} \quad , \quad Q_{0,1} \quad ; \quad Q_{1,0} \quad , \quad Q_{1,1},$$

or more shortly Q_{ι_1, ι_2}.

Each of these gives rise similarly to two cubes in some A_{n_3}, which may be denoted by $Q_{\iota_1, \iota_2, \iota_3}$, where the indices as before have the values 0, 1. In this way we may continue getting the cubes

$$Q_{\iota_1} \quad , \quad Q_{\iota_1\iota_2} \quad , \quad Q_{\iota_1\iota_2\iota_3} \cdots$$

Let α be a point lying in a sequence of these cubes. It obviously does not lie in $\mathfrak{A}$, if the indices are not, after a certain stage, all 0 or all 1. This point α is characterized by the sequence

$$\iota_1\iota_2\iota_3\iota_4 \cdots$$

which may be read as a number in the dyadic system. But these numbers have the cardinal number $\mathfrak{c}$.

309. *Let $\mathfrak{A}$ be a complete apantactic set in a cube $\mathfrak{Q}$. Then there exists an enumerable set of cubical cells $\{\mathfrak{q}\}$ such that each point of $\mathfrak{A}$ lies on a face of one of these $\mathfrak{q}$, or is a limit point of their faces.*

For let $D_1 > D_2 > \cdots$ be a sequence of superimposed divisions of $\mathfrak{Q}$, whose norms $\delta_n \doteq 0$. Let

$$d_{11},\ d_{12},\ d_{13} \cdots \tag{1}$$

be the cells of D_1 containing no point of $\mathfrak{A}$ within them. Let

$$d_{21},\ d_{22},\ d_{23} \cdots \tag{2}$$

denote those cells of D_2 containing no point of $\mathfrak{A}$ within them and not lying in a cell of 1). In this way we may get an infinite sequence of cells $\mathfrak{D} = \{d_{mn}\}$, where for each m, the corresponding n is finite, and $m \doteq \infty$. Each point a of A lies in some $d_{m,n}$. For $\mathfrak{A}$ being complete, Dist $(a, \mathfrak{A}) > 0$. As the norms $\delta_n \doteq 0$, a must lie in some cell of D_n, for a sufficiently large n. The truth of the theorem is now obvious.

310. *Let $\mathfrak{B}$ be pantactic in $\mathfrak{A}$. Then there exists an enumerable set $\mathfrak{E} \leq \mathfrak{B}$ which is pantactic in $\mathfrak{A}$.*

For let $D_1 > D_2 > \cdots$ be a set of superimposed cubical divisions of norms $d_n \doteq 0$. In any cell of D_1 containing within it a point of $\mathfrak{A}$, there is at least one point of $\mathfrak{B}$. If the point of $\mathfrak{A}$ lies on the face of two or more cells, the foregoing statement will hold for at least one of the cells. Let us now take one of these points in each of these cells; this gives an enumerable set $\mathfrak{E}_1$. The same holds for the cells of D_2. Let us take a point in each of these cells, taking when possible points of $\mathfrak{E}_1$. Let $\mathfrak{E}_2$ denote the points of this set not in $\mathfrak{E}_1$. Continuing in this way, let

$$\mathfrak{E} = \mathfrak{E}_1 + \mathfrak{E}_2 + \cdots$$

Then $\mathfrak{E}$ is pantactic in $\mathfrak{A}$, and is enumerable, since each $\mathfrak{E}_n$ is.

Corollary. In any set $\mathfrak{A}$, finite or infinite, there exists an enumerable set $\mathfrak{E}$ which is pantactic in $\mathfrak{A}$.

For we have only to set $\mathfrak{B} = \mathfrak{A}$ in the above theorem.

311. 1. *The points $\mathfrak{C}$ where the continuous function $f(x_1 \cdots x_m)$ takes on a given value g in the complete set $\mathfrak{A}$, form a complete set.*

For let $c_1,\ c_2 \cdots$ be points of $\mathfrak{C}$ which $\doteq c$. We show c is a point of $\mathfrak{C}$. For

$$g = f(c_1) = f(c_2) = \cdots$$

As f is continuous, $f(c_n) \doteq f(c)$.

Hence $f(c) = g$,

and c lies in $\mathfrak{C}$.

2. *Let $f(x_1 \cdots x_m)$ be continuous in the limited or unlimited set $\mathfrak{A}$. If the value of f is known in an enumerable pantactic set $\mathfrak{E}$ in $\mathfrak{A}$, which contains all the isolated points of $\mathfrak{A}$, in case there be such, the value of f is known at every point of $\mathfrak{A}$.*

For let a be a limiting point of $\mathfrak{A}$ not in $\mathfrak{E}$. Since $\mathfrak{E}$ is pantactic in $\mathfrak{A}$, there exists a sequence of points $e_1, e_2 \cdots$ in $\mathfrak{E}$ which $\doteq a$. Since f is continuous, $f(e_n) \doteq f(a)$. As f is known at each e_n, it is known at a.

3. *Let $\mathfrak{F} = \{f\}$ be the class of one-valued continuous functions defined over a limited point set $\mathfrak{A}$. Then*

$$\mathfrak{f} = \text{Card}\, \mathfrak{F} = \mathfrak{c}.$$

For let $\mathfrak{R}_\infty$ be a space of an infinite enumerable number of dimensions, and let

$$y = (y_1, y_2, \cdots)$$

denote one of its points. Let f have the value η_1 at e_1, the value η_2 at $e_2 \cdots$ for the points of $\mathfrak{E}$ defined in 2. Then the complex

$$(\eta_1, \eta_2 \cdots)$$

completely determines f. But this complex determines also a point η in $\mathfrak{R}_\infty$ whose coördinates are η_n. We now associate f with η. Hence

$$\mathfrak{f} \leq \text{Card}\, \mathfrak{R}_\infty = \mathfrak{c}.$$

On the other hand, $\mathfrak{f} \geq \mathfrak{c}$, since in $\mathfrak{F}$ there is the function $f(x_1 \cdots x_m) = g$ in $\mathfrak{A}$, where g is any real number.

312. *Let $\mathfrak{B}$ denote the class of complete or perfect subsets lying in the infinite set $\mathfrak{A}$, which latter contains at least one complete set. Then*

$$\mathfrak{b} = \text{Card}\, \mathfrak{B} = \mathfrak{c}.$$

For let $a_1, a_2, \cdots \doteq a$, all these points lying in $\mathfrak{A}$. Then

$$a_{\iota_1} \quad , \quad a_{\iota_2} \quad , \quad a_{\iota_3} \cdots \doteq a \quad ; \quad \iota_1 < \iota_2 < \iota_3 \cdots \tag{1}$$

But for ι_1 we may take any number in $\mathfrak{J}_1 = (1, 2, 3, \cdots)$; for ι_2 we may take any number in $\mathfrak{J}_2 = (\iota_1 + 1, \iota_1 + 2, \cdots)$, etc.

Obviously the cardinal number of the class of these sequences 1) is $\mathfrak{e}^{\mathfrak{e}} = \mathfrak{c}$. But

$$(a, a_{\iota_1}, a_{\iota_2}, a_{\iota_3} \cdots)$$

is a complete set in $\mathfrak{A}$. Hence $\mathfrak{b} \geq \mathfrak{c}$. On the other hand, $\mathfrak{b} \leq \mathfrak{c}$. For let

$$D_1 > D_2 > \cdots \tag{2}$$

be a sequence of superimposed cubical division of norms $\doteq 0$. Each D_n embraces an enumerable set of cells. Thus the set of divisions gives an enumerable set of cells. Each cell shall have assigned to it, for a given set in $\mathfrak{B}$, the sign $+$ or $-$ according as $\mathfrak{B}$ is exterior to this cell or not. This determines a distribution of two things over an enumerable set of compartments.

The cardinal number of the class of these distributions is $2^{\mathfrak{e}} = \mathfrak{c}$. But each $\mathfrak{B}$ determines a distribution. Hence $\mathfrak{b} \leq \mathfrak{c}$.

Transfinite Derivatives

313. 1. We have seen, I, 266, that

$$\mathfrak{A}' \geq \mathfrak{A}'' \geq \mathfrak{A}''' \geq \cdots$$

Thus

$$\mathfrak{A}^{(n)} = Dv(\mathfrak{A}', \mathfrak{A}'' \cdots \mathfrak{A}^{(n)}). \tag{1}$$

Let now $\mathfrak{A}$ be a limited point aggregate of the second species. It has then derivatives of every finite order. Therefore by 18,

$$Dv(\mathfrak{A}', \mathfrak{A}'', \mathfrak{A}''', \cdots) \tag{2}$$

contains at least one point, and in analogy with 1), we call the set 2) *the derivative of order* ω *of* $\mathfrak{A}$, and denote it by

$$\mathfrak{A}^{(\omega)},$$

or more shortly by

$$\mathfrak{A}^{\omega}.$$

Now we may reason on $\mathfrak{A}^{\omega}$ as on any point set. If it is infinite, it must have at least one limiting point, and may of course have more. In any case its derivative is denoted by

$$\mathfrak{A}^{(\omega+1)} \text{ or } \mathfrak{A}^{\omega+1}.$$

The derivative of $\mathfrak{A}^{\omega+1}$ is denoted by

$$\mathfrak{A}^{(\omega+2)} \text{ or } \mathfrak{A}^{\omega+2}, \quad \text{etc.}$$

Making use of ω we can now state the theorem:

In order that the point set $\mathfrak{A}$ is of the first species it is necessary and sufficient that $\mathfrak{A}^{(\omega)} = 0$.

2. We have seen in 18 that $\mathfrak{A}^{\omega}$ is complete. The reasoning of I, 266 shows *that* $\mathfrak{A}^{\omega+1}, \mathfrak{A}^{\omega+2}, \cdots$, *when they exist, are also complete.* Then 18 shows that, *if* $\mathfrak{A}^{\omega+n}$ $\quad n = 1, 2, \cdots$ *exist,*

$$Dv(\mathfrak{A}^{\omega} \geq \mathfrak{A}^{\omega+1} \geq \mathfrak{A}^{\omega+2} \geq \cdots) \tag{3}$$

exists and is complete. The set 3) is called *the derivative of order* $\omega \cdot 2$ and is denoted by

$$\mathfrak{A}^{(\omega 2)} \text{ or } \mathfrak{A}^{\omega 2}.$$

Obviously we may continue in this way indefinitely until we reach a derivative of order α containing only a finite number of points. Then

$$\mathfrak{A}^{\alpha+1} = 0.$$

That this process of derivation may never stop is illustrated by taking for $\mathfrak{A}$ any limited perfect set, for then

$$\mathfrak{A} = \mathfrak{A}' = \mathfrak{A}'' = \cdots = \mathfrak{A}^{\omega} = \mathfrak{A}^{\omega \cdot 2} = \cdots$$

3. We may generalize as follows: Let α denote a limitary ordinal number. If each $\mathfrak{A}^{\beta} > 0$, $\beta < \alpha$, we set

$$\mathfrak{A}^{(\alpha)} = \mathfrak{A}^{\alpha} = Dv\{\mathfrak{A}^{\beta}\}$$

when it exists.

4. If $\mathfrak{A}^{\alpha} > 0$, while $\mathfrak{A}^{\ +1} = 0$, we say $\mathfrak{A}$ is of *order* α.

314. 1. Let a be a limiting point of $\mathfrak{A}$. Let

$$\alpha_\delta = \text{Card } V_\delta(a).$$

Obviously α_δ is monotone decreasing with δ. Suppose that there exists an α and a $\delta_0 > 0$, such that for all $0 < \delta \leq \delta_0$

$$\alpha = \text{Card } V(a).$$

We shall say that *a is a limiting point of rank α.*

If every $\alpha_\delta \geq \alpha$, we shall say that

$$\text{Rank } a \geq \alpha.$$

If every $\alpha_\delta > \alpha$, we shall say that

$$\text{Rank } a > \alpha.$$

2. *Let $\mathfrak{A}$ be a limited aggregate of cardinal number α. Then there is at least one limiting point of $\mathfrak{A}$, of rank α.*

The demonstration is entirely similar to I, 264. Let $\delta_1 > \delta_2 > \cdots \doteq 0$. Let us effect a cubical division of $\mathfrak{A}$ of norm δ_1. In at least one cell lies an aggregate $\mathfrak{A}_1$ having the cardinal number α. Let us effect a cubical division of $\mathfrak{A}_1$ of norm δ_2. In at least one cell lies an aggregate $\mathfrak{A}_2$ having the cardinal number α, etc. These cells converge to a point a, such that

$$\text{Card } V_\delta(a) = \alpha,$$

however small δ is taken.

3. *If* Card $\mathfrak{A} > \mathfrak{e}$, *there exists a limiting point of $\mathfrak{A}$ of rank $> \mathfrak{e}$.*

The demonstration is similar to that of 2.

4. *If there is no limiting point of $\mathfrak{A}$ of rank $> \mathfrak{e}$, $\mathfrak{A}$ is enumerable.*

This follows from 3.

5. *Let* Card $\mathfrak{A}$ *be* $> \mathfrak{e}$. *Let $\mathfrak{B}$ denote the limiting points of $\mathfrak{A}$ whose ranks are $> \mathfrak{e}$. Then $\mathfrak{B}$ is perfect.*

For obviously $\mathfrak{B}$ is complete. We need therefore only to show that it is dense. To this end let b be a point of $\mathfrak{B}$. About b let us describe a sequence of concentric spheres of radii $r_n \doteq 0$. These spheres determine a sequence of spherical shells $\{S_n\}$, no two of which have a point in common. If $\mathfrak{A}_n$ denote the points of $\mathfrak{A}$ in S_n, we have

$$V = V^*_{r_1}(b) = \mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_3 + \cdots$$

Thus if each $\mathfrak{A}_m$ were enumerable, V is enumerable and hence Rank b is not $> \mathfrak{e}$. Thus there is one set $\mathfrak{A}_m$ which is not enumerable, and hence by 3 there exists a point of $\mathfrak{B}$ in S_m. But then there are points of $\mathfrak{B}$ in any $V_r^*(b)$, and b is not isolated.

6. *A set $\mathfrak{A}$ which contains no dense component is enumerable.*

For suppose $\mathfrak{A}$ were not enumerable. Let $\mathfrak{P}$ denote the proper limiting points of $\mathfrak{A}$. Then $\mathfrak{P}$ contains a point whose rank is $> \mathfrak{e}$. But the set of these points is dense. This contradicts the hypothesis of the theorem.

315. *Let α lie in Z_n. If $\mathfrak{A}^\alpha > 0$, it is complete.*

For if α is non-limitary, reasoning similar to I, 266 shows that $\mathfrak{A}^\alpha$ is complete. Suppose then that α is limitary, and $\mathfrak{A}^\alpha$ is not

complete. The derivatives of $\mathfrak{A}$ of order $\leq \alpha$ which are not complete, form a well ordered set and have therefore a first element $\mathfrak{A}^\beta$, where β is necessarily a limitary number. Then

$$\mathfrak{A}^\beta = Dv(\mathfrak{A}^\gamma) \quad , \quad \gamma < \beta.$$

But every point of $\mathfrak{A}^\beta$ lies in each $\mathfrak{A}^\gamma$. Hence every limiting point of $\mathfrak{A}^\beta$ is a limiting point of each $\mathfrak{A}^\gamma$ and hence lies in $\mathfrak{A}^\beta$. Hence $\mathfrak{A}^\beta$ is complete, which is a contradiction.

316. *Let α be a limitary number in Z_n. If $\mathfrak{A}^\beta > 0$ for each $\beta < \alpha$, $\mathfrak{A}^\alpha$ exists.*

For there exists an $\aleph_m$, $m \leq n-2$, sequence

$$\gamma < \delta < \epsilon < \eta < \cdots \doteq \alpha. \tag{1}$$

Let c be a point of $\mathfrak{A}^\gamma$, d a point of $\mathfrak{A}^\delta$, e a point of $\mathfrak{A}^\epsilon$, etc. Then the set

$$(c, d, e, f, \cdots)$$

has at least one limiting point l of rank $\aleph_m$. Let ϵ be any number in 1). Then l is a limiting point of rank $\aleph_m$ of the set

$$(e, f, \cdots).$$

Thus l is a limiting point of every $\mathfrak{A}^\beta$, $\beta < \alpha$, and hence of $\mathfrak{A}^\alpha$.

317. Let us show how we may form point sets whose order α is any number in Z_1 or Z_2.

We take the unit interval $\mathfrak{A} = (0, 1)$ as the base of our considerations.

In $\mathfrak{A}$, take the points

$$\mathfrak{B}_1 = \tfrac{1}{2} \quad , \quad \tfrac{3}{4} \quad , \quad \tfrac{7}{8} \quad , \quad \tfrac{15}{16} \quad \cdots \tag{1}$$

Obviously $\mathfrak{B}_1' = 1$, $\mathfrak{B}_1'' = 0$. Hence $\mathfrak{B}_1$ is of order 1. The set $\mathfrak{B}_1$ divides $\mathfrak{A}$ into a set of intervals

$$\mathfrak{A}_1 \quad , \quad \mathfrak{A}_2 \quad , \quad \mathfrak{A}_3 \quad \cdots \tag{2}$$

In $\mathfrak{A}_1 = (0, \frac{1}{2})$ take a set of points similar to 1) which has as single limiting point, the point $\frac{1}{2}$. In $\mathfrak{A}_2 = (\frac{1}{2}, \frac{3}{4})$ take a set of points similar to 1) which has as single limiting point, the point $\frac{3}{4}$, etc. Let us call the resulting set of points $\mathfrak{B}_2$.

Obviously
$$\mathfrak{B}_2' = \tfrac{1}{2} \quad , \quad \tfrac{3}{4} \quad , \quad \tfrac{7}{8} \quad , \quad \cdots = \mathfrak{B}_1 .$$
Hence
$$\mathfrak{B}_2'' = \mathfrak{B}_1' = 1 \text{ and } \mathfrak{B}_2''' = 0$$
Thus $\mathfrak{B}_2$ is of order 2.

In each of the intervals 2) we may place a set of points similar to $\mathfrak{B}_2$, such that the right-hand end point of each interval $\mathfrak{A}_n$ is a limiting point of the set. The resulting set $\mathfrak{B}_3$ is of order 3, etc.

This shows that we may form sets of every finite order.

Let us now place a set of order 1 in $\mathfrak{A}_1$, a set of order 2 in $\mathfrak{A}_2$, etc. The resulting set $\mathfrak{B}_\omega$ is of order ω. For $\mathfrak{B}_\omega^{(n)}$ has no points in $\mathfrak{A}_1, \mathfrak{A}_2 \cdots \mathfrak{A}_{n-1}$, while the point 1 lies in every $\mathfrak{B}_\omega^{(n)}$.

Thus
$$\mathfrak{B}_\omega^{(\omega)} = 1.$$
Hence
$$\mathfrak{B}_\omega^{(\omega+1)} = 0,$$
and $\mathfrak{B}_\omega$ is of order ω.

Let us now place in each $\mathfrak{A}_n$ a set similar to $\mathfrak{B}_\omega$, having the right-hand end point of $\mathfrak{A}_n$ as limiting point. The resulting set $\mathfrak{B}_{\omega+1}$ is of order $\omega + 1$. In this way we may proceed to form sets of order $\omega + 2, \omega + 3, \cdots$ just as we did for orders $2, 3, \cdots$ We may also form now a set of order $\omega 2$, as we before formed a set of order ω.

Thus we may form sets of order
$$\omega \quad , \quad \omega \cdot 2 \quad , \quad \omega \cdot 3 \quad , \quad \omega \cdot 4 \quad \cdots$$
and hence of order ω^2, etc.

318. 1. *Let $\mathfrak{A}$ be limited or not, and let $\mathfrak{A}_\iota^{(\beta)}$ denote the isolated points of $\mathfrak{A}^\beta$. Then*
$$\mathfrak{A}' = \sum_\beta \mathfrak{A}_\iota^{(\beta)} + \mathfrak{A}^\Omega \quad , \quad \beta = 1, 2, \cdots < \Omega. \qquad (1$$
For
$$\mathfrak{A}' = \mathfrak{A}_\iota' + \mathfrak{A}'' \quad , \quad \mathfrak{A}'' = \mathfrak{A}_\iota{}' + \mathfrak{A}''' \cdots$$
Thus
$$\mathfrak{A}' = \mathfrak{A}_\iota' + \mathfrak{A}'' + \cdots + \mathfrak{A}^{(n-1)} + \mathfrak{A}^{(n)} ;$$
that is, $\mathfrak{A}'$ is the sum of the points of $\mathfrak{A}'$ not in $\mathfrak{A}''$, of the points of $\mathfrak{A}''$ not in $\mathfrak{A}'''$, etc. If now there are points common to every $\mathfrak{A}^{(n)}$ we have
$$\mathfrak{A}' = \sum_n \mathfrak{A}_\iota{}^{(n)} + \mathfrak{A}^{(\omega)} \quad , \quad n = 1, 2, \cdots < \omega.$$

On $\mathfrak{A}^{\omega}$ we can reason as on $\mathfrak{A}'$, and in general for any $\alpha < \Omega$ we have

$$\mathfrak{A}' = \sum_{\beta<\alpha} \mathfrak{A}_{\iota}^{(\beta)} + \mathfrak{A}^{(\alpha)},$$

which gives 1).

2. *If* $\mathfrak{A}^{\Omega} = 0$, $\mathfrak{A}$ *and* $\mathfrak{A}'$ *are enumerable.*

For not every

$$\mathfrak{A}^{(\alpha)} > 0 \qquad \alpha < \Omega, \text{ by } 316.$$

Hence there is a first α, call it γ, such that $\mathfrak{A}^{\gamma} = 0$. Then 1) reduces to

$$\mathfrak{A}' = \sum_{\beta} \mathfrak{A}_{\iota}^{(\beta)} \quad , \quad \beta = 1, 2, \cdots < \gamma.$$

But the summation extends over an enumerable set of terms, each of which is enumerable by 289. Hence $\mathfrak{A}'$ is enumerable. But then $\mathfrak{A}$ is also enumerable by 237, 2.

3. *Conversely, if* $\mathfrak{A}'$ *is enumerable*, $\mathfrak{A}^{\Omega} = 0$.

For if $\mathfrak{A}^{\Omega} > 0$, there is a non-enumerable set of terms in 1), if no $\mathfrak{A}^{(\beta)}$ is perfect; and as each term contains at least one point, $\mathfrak{A}'$ is not enumerable. If some $\mathfrak{A}^{(\beta)}$ is perfect, $\mathfrak{A}'$ contains a perfect partial set and is therefore not enumerable by 245.

4. From 2, 3, we have:

For $\mathfrak{A}'$ *to be enumerable, it is necessary and sufficient that there exists a number* α *in* Z_1 *or* Z_2 *such that* $\mathfrak{A}^{\alpha} = 0$.

5. *If* $\mathfrak{A}$ *is complete, it is necessary and sufficient in order that* $\mathfrak{A}$ *be enumerable, that there exists an* α *in* Z_1 *or* Z_2 *such that* $\mathfrak{A}^{\alpha} = 0$.

For

$$\mathfrak{A} = \mathfrak{A}_{\iota} + \mathfrak{A}',$$

and the first term is enumerable.

6. If $\mathfrak{A}^{\beta} = 0$ for some $\beta < \Omega$, we say $\mathfrak{A}$ is *reducible*, otherwise it is *irreducible*.

319. *If* $\mathfrak{A}^{\Omega} > 0$, *it is perfect.*

By 315 it is complete. We therefore have only to show that its isolated points $\mathfrak{A}_{\iota}^{\Omega} = 0$. Suppose the contrary; let a be an isolated point of $\mathfrak{A}^{\Omega}$.

Let us describe a sphere S of radius r about a, containing no other point of $\mathfrak{A}^{\Omega}$. Let $\mathfrak{B}$ denote the points of $\mathfrak{A}'$ in S. Let

$$r > r_1 > r_2 > \cdots \doteq 0.$$

Let S_n denote a sphere about a of radius r_n. Let $\mathfrak{B}_n$ denote the points of $\mathfrak{B}$ lying between S_{n-1}, S_n, including those points which may lie on S_{n-1}. Then

$$\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2 + \mathfrak{B}_3 + \cdots + a.$$

Each $\mathfrak{B}_m$ is enumerable. For any point of $\mathfrak{B}_m^\Omega$ is a point of $\mathfrak{B}^\Omega = a$. Hence $\mathfrak{B}_m^\Omega = 0$ and $\mathfrak{B}_m$ is enumerable by 318, 2.

Thus $\mathfrak{B}$ is enumerable. This, however, is impossible since $\bar{\mathfrak{B}}^\Omega = a$, and is thus > 0.

320. 1. *In the relation*

$$\mathfrak{A}' = \sum_\beta \mathfrak{A}_\iota^{(\beta)} + \mathfrak{A}^\Omega \qquad \beta = 1, 2, \cdots < \Omega,$$

the first term on the right is enumerable.

For let us set

$$\mathfrak{B} = \sum_\beta \mathfrak{A}_\iota^{(\beta)};$$

also let

$$r_1 > r_2 > \cdots \doteq 0.$$

Let $\mathfrak{B}_n$ denote the points of $\mathfrak{B}$ whose distance δ from $\mathfrak{A}^\Omega$ satisfies the relation

$$r_n \geq \delta > r_{n+1}.$$

Then the distance of any point of $\mathfrak{B}_n'$ from $\mathfrak{A}^\Omega$ is $\geq r_{n+1}$. If $\mathfrak{B}_0$ includes all points of $\mathfrak{B}$ whose distance from $\mathfrak{A}^\Omega$ is $> r_1$, we have

$$\mathfrak{B} = \mathfrak{B}_0 + \mathfrak{B}_1 + \mathfrak{B}_2 + \cdots$$

Each $\mathfrak{B}_n$ is enumerable. For if not, $\mathfrak{B}_n^\Omega > 0$. Any point of $\mathfrak{B}_n^\Omega$ as b lies in $\mathfrak{A}^\Omega$. Hence

$$\text{Dist}\,(b, \mathfrak{A}^\Omega) = 0.$$

On the other hand, as b lies in $\mathfrak{B}_n'$, its distance from $\mathfrak{A}^\Omega$ is $\geq r_{n+1}$, which is a contradiction.

2. *If $\mathfrak{A}'$ is not enumerable, there exists a first number α in Z_1 or Z_2 such that $\mathfrak{A}^\alpha$ is perfect.*

This is a corollary of 1.

3. *If $\mathfrak{A}$ is complete and not enumerable, there exists a first number α in $Z_1 + Z_2$ such that $\mathfrak{A}^\alpha$ is perfect.*

4. *If $\mathfrak{A}$ is complete,*

$$\mathfrak{A} = \mathfrak{E} + \mathfrak{P};$$

where $\mathfrak{E}$ is enumerable, and $\mathfrak{P}$ is perfect. If $\mathfrak{A}$ is enumerable, $\mathfrak{P} = 0$.

Complete Sets

321. Let us study now some of the properties of complete point sets. We begin by considering limited perfect rectilinear sets. Let $\mathfrak{A}$ be such a set. It has a first point a and a last point b. It therefore lies in the interval $I = (a, b)$. If $\mathfrak{A}$ is pantactic in any partial interval $J = (\alpha, \beta)$ of I, $\mathfrak{A}$ embraces all the points of J, since $\mathfrak{A}$ is perfect. Let us therefore suppose that $\mathfrak{A}$ is apantactic in I. An example of such sets is the Cantor set of I, 272.

Let $D = \{\delta\}$ be a set of intervals no two of which have a point in common. We say D is *pantactic* in an interval I, when I contains no interval which does not contain some interval δ, or at least a part of some δ. Def

It is *separated* when no two of its intervals have a point in common. Def

322. 1. *Every limited rectilinear apantactic perfect set $\mathfrak{A}$ determines an enumerable pantactic set of separated intervals $D = \{\delta\}$, whose end points alone lie in $\mathfrak{A}$.*

For let $\mathfrak{A}$ lie in $I = (\alpha, \beta)$, where α, β are the first and last points of $\mathfrak{A}$. Let $\mathfrak{B} = I - \mathfrak{A}$. Each point b of $\mathfrak{B}$ falls in some interval δ whose end points lie in $\mathfrak{A}$. For otherwise we could approach b as near as we chose, ranging over a set of points of $\mathfrak{A}$. But then b is a point of $\mathfrak{A}$, as this is perfect. Let us therefore take these intervals as large as possible and call them δ.

The intervals δ are pantactic in I, for otherwise $\mathfrak{A}$ could not be apantactic. They are enumerable, for but a finite set can have lengths $> I/n + 1$ and $\leq I/n$, $n = 1, 2 \cdots$

It is separated, since $\mathfrak{A}$ contains no isolated points.

2. The set of intervals $D = \{\delta\}$ just considered are said to be *adjoint to* $\mathfrak{A}$, or *determined by* $\mathfrak{A}$, or *belonging to* $\mathfrak{A}$. Def

323. *Let $\mathfrak{A}$ be an apantactic limited rectilinear perfect point set, to which belongs the set of intervals $D = \{\delta\}$. Then $\mathfrak{A}$ is formed of the end points $E = \{e\}$ of these intervals, and their limiting points E'.*

For we have just seen that the end points e belong to $\mathfrak{A}$. Moreover, $\mathfrak{A}$ being perfect, E' must be a part of $\mathfrak{A}$.

$\mathfrak{A}$ contains no other points. For let a be a point of $\mathfrak{A}$ not in E, E'. Let α be another point of $\mathfrak{A}$. In the interval (a, α) lies an end point e of some interval of D. In the interval (a, e) lies another end point e_1. In the interval (a, e_1) lies another end point e_2, etc. The set of points $e, e_1, e_2 \cdots \doteq a$. Hence a lies in E', which is a contradiction.

324. *Conversely, the end points $E = \{e\}$ and the limiting points of the end points of a pantactic enumerable set of separated intervals $D = \{\delta\}$ form a perfect apantactic set $\mathfrak{A}$.*

For in the first place, $\mathfrak{A}$ is complete, since $\mathfrak{A} = (E, E')$. $\mathfrak{A}$ can contain no isolated points, since the intervals δ are separated. Hence $\mathfrak{A}$ is perfect. It is apantactic, since otherwise $\mathfrak{A}$ would embrace all the points of some interval, which is impossible, as D is pantactic.

325. Since the adjoint set of intervals $D = \{\delta\}$ is enumerable, it can be arranged in a 1, 2, 3, $\cdots$ order *according to size* as follows.

Let δ be the largest interval, or if several are equally large, one of them. The interval δ causes I to fall into two other intervals. The interval to the left of δ, call I_0, that to the right of δ, call I_1. The largest interval in I_0, call δ_0, that in I_1, call δ_1. In this way we may continue without end, getting a sequence of intervals

$$\delta, \delta_0, \delta_1, \delta_{00}, \delta_{01}, \delta_{10}, \delta_{11} \cdots \tag{1}$$

and a similar series of intervals

$$I, I_0, I_1, I_{00}, I_{01} \cdots$$

The lengths of the intervals in 1) form a monotone decreasing sequence which $\doteq 0$.

If ν denote a complex of indices $\iota j \kappa \cdots$

$$D = \{\delta_\nu\} = \{\delta_{\iota j \kappa \cdots}\},$$

and

$$I_\nu = I_{\nu 0} + \delta_\nu + I_{\nu 1}.$$

326. 1. *The cardinal number of every perfect limited rectilinear point set $\mathfrak{A}$ is $\mathfrak{c}$.*

For if $\mathfrak{A}$ is not apantactic, it embraces all the points of some interval, and hence Card $\mathfrak{A} = \mathfrak{c}$. Let it be therefore apantactic.

Let $D = \{\delta_\nu\}$ be its adjoint set of intervals, arranged as in 325. Let $\mathfrak{C}$ be the Cantor set of I, 272. Let its adjoint set of intervals be $H = \{\eta_\nu\}$, arranged also as in 325. If we set $\delta_\nu \sim \eta_\nu$, we have $D \simeq H$. Hence

$$\mathrm{Card}\ \mathfrak{A} = \mathrm{Card}\ \mathfrak{C}.$$

But Card $\mathfrak{C} = \mathfrak{c}$ by 244, 4.

2. *The cardinal number of every limited rectilinear complete set* $\mathfrak{A}$
p. 280. *is either* $\mathfrak{e}$ *or* $\mathfrak{c}$.

For we have seen, 320, 4, that

$$\mathfrak{A} = \mathfrak{C} + \mathfrak{P}, \qquad \mathfrak{P} \gtreqless 0,$$

where $\mathfrak{C}$ is enumerable and $\mathfrak{P}$ is perfect,

$$\text{If } \mathfrak{P} = 0, \qquad \mathrm{Card}\ \mathfrak{A} = \mathfrak{e}.$$

$$\text{If } \mathfrak{P} > 0, \qquad \mathrm{Card}\ \mathfrak{A} = \mathfrak{c}.$$

For Card $\mathfrak{A} =$ Card $\mathfrak{C} +$ Card $\mathfrak{P} = \mathfrak{e} + \mathfrak{c} = \mathfrak{c}$.

327. *The cardinal number of every limited complete set* $\mathfrak{A}$ *in* $\mathfrak{R}_n$ *is either* $\mathfrak{e}$ *or* $\mathfrak{c}$. *It is* $\mathfrak{c}$, *if* $\mathfrak{A}$ *has a perfect component.*

The proof may be made by induction.

For simplicity take $m = 2$. By a transformation of space [242], we may bring $\mathfrak{A}$ into a unit square S. Let us therefore suppose $\mathfrak{A}$ were in S originally. Then Card $\mathfrak{A} \leq \mathfrak{c}$ by 247, 2.

Let $\mathfrak{C}$ be the projection of $\mathfrak{A}$ on one of the sides of S, and $\mathfrak{B}$ the points of $\mathfrak{A}$ lying on a parallel to the other side passing through a point of $\mathfrak{C}$. If $\mathfrak{B}$ has a perfect component, Card $\mathfrak{B} = \mathfrak{c}$, and hence Card $\mathfrak{A} = \mathfrak{c}$. If $\mathfrak{B}$ does not have a perfect component, the cardinal number of each $\mathfrak{B}$ is $\mathfrak{e}$. Now $\mathfrak{C}$ is complete by I, 717, 4. Hence if $\mathfrak{C}$ contains a perfect component, Card $\mathfrak{C} = \mathfrak{c}$, otherwise Card $\mathfrak{C} = \mathfrak{e}$. In the first case Card $\mathfrak{A} = \mathfrak{c}$, in the second it is $\mathfrak{e}$.

328. 1. Let $\mathfrak{A}$ be a complete set lying within the cube $\mathfrak{Q}$. Let $D_1 > D_2 > \cdots$ denote a set of superimposed cubical divisions of $\mathfrak{Q}$ of norms $\doteq 0$. Let d_1 be the set of those cubes of D_1 containing no point of $\mathfrak{A}$. Let d_2 be the set of those cubes of D_2 not in d_1, which contain no point of $\mathfrak{A}$. In this way we may continue. Let $\mathfrak{B} = \{d_n\}$. Then every point of $A = \mathfrak{Q} - \mathfrak{A}$ lies in $\mathfrak{B}$. For $\mathfrak{A}$ being

complete, any point a of A is an inner point of A. Hence $D_\rho(a)$ lies in A, for some ρ sufficiently small. Hence a lies in some d_m.

We have thus the result:

Any limited complete set is uniquely determined by an enumerable set of cubes $\{d_n\}$, *each of which is exterior to it.*

We may call $\mathfrak{B} = \{d_n\}$ the *border of* $\mathfrak{A}$, and the cells d_n, *border cells.*

2. *The totality of all limited perfect or complete sets has the cardinal number* $\mathfrak{c}$.

For any limited complete set $\mathfrak{C}$ is completely determined by its border $\{d_n\}$. The totality of such sets has a cardinal number $\leq \mathfrak{c}^{\mathfrak{e}} = \mathfrak{c}$. Hence Card $\{\mathfrak{C}\} \leq \mathfrak{c}$. Since among the sets $\mathfrak{C}$ is a $\mathfrak{c}$-set of segments, Card $\mathfrak{C} \geq \mathfrak{c}$.

329. If $\mathfrak{A}_\iota$ denote the isolated points of $\mathfrak{A}$, and $\mathfrak{A}_\lambda$ its proper limiting points, we may write

$$\mathfrak{A} = \mathfrak{A}_\iota + \mathfrak{A}_\lambda.$$

Similarly we have

$$\mathfrak{A}_\lambda = \mathfrak{A}_{\lambda\iota} + \mathfrak{A}_{\lambda^2},$$

$$\mathfrak{A}_{\lambda^2} = \mathfrak{A}_{\lambda^2\iota} + \mathfrak{A}_{\lambda^3}, \text{ etc.}$$

We thus have

$$\mathfrak{A} = \mathfrak{A}_\iota + \mathfrak{A}_{\lambda_\iota} + \mathfrak{A}_{\lambda^2_\iota} + \cdots + \mathfrak{A}_{\lambda^{n-1}_\iota} + \mathfrak{A}_{\lambda^n}.$$

At the end of each step, certain points of $\mathfrak{A}$ are *sifted* out. They may be considered as *adhering* loosely to $\mathfrak{A}$, while the part which remains may be regarded as *cohering* more closely to the set. Thus we may call $\mathfrak{A}_{\lambda^{n-1}_\iota}$, the n^{th} *adherent*, and $\mathfrak{A}_{\lambda^n}$ the n^{th} *coherent.*

If the n^{th} *coherent is* 0, $\mathfrak{A}$ *is enumerable.*

If the above process does not stop after a finite number of steps,

let $$\mathfrak{A}_\omega = Dv(\mathfrak{A}_\lambda, \mathfrak{A}_{\lambda^2}, \mathfrak{A}_{\lambda^3} \cdots).$$

If $\mathfrak{A}_\omega > 0$, we call it the *coherent of order* ω.

Then obviously $$\mathfrak{A} = \Sigma \mathfrak{A}_{\lambda^n_\iota} + \mathfrak{A}_\omega.$$

We may now sift $\mathfrak{A}_\omega$ as we did $\mathfrak{A}$.

If α is a limitary number, defined by

$$\alpha_1 < \alpha_2 < \alpha_3 \cdots \doteq \alpha,$$

we set
$$\mathfrak{A}_\alpha = Dv\{\mathfrak{A}_\lambda{}^{\alpha n}\}$$

and call it, when it exists, the *coherent of order* α. Thus we can write
$$\mathfrak{A} = \sum_\alpha \mathfrak{A}_{\lambda^{\alpha_\iota}} + \mathfrak{A}_{\lambda^\beta} \qquad \alpha = 1, 2, \cdots < \beta \tag{1}$$

where β is a number in Z_2.

330. 1. *When* $\mathfrak{A}$ *is enumerable,*

$$\mathfrak{A} = \sum_\alpha \mathfrak{A}_{\lambda^{\alpha_\iota}} + \mathfrak{A}_{\lambda^\beta} \qquad \alpha = 1, 2, \cdots \beta$$
$$= \mathfrak{J} + \mathfrak{D}\,; \tag{1}$$

where $\mathfrak{J}$ *is the sum of an enumerable set of isolated sets, and* $\mathfrak{D}$, *when it exists, is dense.*

For the adherences of different orders have no point in common with those of any other order. They are thus distinct. Thus the sum $\mathfrak{J}$ can contain but an enumerable set of adherents, for otherwise $\mathfrak{A}$ could not be enumerable. Thus there is a first ordinal number β for which

$$\mathfrak{A}_{\lambda^{\beta_\iota}} = 0.$$

As now in general

$$\mathfrak{A}_{\lambda^\beta} = \mathfrak{A}_{\lambda^{\beta_\iota}} + \mathfrak{A}_{\lambda^{\beta+1}},$$

we have
$$\mathfrak{A}_{\lambda^\beta} = \mathfrak{A}_\lambda{}^{\beta+1} = \mathfrak{A}_\lambda{}^{\beta+2} = \cdots$$

As $\mathfrak{A}_{\lambda^\beta}$ thus contains no isolated points, it is dense, when not 0, by I, 270.

2. *When* $\mathfrak{A}$ *is not enumerable,* $\mathfrak{D} > 0$. For if not, $\mathfrak{A} = \mathfrak{J}$, and $\mathfrak{J}$ is enumerable.

331.
$$\overline{\mathfrak{A}} = \overline{\mathfrak{A}}'. \tag{1}$$

For let D be a cubical division of space. As usual let

$$\overline{\mathfrak{A}}_D \quad , \quad \overline{\mathfrak{A}}'_D$$

denote those cells of D containing a point of $\mathfrak{A}$, $\mathfrak{A}'$ respectively. The cells of $\overline{\mathfrak{A}}'_D$ not in $\overline{\mathfrak{A}}_D$ will be adjacent to those of $\overline{\mathfrak{A}}_D$, and

these may be consolidated with the cells of D, forming a new division Δ of norm δ which in general will not be cubical. Then

$$\overline{\mathfrak{A}}_\Delta = \overline{\mathfrak{A}}'_\Delta + \overline{\mathfrak{A}}_\Delta{}^*.$$

The last term is formed of cells that contain only a finite number of points of $\mathfrak{A}$. These cells may be subdivided, forming a new division E such that in

$$\overline{\mathfrak{A}}_E = \overline{\mathfrak{A}}'_E + \overline{\mathfrak{A}}_E{}^* \tag{2}$$

the last term is $< \epsilon/3$. Now if δ is sufficiently small,

$$\overline{\mathfrak{A}}_\Delta - \overline{\mathfrak{A}} < \frac{\epsilon}{3} \quad , \quad \overline{\mathfrak{A}}'_\Delta - \overline{\mathfrak{A}}' < \frac{\epsilon}{3}. \tag{3}$$

Hence from 2), 3) we have 1).

332. *If* $\overline{\mathfrak{A}} > 0$, Card $\mathfrak{A} = \mathfrak{c}$.

For let $\mathfrak{B}$ denote the sifted set of $\mathfrak{A}$ [I, 712]. Then $\mathfrak{B}$ is perfect. Hence Card $\mathfrak{B} = \mathfrak{c}$, hence Card $\mathfrak{A} = \mathfrak{c}$.

333. *Let* $\mathfrak{A} = \{\mathfrak{a}\}$, *where each* $\mathfrak{a}$ *is metric and not discrete. If no two of the* $\mathfrak{a}$*'s have more than their frontiers in common,* $\mathfrak{A}$ *is an enumerable set in the* $\mathfrak{a}$*'s.* $\mathfrak{A}$ *may be unlimited.*

Let us first suppose that $\mathfrak{A}$ lies in a cube $\mathfrak{Q}$. Let α denote $\mathfrak{a}$ on removing its proper frontier points. Then no two of the α's have a point in common. Let

$$q_1 > q_2 > \cdots \doteq 0,$$

where the first term $q_1 = \widehat{\mathfrak{Q}}$. There can be but a finite number of sets α, such that their contents lie between two successive q's. For if

$$\widehat{\alpha}_{\iota_1} \quad , \quad \widehat{\alpha}_{\iota_2} \cdots \geq q_s$$

we have

$$\widehat{\alpha}_{\iota_1} + \widehat{\alpha}_{\iota_2} + \cdots + \alpha_{\iota_n} \geq n q_s.$$

But the sum on the left is $\overline{<} \widehat{\mathfrak{Q}}$, for any n.

As n may $\doteq \infty$, this makes $\widehat{\mathfrak{Q}} = \infty$, which is absurd.

If $\mathfrak{A}$ is not limited, we may effect a cubical division of $\mathfrak{R}_m$. This in general will split some of the $\mathfrak{a}$'s into smaller sets $\mathfrak{b}$. In each cube of this division there is but an enumerable set of the $\mathfrak{b}$'s by what has just been proved.

CHAPTER XI

MEASURE

Upper Measure

334. 1. Let $\mathfrak{A}$ be a limited point set. An enumerable set of metric sets $D = \{d_\iota\}$, such that each point of $\mathfrak{A}$ lies in some d_ι, is called an *enclosure* of $\mathfrak{A}$. If each point of $\mathfrak{A}$ lies *within* some d_ι, D is called an *outer* enclosure. The sets d_ι are called *cells*. To each enclosure corresponds the finite or infinite series

$$\Sigma \widehat{d}_\iota \tag{1}$$

which may or may not converge. In any case the minimum of all the numbers 1) is finite and ≤ 0. For let Δ be a cubical division of space, $\overline{\mathfrak{A}}_\Delta$ is obviously an enclosure and the corresponding sum 1) is also $\overline{\mathfrak{A}}_\Delta$, since we have agreed to read this last symbol either as a point set or as its content.

We call

$$\text{Min}\, \Sigma d_\iota,$$

with respect to the class of all possible enclosures D, the *upper measure of* $\mathfrak{A}$, and write

$$\overline{\overline{\mathfrak{A}}} = \overline{\text{Meas}}\, \mathfrak{A} = \underset{D}{\text{Min}}\, \Sigma d_\iota.$$

2. *The minimum of the sums* 1) *is the same when we restrict ourselves to the class of all outer enclosures.*

For let $D = \{d_\iota\}$ be any enclosure. For each d_ι, there exists a cubical division of space such that those of its cells, call them $d_{\iota\kappa}$, containing points of d_ι have a content differing from $\widehat{d}_\iota$ by $< \frac{\epsilon}{2^\iota}$. Obviously the cells $\{d_{\iota\kappa}\}$ form an outer enclosure of $\mathfrak{A}$, and

$$\Sigma \widehat{d}_{\iota\kappa} - \Sigma d_\iota < \Sigma \frac{\epsilon}{2^\iota} = \epsilon.$$

As ϵ is small at pleasure, Min Σd_ι over the class of outer enclosures $=$ Min Σd_ι over the class of all enclosures.

3. Two metric sets whose common points lie on their frontiers are called *non-overlapping.* The enclosure $D = \Sigma d_\iota$ is called non-overlapping, when any two of its cells are non-overlapping.

Any enclosure D may be replaced by a non-overlapping enclosure.

For let
$$U(d_1, d_2) = d_1 + e_2,$$
$$U(d_1, d_2, d_3) = d_1 + e_2 + e_3,$$
$$U(d_1\, d_2\, d_3\, d_4) = d_1 + e_2 + e_3 + e_4, \text{ etc.}$$

Obviously each e_n is metric. For uniformity let us set $d_1 = e_1$. Then $E = \{e_n\}$ is a non-overlapping enclosure of $\mathfrak{A}$. As

$$\Sigma \widehat{e}_n \leq \Sigma \widehat{d}_n$$

we see that *the minimum of the sums* 1) *is the same, when we restrict ourselves to the class of non-overlapping enclosures.*

Obviously we may adjoin to any cell e_n, any or all of its improper limiting points.

4. In the enclosure $E = \{e_n\}$ found in 3, no two of its cells have a point in common. Such enclosures may be called *distinct.*

335. 1. *Let $D = \{d_\iota\}$, $E = \{e_\kappa\}$ be two non-overlapping enclosures of $\mathfrak{A}$. Let*

$$\delta_{\iota\kappa} = Dv\,(d_\iota, e_\kappa).$$

Then

$$\Delta = \{\delta_{\iota\kappa}\}, \qquad \iota, \kappa = 1, 2, \cdots$$

is a non-overlapping enclosure of $\mathfrak{A}$.

For $\delta_{\iota\kappa}$ is metric by 22, 2. Two of the δ's are obviously non-overlapping. Each point of $\mathfrak{A}$ lies in some d_ι and in some e_κ, hence a lies in $\delta_{\iota\kappa}$.

2. We say Δ is the *divisor of the enclosures D, E.*

336. *If* $\mathfrak{A} < \mathfrak{B}$, $$\overline{\overline{\mathfrak{A}}} \leq \overline{\overline{\mathfrak{B}}}. \qquad (1$$

For let $E = \{e_\iota\}$ be an enclosure of $\mathfrak{B}$. Those of its cells d_ι containing a point of $\mathfrak{A}$ form an enclosure $D = \{d_\iota\}$ of $\mathfrak{A}$. Now the class of all enclosures $\Delta = \{\delta_\iota\}$ of $\mathfrak{A}$ contains the class D as a subclass.

As
$$\Sigma\hat{d}_\iota \leq \Sigma\hat{e}_\iota,$$
we have
$$\underset{\Delta}{\text{Min}}\ \Sigma\hat{\delta}_\iota \leq \underset{D}{\text{Min}}\ \Sigma d_\iota \leq \underset{E}{\text{Min}}\ \Sigma\hat{e}_\iota,$$
from which 1) follows at once.

337. *If $\mathfrak{A}$ is metric,*
$$\overline{\overline{\mathfrak{A}}} = \hat{\mathfrak{A}}. \tag{1}$$

For let D be a cubical division of space such that
$$\overline{\mathfrak{A}}_D - \hat{\mathfrak{A}} < \epsilon \quad , \quad \hat{\mathfrak{A}} - \underline{\mathfrak{A}}_D < \epsilon. \tag{2}$$

Let us set $\mathfrak{B} = \underline{\mathfrak{A}}_D$. Let $E = \{e_\iota\}$ be an outer enclosure of $\mathfrak{B}$. Since $\mathfrak{B}$ is complete, there exists a finite set of cells in E which contain all the points of $\mathfrak{B}$ by 301. The volume of this set is obviously $> \hat{\mathfrak{B}}$; hence *a fortiori*
$$\Sigma\hat{e}_\iota > \hat{\mathfrak{B}}.$$
Hence
$$\overline{\overline{\mathfrak{B}}} \geq \hat{\mathfrak{B}}.$$
But
$$\overline{\overline{\mathfrak{A}}} > \overline{\overline{\mathfrak{B}}}, \text{ by } 336,$$
$$\geq \hat{\mathfrak{B}} = \underline{\mathfrak{A}}_D$$
$$> \hat{\mathfrak{A}} - \epsilon, \text{ by } 2). \tag{3}$$
On the other hand,
$$\overline{\overline{\mathfrak{A}}} \leq \overline{\mathfrak{A}}_D < \hat{\mathfrak{A}} + \epsilon, \text{ by } 2). \tag{4}$$

From 3), 4) we have 1), since ϵ is arbitrarily small.

338. *If $\mathfrak{A}$ is complete,*
$$\overline{\overline{\mathfrak{A}}} = \overline{\mathfrak{A}}.$$
For by definition
$$\overline{\overline{\mathfrak{A}}} = \text{Min}\ \Sigma d_\iota,$$
with respect to all outer enclosures $D = \{d_\iota\}$. But $\mathfrak{A}$ being complete, we can replace D by a finite set of cells $F = \{f_\iota\}$ lying in D, such that F is an enclosure of $\mathfrak{A}$. Finally the enclosure F can be replaced by a non-overlapping enclosure $G = \{g_\iota\}$ by 334, 3.

Thus
$$\overline{\overline{\mathfrak{A}}} = \text{Min}\ \Sigma\hat{g}_\iota,$$
with respect to the class of enclosures G. But this minimum value is also $\overline{\mathfrak{A}}$ by 2, 8.

339. *Let the limited set $\mathfrak{A} = \{\mathfrak{A}_n\}$ be the union of a finite or infinite enumerable set of sets $\mathfrak{A}_n$. Then*

$$\overline{\overline{\mathfrak{A}}} \leq \Sigma \overline{\overline{\mathfrak{A}}}_n. \tag{1}$$

For to each $\mathfrak{A}_n$ corresponds an enclosure $D_n = \{d_{n\iota}\}$ such that

$$\sum_\iota \widehat{d}_{n\iota} < \overline{\overline{\mathfrak{A}}}_n + \frac{\epsilon}{2^n} \quad , \quad \epsilon > 0, \text{ arbitrarily small.}$$

But the cells of all the enclosures D_n, also form an enclosure. Hence

$$\overline{\overline{\mathfrak{A}}} \leq \sum_{\iota, n} \widehat{d}_{n\iota} < \Sigma\left(\overline{\overline{\mathfrak{A}}}_n + \frac{\epsilon}{2^n}\right)$$

$$\leq \Sigma \overline{\overline{\mathfrak{A}}}_n + \epsilon.$$

This gives 1), as ϵ is small at pleasure.

340. *Let $\mathfrak{A}$ lie in the metric set $\mathfrak{M}$. Let $A = \mathfrak{M} - \mathfrak{A}$, be the complementary set. Then*

$$\overline{\overline{\mathfrak{A}}} + \overline{\overline{A}} \geq \widehat{\mathfrak{M}}.$$

For from

$$\mathfrak{M} = \mathfrak{A} + A,$$

follows

$$\overline{\overline{\mathfrak{M}}} \leq \overline{\overline{\mathfrak{A}}} + \overline{\overline{A}}, \qquad \text{by 339.}$$

But

$$\overline{\overline{\mathfrak{M}}} = \widehat{\mathfrak{M}}, \qquad \text{by 337.}$$

341. *If $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$, and $\mathfrak{B}$, $\mathfrak{C}$ are exterior to each other,*

$$\overline{\overline{\mathfrak{A}}} = \overline{\overline{\mathfrak{B}}} + \overline{\overline{\mathfrak{C}}}. \tag{1}$$

For, if any enclosure $D = \{d_\iota\}$ of $\mathfrak{A}$ embraces a cell containing a point of $\mathfrak{B}$ and $\mathfrak{C}$, it may be split up into two metric cells d'_ι, d''_ι, each containing points of $\mathfrak{B}$ only, or of $\mathfrak{C}$ only. Then

$$\widehat{d}_\iota = \widehat{d}'_\iota + \widehat{d}''_\iota.$$

Thus we may suppose the cells of D embrace only cells $D' = \{d'_\iota\}$ containing no point of $\mathfrak{C}$, and cells $D'' = \{d''_\iota\}$ containing no point of $\mathfrak{B}$. Then

$$\Sigma \widehat{d}_\iota = \Sigma \widehat{d}'_\iota + \Sigma \widehat{d}''_\iota. \tag{2}$$

By properly choosing D, we may crowd the sum on the left down toward its minimum. Now the class of enclosures D' is included in the class of all enclosures of $\mathfrak{B}$, and a similar remark holds for D''.

Thus from 2) follows that

$$\overline{\overline{\mathfrak{A}}} \geq \overline{\overline{\mathfrak{B}}} + \overline{\overline{\mathfrak{C}}}.$$

This with 339 gives 1).

342. *If* $\mathfrak{A} = \mathfrak{B} + \mathfrak{M}$, $\mathfrak{M}$ *being metric,*

$$\overline{\overline{\mathfrak{A}}} = \overline{\overline{\mathfrak{B}}} + \widehat{\mathfrak{M}}. \tag{1}$$

For let D be a cubical division of norm d. Let $\mathfrak{n}$ denote points of $\mathfrak{M}$ in the cells containing points of Front $\mathfrak{M}$. Let $\mathfrak{m}$ denote the other points of $\mathfrak{M}$. Then $\mathfrak{m}$ and $\mathfrak{B}$ are exterior to each other, and by 337 and 341,

$$\mathrm{M}\overline{\mathrm{eas}}\,(\mathfrak{B} + \mathfrak{m}) = \overline{\overline{\mathfrak{B}}} + \widehat{\mathfrak{m}}.$$

As

$$\mathfrak{A} = \mathfrak{B} + \mathfrak{m} + \mathfrak{n},$$

$$\mathrm{M}\overline{\mathrm{eas}}\,(\mathfrak{B} + \mathfrak{m}) < \overline{\overline{\mathfrak{A}}} \qquad \text{by 336.}$$

Also

$$\overline{\overline{\mathfrak{A}}} \leq \overline{\overline{\mathfrak{B}}} + \widehat{\mathfrak{m}} + \widehat{\mathfrak{n}} \qquad \text{by 339.}$$

Thus

$$\overline{\overline{\mathfrak{B}}} + \widehat{\mathfrak{m}} \leq \overline{\overline{\mathfrak{A}}} \leq \overline{\overline{\mathfrak{B}}} + \widehat{\mathfrak{m}} + \widehat{\mathfrak{n}}. \tag{2}$$

Now if d is sufficiently small,

$$\widehat{\mathfrak{M}} - \epsilon < \widehat{\mathfrak{m}} \quad ; \quad \mathfrak{n} < \epsilon.$$

Thus 2) gives, as $\widehat{\mathfrak{m}} \leq \widehat{\mathfrak{M}}$,

$$\overline{\overline{\mathfrak{B}}} + \widehat{\mathfrak{M}} - \epsilon \leq \overline{\overline{\mathfrak{A}}} \leq \overline{\overline{\mathfrak{B}}} + \widehat{\mathfrak{M}} + \epsilon,$$

which gives 1), as $\epsilon > 0$ is arbitrarily small.

343. 1. *Let* $\mathfrak{A}$ *lie in the metric set* $\mathfrak{B}$, *and also in the metric set* $\mathfrak{C}$. *Let*

$$B = \mathfrak{B} - \mathfrak{A} \quad , \quad C = \mathfrak{C} - \mathfrak{A}.$$

Then

$$\widehat{\mathfrak{B}} - \overline{\overline{B}} = \widehat{\mathfrak{C}} - \overline{\overline{C}}.$$

For let

$$\mathfrak{D} = Dv\,(\mathfrak{B}, \mathfrak{C}) \quad , \quad \mathfrak{B} = \mathfrak{D} + \mathfrak{B}_1 \quad , \quad \mathfrak{C} = \mathfrak{D} + \mathfrak{C}_1$$

$$B = \mathfrak{B}_1 + D \quad , \quad C = \mathfrak{C}_1 + D.$$

Thus

$$\widehat{\mathfrak{B}} - \overline{\overline{B}} = \widehat{\mathfrak{D}} + \widehat{\mathfrak{B}}_1 - (\widehat{\mathfrak{B}}_1 + \overline{\overline{D}}) = \widehat{\mathfrak{D}} - \overline{\overline{D}}$$

$$\widehat{\mathfrak{C}} - \overline{\overline{C}} = \widehat{\mathfrak{D}} + \widehat{\mathfrak{C}}_1 - (\widehat{\mathfrak{C}}_1 + \overline{\overline{D}}) = \widehat{\mathfrak{D}} - \overline{\overline{D}}.$$

2. If $\mathfrak{A} < \mathfrak{B}$, the complement of $\mathfrak{A}$ with respect to $\mathfrak{B}$ will frequently be denoted by the corresponding English letter. Thus

$$A = C(\mathfrak{A}), \qquad \text{Mod } \mathfrak{B}$$
$$= \mathfrak{B} - \mathfrak{A}.$$

Lower Measure

344. 1. We are now in position to define the notion of lower measure. Let $\mathfrak{A}$ lie in a metric set $\mathfrak{M}$. The complementary set $A = \mathfrak{M} - \mathfrak{A}$ has an upper measure $\overline{\overline{A}}$. We say now that $\widehat{\mathfrak{M}} - \overline{\overline{A}}$ is the *lower measure* of $\mathfrak{A}$, and write

$$\underline{\underline{\mathfrak{A}}} = \underline{\text{Meas}}\, \mathfrak{A} = \widehat{\mathfrak{M}} - \overline{\overline{A}}.$$

By 343 this definition is independent of the set $\mathfrak{M}$ chosen.

When

$$\overline{\overline{\mathfrak{A}}} = \underline{\underline{\mathfrak{A}}}$$

we say $\mathfrak{A}$ is *measurable*, and write

$$\widehat{\mathfrak{A}} = \overline{\overline{\mathfrak{A}}} = \underline{\underline{\mathfrak{A}}}.$$

A set whose measure is 0 is called a *null* set.

2. *Let $E = \{e_\iota\}$ be an enclosure of A.*

Then
$$\underline{\underline{\mathfrak{A}}} = \text{Max}\,(\widehat{\mathfrak{M}} - \Sigma \widehat{e}_\iota),$$

with respect to the class of all enclosures E.

3. If $\mathfrak{E} = \{\mathfrak{e}_\iota\}$ is an enclosure of $\mathfrak{A}$, the enclosures E and $\mathfrak{E}$ may obviously, without loss of generality, be restricted to metric cells which contain no points not in $\mathfrak{M}$. If this is the case, and if $\mathfrak{E}$, E are each non-overlapping, we shall say they are *normal enclosures*.

If $\mathfrak{E}$, $\mathfrak{F}$ are two normal enclosures of a set $\mathfrak{A}$, obviously their divisor is also normal.

345. 1.
$$\underline{\underline{\mathfrak{A}}} \geq 0.$$

For let $\mathfrak{A}$ lie in the metric set $\mathfrak{M}$.

Then
$$\underline{\underline{\mathfrak{A}}} = \widehat{\mathfrak{M}} - \bar{\bar{A}}.$$

But by 336,
$$\bar{\bar{A}} \leq \widehat{\mathfrak{M}},$$

hence
$$\widehat{\mathfrak{M}} - \bar{\bar{A}} \geq 0.$$

2.
$$\underline{\underline{\mathfrak{A}}} \leq \bar{\bar{\mathfrak{A}}}.$$

For let $\mathfrak{A}$ lie in the metric set $\mathfrak{M}$.

Then
$$\bar{\bar{\mathfrak{A}}} + \bar{\bar{A}} \geq \widehat{\mathfrak{M}} \qquad \text{by 340.}$$

Hence
$$\underline{\underline{\mathfrak{A}}} = \widehat{\mathfrak{M}} - \bar{\bar{A}} \leq \bar{\bar{\mathfrak{A}}}.$$

346. 1. *For any limited set* $\mathfrak{A}$,
$$\underline{\mathfrak{A}} \leq \underline{\underline{\mathfrak{A}}} \leq \bar{\bar{\mathfrak{A}}} \leq \bar{\mathfrak{A}}. \tag{1}$$

For let $D = \{d_\iota\}$ be an enclosure of $\mathfrak{A}$. Then
$$\bar{\mathfrak{A}} = \operatorname*{Min}_D \Sigma \hat{d}_\iota,$$

when D ranges over the class F of all *finite* enclosures. On the other hand,
$$\bar{\bar{\mathfrak{A}}} = \operatorname*{Min}_D \Sigma \hat{d}_\iota$$

when D ranges over the class E of all enumerable enclosures. But the class E includes the class F. Hence $\bar{\bar{\mathfrak{A}}} \leq \bar{\mathfrak{A}}$.

To show that
$$\underline{\mathfrak{A}} \leq \underline{\underline{\mathfrak{A}}}, \tag{2}$$

we observe that as just shown
$$\bar{A} \geq \bar{\bar{A}}.$$

Hence,
$$\widehat{\mathfrak{M}} - \bar{A} \leq \widehat{\mathfrak{M}} - \bar{\bar{A}} = \underline{\underline{\mathfrak{A}}}. \tag{3}$$

But
$$\bar{A} + \underline{\mathfrak{A}} = \widehat{\mathfrak{M}}, \qquad \text{by 16.}$$

This with 3) gives 2).

2. *If* $\mathfrak{A}$ *is metric, it is measurable, and*

$$\widehat{\mathfrak{A}} = \hat{\hat{\mathfrak{A}}}.$$

This follows at once from 1).

347. *Let* $\mathfrak{A}$ *be measurable and lie in the metric set* $\mathfrak{M}$. *Then* A *is measurable, and*

$$\hat{\hat{\mathfrak{A}}} + \hat{\hat{A}} = \widehat{\mathfrak{M}}. \tag{1}$$

For

$$\underline{\underline{A}} = \widehat{\mathfrak{M}} - \hat{\hat{\mathfrak{A}}}. \tag{2}$$

$$\underline{\underline{\mathfrak{A}}} = \widehat{\mathfrak{M}} - \overline{\overline{A}} = \hat{\hat{\mathfrak{A}}},$$

since $\mathfrak{A}$ is measurable. This last gives

$$\overline{\overline{A}} = \widehat{\mathfrak{M}} - \hat{\hat{\mathfrak{A}}}.$$

This with 2) shows that $\overline{\overline{A}} = \underline{\underline{A}}$; hence A is measurable. From 2) now follows 1).

348. *If* $\mathfrak{A} < \mathfrak{B}$, *then*

$$\underline{\underline{\mathfrak{A}}} \leq \underline{\underline{\mathfrak{B}}}. \tag{1}$$

For as usual let A, B be the complements of $\mathfrak{A}$, $\mathfrak{B}$ with respect to a metric set $\mathfrak{M}$. Since $\mathfrak{A} < \mathfrak{B}$, $A > B$.

Hence, by 336,

$$\overline{\overline{A}} \geq \overline{\overline{B}}.$$

Thus,

$$\widehat{\mathfrak{M}} - \overline{\overline{A}} \leq \widehat{\mathfrak{M}} - \overline{\overline{B}},$$

which gives 1).

349. *For* $\mathfrak{A}$ *to be measurable, it is necessary and sufficient that*

$$\overline{\overline{\mathfrak{A}}} + \overline{\overline{A}} = \widehat{\mathfrak{M}} \tag{1}$$

where $\mathfrak{M}$ *is any metric set* $> \mathfrak{A}$, *and* $A = \mathfrak{M} - \mathfrak{A}$.

It is sufficient, for then 1) shows that

$$\overline{\overline{\mathfrak{A}}} = \widehat{\mathfrak{M}} - \overline{\overline{A}}.$$

But the right side is by definition $\underline{\underline{\mathfrak{A}}}$; hence $\overline{\overline{\mathfrak{A}}} = \underline{\underline{\mathfrak{A}}}$.
It is necessary as 347 shows.

350. *Let* $\mathfrak{A} = \{\mathfrak{a}_n\}$ *be the union of an enumerable set of non-overlapping metric sets. Then* $\mathfrak{A}$ *is measurable, and*

$$\hat{\hat{\mathfrak{A}}} = \Sigma \widehat{\mathfrak{a}}_n. \tag{1}$$

Let S denote the infinite series on the right of 1). As usual let S_n denote the sum of the first n terms. Let $\mathfrak{A}_n = (\mathfrak{a}_1, \cdots \mathfrak{a}_n)$. Then $\mathfrak{A}_n \leq \mathfrak{A}$ and by 336,

$$\widehat{\mathfrak{A}}_n = S_n \leq \overline{\overline{\mathfrak{A}}} \quad , \quad \text{for any } n. \tag{2}$$

Thus S is convergent and

$$S \leq \overline{\overline{\mathfrak{A}}}. \tag{3}$$

On the other hand, by 339,

$$\overline{\overline{\mathfrak{A}}} \leq S. \tag{4}$$

From 3), 4) follows that

$$S = \overline{\overline{\mathfrak{A}}} = \lim S_n = \lim \widehat{\mathfrak{A}}_n. \tag{5}$$

We show now that $\mathfrak{A}$ is measurable. To this end, let $\mathfrak{M}$ be a metric set $> \mathfrak{A}$, and $\mathfrak{A}_n + A_n = \mathfrak{M}$ as usual.

Then

$$\widehat{\mathfrak{A}}_n + \widehat{A}_n = \widehat{\mathfrak{M}}. \tag{6}$$

But $\quad A \leq A_n \quad , \quad$ hence $\overline{\overline{A}} \leq \widehat{A}_n$.

Thus 6) gives

$$\overline{\overline{A}} + \widehat{\mathfrak{A}}_n \leq \widehat{\mathfrak{M}},$$

for any n. Hence

$$\overline{\overline{A}} + \lim \widehat{\mathfrak{A}}_n \leq \widehat{\mathfrak{M}};$$

or using 5),

$$\overline{\overline{A}} + \overline{\overline{\mathfrak{A}}} \leq \widehat{\mathfrak{M}}.$$

Hence by 339,

$$\overline{\overline{A}} + \overline{\overline{\mathfrak{A}}} = \widehat{\mathfrak{M}}.$$

Thus by 349, $\mathfrak{A}$ is measurable.

351. *Let*

$$\mathfrak{A} = \mathfrak{B} + \mathfrak{C};$$

then

$$\underline{\underline{\mathfrak{B}}} + \underline{\underline{\mathfrak{C}}} \leq \underline{\underline{\mathfrak{A}}}. \tag{1}$$

For let $\mathfrak{M}$ be a metric set $> \mathfrak{A}$. Let A, B, C be the complements of $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, with reference to $\mathfrak{M}$.

Let $\quad E = \{e_m\} \quad , \quad F = \{f_n\}$

be normal enclosures of B, C. Let

$$d_{mn} = Dv\,(e_m, f_n),$$

and $D = \{d_{mn}\}$ the divisor of E, F.

As all the points of A are in B, and also in C, they are in both E and F, and hence in the cells of D, which thus forms a normal enclosure of A. Let

$$\gamma_m = (d_{m1}, d_{m2} \cdots) \quad , \quad \eta_n = (d_{1n}, d_{2n} \cdots).$$

Let us set

$$e_m = \gamma_m + g_m \quad , \quad f_n = \eta_n + h_n.$$

Then by 350,

$$\hat{\hat{\gamma}}_m = \sum_n \widehat{d}_{mn} \quad , \quad \hat{\hat{\eta}}_n = \sum_m \widehat{d}_{mn}.$$

By 347,

$$\widehat{e}_m = \hat{\hat{\gamma}}_m + \hat{\hat{g}}_m \quad , \quad \widehat{f}_n = \hat{\hat{\eta}}_n + \hat{\hat{h}}_n.$$

Hence

$$\widehat{\mathfrak{M}} - \sum_m \widehat{e}_m = \widehat{\mathfrak{M}} - \sum_{m,n} \widehat{d}_{mn} - \sum_m \hat{\hat{g}}_m,$$

$$\widehat{\mathfrak{M}} - \sum_n \widehat{f}_n = \widehat{\mathfrak{M}} - \sum_{m,n} \widehat{d}_{mn} - \sum_n \hat{\hat{h}}_n.$$

Hence adding,

$$(\widehat{\mathfrak{M}} - \Sigma \widehat{e}_m) + (\widehat{\mathfrak{M}} - \Sigma \widehat{f}_n)$$

$$= \widehat{\mathfrak{M}} - \Sigma \widehat{d}_{mn} + [\widehat{\mathfrak{M}} - (\Sigma \hat{\hat{g}}_m + \Sigma \hat{\hat{h}}_n + \Sigma \widehat{d}_{mn})]. \tag{2}$$

Now

$$\mathfrak{M} = U\{g_m, h_n, d_{mn}\} \qquad m, n = 1, 2, \cdots$$

Thus by 339, the term in [ ] is ≤ 0. Thus 2) gives

$$(\widehat{\mathfrak{M}} - \Sigma \widehat{e}_m) + (\widehat{\mathfrak{M}} - \Sigma \widehat{f}_n) \leq \widehat{\mathfrak{M}} - \Sigma \widehat{d}_{mn} \leq \underline{\underline{\mathfrak{A}}}. \tag{3}$$

But

$$\underline{\underline{\mathfrak{B}}} = \text{Max}\,(\widehat{\mathfrak{M}} - \Sigma \widehat{e}_m)$$

$$\underline{\underline{\mathfrak{C}}} = \text{Max}\,(\widehat{\mathfrak{M}} - \Sigma \widehat{f}_n).$$

Thus 3) gives 1) at once.

Measurable Sets

352. 1. *Let* $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$. *If* $\mathfrak{B}$, $\mathfrak{C}$ *are measurable, then* $\mathfrak{A}$ *is measurable, and*

$$\hat{\hat{\mathfrak{A}}} = \hat{\hat{\mathfrak{B}}} + \hat{\hat{\mathfrak{C}}}. \tag{1}$$

For

$$\underline{\underline{\mathfrak{B}}} + \underline{\underline{\mathfrak{C}}} \leq \underline{\underline{\mathfrak{A}}} \quad , \quad \text{by } 351$$

$$\leq \bar{\bar{\mathfrak{A}}} \leq \bar{\bar{\mathfrak{B}}} + \bar{\bar{\mathfrak{C}}} \quad , \quad \text{by } 339.$$

But

$$\underline{\underline{\mathfrak{B}}} = \bar{\bar{\mathfrak{B}}} = \hat{\hat{\mathfrak{B}}} \quad , \quad \underline{\underline{\mathfrak{C}}} = \bar{\bar{\mathfrak{C}}} = \hat{\hat{\mathfrak{C}}}.$$

2. *Let* $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$. *If* $\mathfrak{A}$, $\mathfrak{B}$ *are measurable, so is* $\mathfrak{C}$ *and*

$$\widehat{\widehat{\mathfrak{C}}} = \widehat{\widehat{\mathfrak{A}}} - \widehat{\widehat{\mathfrak{B}}}. \tag{2}$$

For let $\mathfrak{A}$ lie in the metric set $\mathfrak{M}$. Then

$$\mathfrak{M} - \mathfrak{A} = \mathfrak{M} - (\mathfrak{B} + \mathfrak{C}) = (\mathfrak{M} - \mathfrak{C}) - \mathfrak{B}.$$

Thus

$$A = C - \mathfrak{B};$$

Hence

$$C = \mathfrak{B} + A.$$

Thus C is measurable by 1. Hence $\mathfrak{C}$ is measurable by 347, and

$$\widehat{\widehat{\mathfrak{A}}} = \widehat{\widehat{\mathfrak{B}}} + \widehat{\widehat{\mathfrak{C}}}.$$

From this follows 2) at once.

353. 1. *Let* $\mathfrak{A} = \Sigma\mathfrak{A}_n$ *be the sum of an enumerable set of measurable sets. Then* $\mathfrak{A}$ *is measurable and*

$$\widehat{\widehat{\mathfrak{A}}} = \Sigma\widehat{\widehat{\mathfrak{A}}}_n.$$

If $\mathfrak{A}$ is the sum of a finite number of sets, the theorem is obviously true by 352, 1. In case $\mathfrak{A}$ embraces an infinite number of sets, the reasoning of 350 may be employed.

2. *Let* $\mathfrak{N} = \{\mathfrak{N}_n\}$ *be the union of an enumerable set of null sets. Then* $\mathfrak{N}$ *is a null set.*

Follows at once from 1.

3. *Let* $\mathfrak{A} = \{\mathfrak{A}_n\}$ *be the union of an enumerable set of measurable sets whose common points two and two, form null sets. Then* $\mathfrak{A}$ *is measurable and*

$$\widehat{\widehat{\mathfrak{A}}} = \Sigma\widehat{\widehat{\mathfrak{A}}}_n.$$

4. *Let* $\mathfrak{E} = \{\mathfrak{e}_n\}$ *be a non-overlapping enclosure of* $\mathfrak{A}$. *Then* $\mathfrak{E}$ *is measurable, and*

$$\widehat{\widehat{\mathfrak{E}}} = \Sigma\widehat{\mathfrak{e}}_n.$$

5. Let $\mathfrak{B} \leq \mathfrak{A}$. Those cells of $\mathfrak{E}$ containing a point of $\mathfrak{B}$ may be denoted by $\mathfrak{B}_{\mathfrak{E}}$, and their measure will then be of course

$$\widehat{\widehat{\mathfrak{B}}}_{\mathfrak{E}}.$$

If $\mathfrak{B} = \mathfrak{A}$, this will be $\widehat{\widehat{\mathfrak{E}}}$. This notation is analogous to that used in volume I when treating content.

6. *If* $\mathfrak{F} = \{\mathfrak{f}_n\}$ *is another non-overlapping enclosure of some set then*

$$\mathfrak{D} = Dv(\mathfrak{E}, \mathfrak{F})$$

is measurable.

For the cells of $\mathfrak{D}$ are

$$\delta_{\iota\kappa} = Dv(\mathfrak{e}_\iota, \mathfrak{f}_k).$$

Thus $\delta_{\iota\kappa}$ is metric, and

$$\widehat{\mathfrak{D}} = \Sigma \widehat{\delta}_{\iota\kappa}.$$

354. 1. *Harnack Sets.* Let $\mathfrak{A}$ be an interval of length l. Let

$$\lambda = l_1 + l_2 + \cdots$$

be a positive term series whose sum $\lambda > 0$ is $\leq l$. As in defining Cantor's set, I, 272, let us place a *black* interval of length l_1 in the middle of $\mathfrak{A}$. In a similar manner let us place in each of the remaining or *white* intervals, a black interval, whose total lengths $= l_2$. Let us continue in this way; we get an enumerable set of black intervals $\mathfrak{B}$, and obviously

$$\widehat{\mathfrak{B}} = \lambda.$$

If we omit the end points from each of the black intervals we get a set $\mathfrak{B}^*$, and obviously

$$\widehat{\mathfrak{B}}^* = \lambda.$$

The set

$$\mathfrak{H} = \mathfrak{A} - \mathfrak{B}^*$$

we call a *Harnack set.* This is complete by 324; and by 338, 347,

$$\overline{\mathfrak{H}} = \widehat{\mathfrak{H}} = l - \lambda.$$

When $\lambda = l$, $\mathfrak{H}$ is discrete, and the set reduces to a set similar to Cantor's set. When $\lambda < l$, we get an apantactic perfect set whose upper content is $l - \lambda > 0$, and whose lower content is 0.

2. Within each of the black intervals let us put a set of points having the end points for its first derivative. The totality of these points form an isolated set $\mathfrak{J}$ and $\mathfrak{J}' = \mathfrak{H}$. But by 331, $\overline{\mathfrak{J}} = \overline{\mathfrak{J}'}$. If now $\mathfrak{H}$ is not discrete, $\mathfrak{J}$ is not. We have thus the theorem :

There exist isolated point sets which are not discrete.

3. It is easy to extend Harnack sets to $\mathfrak{R}_n$. For example, in $\mathfrak{R}_2$, let S be the unit square. On two of its adjacent sides let us place congruent Harnack sets $\mathfrak{H}$. We now draw lines through the end points of the black intervals parallel to the sides. There results an enumerable set of black squares $\mathfrak{S} = \{S_n\}$. The sides of the squares $\mathfrak{S}$ and their limiting points form obviously an apantactic perfect set $\mathfrak{K}$.

Let

$$a_1^2 + a_2^2 + \cdots = m$$

be a series whose sum $0 < m \leq 1$.

We can choose $\mathfrak{H}$ such that the square corresponding to its largest black interval has the area a_1^2; the four squares corresponding to the next two largest black intervals have the total area a_2^2, etc.

Then

$$\overline{\overline{\mathfrak{S}}} = \Sigma a_n^2 = m.$$

Hence

$$\overline{\overline{\mathfrak{K}}} = 1 - m = \overline{\mathfrak{K}}.$$

355. 1. If $\mathfrak{E} = \{\mathfrak{e}_m\}$ is an enclosure of $\mathfrak{A}$ such that

$$\Sigma\widehat{\mathfrak{e}}_m - \overline{\overline{\mathfrak{A}}} < \epsilon,$$

it is called an *ϵ-enclosure*. Let A be the complement of $\mathfrak{A}$ with respect to the metric set $\mathfrak{M}$. Let $E = \{e_n\}$ be an ϵ-enclosure of A. We call $\mathfrak{E}$, E *complementarg ϵ-enclosures belonging to* $\mathfrak{A}$.

2. *If $\mathfrak{A}$ is measurable, then each pair of complementary $\epsilon/2$ normal enclosures $\mathfrak{E}$, E, whose divisor $\mathfrak{D} = Dv(\mathfrak{E}, E)$, is such that*

$$\widehat{\widehat{\mathfrak{D}}} < \epsilon, \qquad \epsilon \text{ small at pleasure.} \tag{1}$$

For let $\mathfrak{E}$, E be any pair of complementary $\epsilon/2$ normal enclosures. Then

$$\widehat{\widehat{\mathfrak{E}}} - \widehat{\mathfrak{A}} < \frac{\epsilon}{2} \quad , \quad \widehat{\widehat{E}} - \widehat{A} < \frac{\epsilon}{2}.$$

Adding, we get

$$0 \leq \widehat{\widehat{\mathfrak{E}}} + \widehat{\widehat{E}} - (\widehat{\mathfrak{A}} + \widehat{A}) < \epsilon;$$

or

$$0 \leq \widehat{\widehat{\mathfrak{E}}} + \widehat{\widehat{E}} - \widehat{\mathfrak{M}} < \epsilon. \tag{2}$$

But the points of $\mathfrak{M}$ fall into one of three classes: 1° the points of $\mathfrak{D}$; 2° those of $\mathfrak{E}$ not in $\mathfrak{D}$; 3° those of E not in $\mathfrak{D}$. Thus

$$\widehat{\widehat{\mathfrak{E}}} + \widehat{\widehat{E}} = \widehat{\mathfrak{M}} + \widehat{\widehat{\mathfrak{D}}}.$$

This in 2) gives 1).

356. 1. Up to the present we have used only metric enclosures of a set $\mathfrak{A}$. If the cells enclosing $\mathfrak{A}$ are measurable, we call the enclosure *measurable.*

Let $\mathfrak{E} = \{\mathfrak{e}_n\}$ be a measurable enclosure. If the points common to any two of its cells form a null set, we say $\mathfrak{E}$ is *non-overlapping.* The terms distinct, normal, go over without change.

2. *We prove now that*
$$\overline{\overline{\mathfrak{A}}} = \text{Min } \Sigma\widehat{\widehat{\mathfrak{e}}}_n, \tag{1}$$
with respect to the class of non-overlapping measurable enclosures.

For, as in 339, there exists a metric enclosure $\mathfrak{m}_n = \{d_{n\kappa}\}$ of each $\mathfrak{e}_n$ such that $\sum_\kappa \widehat{d}_{n\kappa}$ differs from $\widehat{\widehat{\mathfrak{e}}}_n$ by $< \epsilon/2^n$. But the set $\{\mathfrak{m}_n\}$ forms a metric enclosure of $\mathfrak{A}$. Thus
$$\overline{\overline{\mathfrak{A}}} \leq \sum_{n,\kappa} \widehat{\widehat{d}}_{n,\kappa} < \Sigma\left(\widehat{\widehat{\mathfrak{e}}}_n + \frac{\epsilon}{2^n}\right) = \Sigma\widehat{\widehat{\mathfrak{e}}}_n + \epsilon,$$
which establishes 1).

357. *Let $\mathfrak{E}$ be a distinct measurable enclosure of $\mathfrak{A}$. Let $\mathfrak{f}$ denote those cells containing points of the complement A. If for each $\epsilon > 0$ there exists an $\mathfrak{E}$ such that $\widehat{\widehat{\mathfrak{f}}} < \epsilon$, then $\mathfrak{A}$ is measurable.*

For let $\mathfrak{E} = \mathfrak{e} + \mathfrak{f}$. Then $\mathfrak{e} \leq \mathfrak{A}$. Hence $\mathfrak{e} \leq \underline{\underline{\mathfrak{A}}}$ by 348. But
$$\overline{\overline{\mathfrak{A}}} \leq \widehat{\widehat{\mathfrak{E}}} = \widehat{\widehat{\mathfrak{e}}} + \widehat{\widehat{\mathfrak{f}}} \leq \underline{\underline{\mathfrak{A}}} + \epsilon.$$
Hence
$$\overline{\overline{\mathfrak{A}}} - \underline{\underline{\mathfrak{A}}} < \epsilon,$$
and thus
$$\overline{\overline{\mathfrak{A}}} = \underline{\underline{\mathfrak{A}}}.$$

358. 1. *The divisor $\mathfrak{D}$ of two measurable sets $\mathfrak{A}$, $\mathfrak{B}$ is also measurable.*

For let $\mathfrak{E}$, E be a pair of complementary $\epsilon/4$ normal enclosures belonging to $\mathfrak{A}$; let $\mathfrak{F}$, F be similar enclosures of $\mathfrak{B}$. Let
$$\mathfrak{e} = Dv(\mathfrak{E}, E) \quad , \quad \mathfrak{f} = Dv(\mathfrak{F}, F).$$
Then
$$\widehat{\widehat{\mathfrak{e}}} < \epsilon/2 \quad , \quad \widehat{\widehat{\mathfrak{f}}} < \epsilon/2, \qquad \text{by 355, 2.}$$

Now $\mathfrak{G} = Dv(\mathfrak{E}, \mathfrak{F})$ is a normal metric enclosure of $\mathfrak{D}$. Moreover its cells $\mathfrak{g}$ which contain points of $\mathfrak{D}$ and $C(\mathfrak{D})$ lie among the cells of $\mathfrak{e}, \mathfrak{f}$. Hence

$$\widehat{\mathfrak{g}} \leq \widehat{\mathfrak{e}} + \widehat{\mathfrak{f}} < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

Thus by 357, $\mathfrak{D}$ is measurable.

2. *Let $\mathfrak{A}, \mathfrak{B}$ be measurable.*

Let

$$\mathfrak{D} = Dv(\mathfrak{A}, \mathfrak{B}) \quad , \quad \mathfrak{U} = (\mathfrak{A}, \mathfrak{B}).$$

Then

$$\widehat{\mathfrak{A}} + \widehat{\mathfrak{B}} = \widehat{\mathfrak{U}} + \widehat{\mathfrak{D}}.$$

For

$$\mathfrak{U} = \mathfrak{A} + (\mathfrak{B} - \mathfrak{D}).$$

Hence

$$\widehat{\mathfrak{U}} = \widehat{\mathfrak{A}} + \text{Meas}\,(\mathfrak{B} - \mathfrak{D})$$
$$= \widehat{\mathfrak{A}} + \widehat{\mathfrak{B}} - \widehat{\mathfrak{D}}.$$

359. *Let $\mathfrak{A} = U\{\mathfrak{A}_m\}$ be the union of an enumerable set of measurable cells; moreover let $\mathfrak{A}$ be limited. Then $\mathfrak{A}$ is measurable. If we set*

$$\mathfrak{B}_1 = \mathfrak{A}_1 \quad , \quad (\mathfrak{A}_1, \mathfrak{A}_2) = \mathfrak{B}_1 + \mathfrak{B}_2,$$

$$(\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3) = \mathfrak{B}_1 + \mathfrak{B}_2 + \mathfrak{B}_3, \; etc.,$$

then

$$\widehat{\mathfrak{A}} = \Sigma \widehat{\mathfrak{B}}_m. \tag{1}$$

For $\mathfrak{D} = Dv(\mathfrak{A}_1, \mathfrak{A}_2)$ is measurable by 358.

Let

$$\mathfrak{A}_1 = \mathfrak{D} + \mathfrak{a}_1 \quad , \quad \mathfrak{A}_2 = \mathfrak{D} + \mathfrak{a}_2.$$

Then $\mathfrak{a}_1, \mathfrak{a}_2$ are measurable by 352, 2.

As

$$\mathfrak{U} = (\mathfrak{A}_1, \mathfrak{A}_2) = \mathfrak{D} + \mathfrak{a}_1 + \mathfrak{a}_2,$$

$\mathfrak{U}$ is measurable. As $\mathfrak{U}$ and $\mathfrak{B}_1$ are measurable, so is $\mathfrak{B}_2$. In a similar manner we show that $\mathfrak{B}_3, \mathfrak{B}_4 \cdots$ are measurable. As

$$\mathfrak{A} = \Sigma \mathfrak{B}_m,$$

$\mathfrak{A}$ is measurable by 353, 1, and the relation 1) holds by the same theorem.

360. *Let $\mathfrak{A}_1 \leq \mathfrak{A}_2 \leq \cdots$ be a set of measurable aggregates whose union $\mathfrak{A}$ is limited. Then $\mathfrak{A}$ is measurable, and*

$$\widehat{\mathfrak{A}} = \lim_{n=\infty} \widehat{\mathfrak{A}}_n.$$

For let
$$\mathfrak{a}_2 = \mathfrak{A}_2 - \mathfrak{A}_1 \quad , \quad \mathfrak{a}_3 = \mathfrak{A}_3 - \mathfrak{A}_2 \cdots$$

For uniformity let us set $\mathfrak{a}_1 = \mathfrak{A}$. Then
$$\mathfrak{A} = \Sigma \mathfrak{a}_m.$$

As each $\mathfrak{a}_n$ is measurable
$$\widehat{\mathfrak{A}} = \Sigma \widehat{\mathfrak{a}}_m$$
$$= \lim_{n=\infty} (\widehat{\mathfrak{a}}_1 + \cdots + \widehat{\mathfrak{a}}_n)$$
$$= \lim \widehat{\mathfrak{A}}_n .$$

361. *Let $\mathfrak{A}_1, \mathfrak{A}_2 \cdots$ be measurable and their union $\mathfrak{A}$ limited. If $\mathfrak{D} = Dv\,\{\mathfrak{A}_n\} > 0$, it is measurable.*

For let $\mathfrak{A}$ lie in the metric set $\mathfrak{M}$;

let
$$\mathfrak{D} + D = \mathfrak{M} \quad , \quad \mathfrak{A}_n + A_n = \mathfrak{M}$$
as usual.

Now $\mathfrak{D}$ denoting the points common to all the $\mathfrak{A}_n$, no point of D can lie in all of the $\mathfrak{A}_n$, hence it lies in some one or more of the A_n. Thus
$$D \leq \{A_n\}. \tag{1}$$

On the other hand, a point of $\{A_n\}$ lies in some A_m, hence it does not lie in $\mathfrak{A}_m$. Hence it does not lie in $\mathfrak{D}$. Thus it lies in D. Hence
$$\{A_n\} \leq D. \tag{2}$$

From 1), 2) we have
$$D = \{A_n\}.$$

As each A_n is measurable, so is D. Hence $\mathfrak{D}$ is.

362. *If $\mathfrak{A}_1 \geq \mathfrak{A}_2 \geq \cdots$ is an enumerable set of measurable aggregates, their divisor $\mathfrak{D}$ is measurable, and*
$$\widehat{\mathfrak{D}} = \lim_{n=\infty} \widehat{\mathfrak{A}}_n .$$

For as usual let D, A_n be the complements of $\mathfrak{D}$, $\mathfrak{A}_n$ with respect to some metric set $\mathfrak{M}$.

Then
$$D = \{A_n\} \quad , \quad A_n \leq A_{n+1} .$$

Hence by 360,
$$\widehat{D} = \lim \widehat{A}_n .$$

As
$$\mathfrak{D} = \mathfrak{M} - D,$$
we have
$$\widehat{\mathfrak{D}} = \widehat{\mathfrak{M}} - \widehat{D}$$
$$= \lim (\widehat{\mathfrak{M}} - \widehat{A}_n)$$
$$= \lim \widehat{\mathfrak{A}}_n.$$

363. 1. The points $x = (x_1 \cdots x_m)$ such that
$$a_1 \leq x_1 \leq b_1 \quad , \quad \cdots \quad , \quad a_m \leq x_m \leq b_m \tag{1}$$
form a *standard rectangular cell*, whose edges have the lengths
$$e_1 = b_1 - a_1 \quad , \quad \cdots \quad , \quad e_m = b_m - a_m.$$
When $e_1 = e_2 = \cdots = e_m$, the cell is a *standard cube*. A normal enclosure of the limited set $\mathfrak{A}$, whose cells $\mathfrak{E} = \{\mathfrak{e}_n\}$ are standard cells, is called a *standard enclosure.*

2. *For each $\epsilon > 0$, there are standard ϵ-enclosures of any limited set $\mathfrak{A}$.*

For let $\mathfrak{E} = \{\mathfrak{e}_n\}$ be any η-enclosure of $\mathfrak{A}$. Then
$$\Sigma \widehat{\mathfrak{e}}_n - \overline{\overline{\mathfrak{A}}} < \eta. \tag{2}$$
Each $\mathfrak{e}_n$ being metric, may be enclosed in the cells of a finite standard outer enclosure F_n, such that
$$\widehat{F}_n - \widehat{\mathfrak{e}}_n < \eta/2^n \quad , \quad n = 1, 2, \cdots$$
Then $\mathfrak{F} = \{F_n\}$ is an enclosure of $\mathfrak{A}$, and
$$\Sigma \widehat{F}_n < \Sigma (\widehat{\mathfrak{e}}_n + \eta/2^n) = \Sigma \widehat{\mathfrak{e}}_n + \eta$$
$$< \overline{\overline{\mathfrak{A}}} + 2\eta, \qquad \text{by 2)}.$$
But the enclosure F can be replaced by a non-overlapping standard enclosure $\mathfrak{G} = \{\mathfrak{g}_n\}$, as in 334, 3. But $\mathfrak{G} \leq \Sigma \widehat{F}_n$.
Hence if 2η is taken $< \epsilon$,
$$\widehat{\mathfrak{G}} - \overline{\overline{\mathfrak{A}}} < \epsilon,$$
and $\mathfrak{G}$ is an ϵ-enclosure.

3. Let
$$\mathfrak{E} = \{\mathfrak{e}_m\}, \qquad \mathfrak{F} = \{\mathfrak{f}_n\}$$
be two non-overlapping enclosures of the same or of different sets. Let $\mathfrak{e}_{mn} = Dv(\mathfrak{e}_m, \mathfrak{f}_n)$.

Let
$$\mathfrak{e}_m = (\mathfrak{e}_{m,1}, \mathfrak{e}_{m,2}, \mathfrak{e}_{m,3} \cdots) + e_m, \tag{3}$$

then e_m is measurable. By this process the metric or measurable cell $\mathfrak{e}_m$ falls into an enumerable set of non-overlapping measurable cells, as indicated in 3). If we suppose this decomposition to take place for each cell of $\mathfrak{E}$, we shall say we have *superimposed* $\mathfrak{F}$ *on* $\mathfrak{E}$.

364. (*W. H. Young.*) *Let* $\mathfrak{C}$ *be any complete set in limited* $\mathfrak{A}$. *Then*
$$\underline{\underline{\mathfrak{A}}} = \text{Max } \overline{\mathfrak{C}}. \tag{1}$$

For let $\mathfrak{A}$ lie within a cube $\mathfrak{M}$, and let $A = \mathfrak{M} - \mathfrak{A}$, $C = \mathfrak{M} - \mathfrak{C}$ be as usual the complementary sets.

Let $\mathfrak{B} = \{\mathfrak{b}_n\}$ be a border set of $\mathfrak{C}$ [328]. It is also a non-overlapping enclosure of C; we may suppose it is a standard enclosure of C. Let E be a standard ϵ-enclosure of A. Let us superimpose E on $\mathfrak{B}$, getting a measurable enclosure Δ of both C and A. Then
$$C = C_\Delta \geq A_\Delta.$$

Hence
$$\mathfrak{C} = \mathfrak{M} - C = \mathfrak{M} - C_\Delta \leq \mathfrak{M} - A_\Delta.$$

Thus
$$\begin{aligned} \overline{\mathfrak{C}} &= \overline{\overline{\mathfrak{C}}}, \qquad \text{by } 338 \\ &\leq \overline{\text{Meas}}\ (\mathfrak{M} - A_\Delta) \\ &\leq \widehat{\mathfrak{M}} - \widehat{A}_\Delta, \qquad \text{by } 352,\ 2 \\ &\leq \widehat{\mathfrak{M}} - \overline{\overline{A}}. \end{aligned}$$

Hence
$$\overline{\mathfrak{C}} \leq \underline{\underline{\mathfrak{A}}},$$

and thus
$$\text{Max } \overline{\mathfrak{C}} \leq \underline{\underline{\mathfrak{A}}}. \tag{2}$$

On the other hand, it is easy to show that
$$\text{Max } \overline{\mathfrak{C}} \geq \underline{\underline{\mathfrak{A}}}. \tag{3}$$

For let A_D be an ϵ-outer enclosure of A, formed of standard non-overlapping cells all of which, after having discarded certain parts, lie in $\mathfrak{M}$.

Let $$\mathfrak{K} = \mathfrak{M} - A_D + \mathfrak{F}, \tag{4}$$

where $\mathfrak{F}$ denotes the frontier points of A_D lying in $\mathfrak{A}$. Obviously $\mathfrak{K}$ is complete. Since each face of D is a null set, $\mathfrak{F}$ is a null set. Thus each set on the right of 4) is measurable, hence

$$\begin{aligned}\widehat{\mathfrak{K}} &= \widehat{\mathfrak{M}} - \widehat{A}_D + \widehat{\mathfrak{F}} \\ &= \widehat{\mathfrak{M}} - \widehat{A}_D \\ &= \widehat{\mathfrak{M}} - \overline{\overline{A}} - \epsilon' \quad , \quad 0 \leq \epsilon' < \epsilon \\ &= \underline{\underline{\mathfrak{A}}} - \epsilon'.\end{aligned}$$

Thus $$\text{Max } \overline{\mathfrak{C}} \geq \overline{\mathfrak{K}} = \overline{\overline{\mathfrak{K}}} > \underline{\underline{\mathfrak{A}}} - \epsilon,$$

from which follows 3), since ϵ is small at pleasure.

365. 1. *If $\mathfrak{A}$ is complete, it is measurable, and*

$$\widehat{\mathfrak{A}} = \overline{\mathfrak{A}}.$$

For by 364,

$$\underline{\underline{\mathfrak{A}}} = \overline{\mathfrak{A}}.$$

On the other hand,

$$\overline{\overline{\mathfrak{A}}} = \overline{\mathfrak{A}}, \qquad \text{by 338.}$$

2. *Let $\mathfrak{B}$ be any measurable set in the limited set $\mathfrak{A}$. Then*

$$\underline{\underline{\mathfrak{A}}} = \text{Max } \widehat{\mathfrak{B}}. \tag{1}$$

For $$\underline{\underline{\mathfrak{A}}} \geq \underline{\underline{\mathfrak{B}}} = \widehat{\mathfrak{B}}.$$

Hence, $$\underline{\underline{\mathfrak{A}}} \geq \text{Max } \widehat{\mathfrak{B}}. \tag{2}$$

But the class of measurable components of $\mathfrak{A}$ embraces the class of complete components $\mathfrak{C}$, since each $\mathfrak{C}$ is measurable by 1.

Thus $$\text{Max } \widehat{\mathfrak{B}} \geq \text{Max } \widehat{\mathfrak{C}}. \tag{3}$$

From 2), 3) we have 1), on using 364.

366. *Van Vleck Sets.* Let $\mathfrak{E}$ denote the unit interval $(0, 1)$, whose middle point call M. Let $\mathfrak{J}$ denote the irrational points of $\mathfrak{E}$. Let the division D_n, $n = 1, 2, \cdots$ divide $\mathfrak{E}$ into equal intervals δ_n of length $1/2^n$.

We throw the points $\mathfrak{J}$ into two classes $\mathfrak{A} = \{a\}$, $\mathfrak{B} = \{b\}$ having the following properties:

1° To each a corresponds a point b symmetrical with respect to M, and conversely.

2° If a falls in the segment δ of D_n, each of the other segments δ' of D_n shall contain a point a' of $\mathfrak{A}$ such that a' is situated in δ' as a is situated in δ.

3° Each δ of D_n shall contain a point a' of $\mathfrak{A}$ such that it is situated in δ, as any given point a of $\mathfrak{A}$ is situated in $\mathfrak{E}$.

4° $\mathfrak{A}$ shall contain a point a situated in $\mathfrak{E}$ as any given point a' of $\mathfrak{A}$ is in any δ_n.

The 1° condition states that $\mathfrak{A}$ goes over into $\mathfrak{B}$ on rotating $\mathfrak{E}$ about M. The 2° condition states that $\mathfrak{A}$ falls into $n = 1, 2, 2^2, 2^3, \cdots$ congruent subsets. The 3° condition states that the subset $\mathfrak{A}_n$ of $\mathfrak{A}$ in δ_n goes over into $\mathfrak{A}$ on stretching it in the ratio $2^n : 1$. The condition 4° states that $\mathfrak{A}$ goes over into $\mathfrak{A}_n$ on contracting it in the ratio $1 : 2^n$.

We show now that $\mathfrak{A}$, and therefore $\mathfrak{B}$ are not measurable. In the first place, we note that

$$\overline{\overline{\mathfrak{A}}} = \overline{\overline{\mathfrak{B}}},$$

by 1°. As $\mathfrak{J} = \mathfrak{A} + \mathfrak{B}$, if $\mathfrak{A}$ or $\mathfrak{B}$ were measurable, the other would be, and

$$\widehat{\mathfrak{A}} = \widehat{\mathfrak{B}} = \tfrac{1}{2}.$$

Thus if we show $\overline{\overline{\mathfrak{A}}}$ or $\overline{\overline{\mathfrak{B}}} = 1$, neither $\mathfrak{A}$ nor $\mathfrak{B}$ is measurable. We show this by proving that if $\overline{\overline{\mathfrak{A}}} = \alpha < 1$, then $\mathfrak{B}$ is a measurable set, and $\widehat{\mathfrak{B}} = 1$. But when $\mathfrak{B}$ is measurable, $\widehat{\mathfrak{B}} = \frac{1}{2}$ as we saw, and we are led to a contradiction.

Let $\epsilon = \epsilon_1 + \epsilon_2 + \cdots$ be a positive term series whose sum ϵ is small at pleasure. Let $\mathfrak{E}_1 = \{e_n\}$ be a non-overlapping ϵ_1-enclosure of $\mathfrak{A}$, lying in $\mathfrak{E}$. Then

$$\widehat{\mathfrak{E}}_1 = \Sigma \widehat{e}_n = \alpha + \epsilon_1' = \alpha_1 \quad , \quad 0 \leq \epsilon_1' < \epsilon_1 .$$

Let $\mathfrak{B}_1 = \mathfrak{J} - \mathfrak{E}_1$; then $\mathfrak{B}_1 \leq \mathfrak{B}$, and

$$\widehat{\mathfrak{B}}_1 = \widehat{\mathfrak{E}} - \widehat{\mathfrak{E}}_1 = 1 - \alpha_1$$
$$= 1 - \alpha - \epsilon_1' > 1 - \alpha - \epsilon_1 .$$

Each interval e_n contains one or more intervals $\eta_{n1}, \eta_{n2}, \cdots$ of some D_s, such that

$$\sum_m \widehat{\eta}_{nm} = \widehat{e}_n - \sigma_n \quad , \quad 0 \leq \sigma_n$$

where

$$\sigma = \Sigma \sigma_n$$

may be taken small at pleasure.

Now each η_{nm} has a subset $\mathfrak{A}_{nm}$ of $\mathfrak{A}$ entirely similar to $\mathfrak{A}$. Hence there exists an enclosure $\mathfrak{E}_{nm}$ of $\mathfrak{A}_{nm}$, whose measure α_{nm} is such that

$$\frac{\alpha_{nm}}{\alpha_1} = \frac{\widehat{\eta}_{nm}}{1} \quad , \quad \text{or } \alpha_{nm} = \alpha_1 \widehat{\eta}_{nm}.$$

But $\mathfrak{E}_2 = \{\mathfrak{E}_{nm}\}$ is a non-overlapping enclosure of $\mathfrak{A}$, whose measure

$$\alpha_2 = \alpha_1 \sum_{n, m} \widehat{\eta}_{nm} = \alpha_1 \sum_n (\widehat{e}_n - \sigma_n)$$

$$= \alpha_1^2 - \sigma\alpha_1 = \alpha^2 + \epsilon_2' \quad , \quad 0 \leq \epsilon_2' < \epsilon_2$$

if σ is taken sufficiently small.

Let $\mathfrak{B}_2$ denote the irrational points in $\mathfrak{E}_1 - \mathfrak{E}_2$. It is a part of $\mathfrak{B}$, and $\mathfrak{B}_2$ has no point in common with $\mathfrak{B}_1$. We have

$$\widehat{\widehat{\mathfrak{B}}}_2 = \widehat{\widehat{\mathfrak{E}}}_1 - \widehat{\widehat{\mathfrak{E}}}_2 = \alpha_1 - \alpha_2$$

$$= \alpha + \epsilon_1' - \alpha^2 - \epsilon_2'$$

$$> \alpha(1 - \alpha) - \epsilon_2.$$

In this way we may continue. Thus $\mathfrak{B}$ contains the measurable component

$$\mathfrak{B}_1 + \mathfrak{B}_2 + \cdots$$

whose measure is

$$> (1 - \alpha)\{1 + \alpha + \alpha^2 + \cdots\} - \Sigma\epsilon_n$$

$$> 1 - \epsilon.$$

As ϵ is small at pleasure, $\overline{\overline{\mathfrak{B}}} = 1$.

367. (*W. H. Young.*) *Let*

$$\mathfrak{A}_1 \quad , \quad \mathfrak{A}_2 \quad , \quad \mathfrak{A}_3 \cdots \qquad (1$$

be an infinite enumerable set of point sets whose union $\mathfrak{A}$ is limited. Let $\underline{\underline{\mathfrak{A}}}_n > \alpha > 0$, $n = 1, 2 \cdots$ Then there exists a set of points each of which belongs to an infinity of the sets 1) *and of lower measure $\geq \alpha$.*

For by 365, 2, there exists in the sets 1), measurable sets

$$\mathfrak{E}_1 \quad , \quad \mathfrak{E}_2 \quad , \quad \mathfrak{E}_3 \cdots \tag{2}$$

each of whose measures $\mathfrak{E}_n > \alpha$. Let us consider the first n of these sets, viz.:

$$\mathfrak{E}_1 \quad , \quad \mathfrak{E}_2 \quad \cdots \quad \mathfrak{E}_n. \tag{3}$$

The points common to any two of the sets 3) form a measurable set $\mathfrak{D}_{\iota\kappa}$ by 358, 1. Hence the union $\mathfrak{E}_{1n} = \{\mathfrak{D}_{\iota\kappa}\}$ is measurable, by 359. The difference of one of the sets 3), as $\mathfrak{E}_1$ and $Dv(\mathfrak{E}_1, \mathfrak{E}_{1n})$, is a measurable set $\mathfrak{c}_1$ which contains no point in common with the remaining sets of 3). Moreover

$$\widehat{\mathfrak{c}}_1 > \alpha - \widehat{\mathfrak{E}}_{1n}.$$

In the same way we may reason with the other sets $\mathfrak{E}_2, \mathfrak{E}_3 \cdots$ of 3). Thus $\mathfrak{A}$ contains n measurable sets $\mathfrak{c}_1, \mathfrak{c}_2 \cdots \mathfrak{c}_n$ no two of which have a common point.

Hence

$$\mathfrak{c} = \mathfrak{c}_1 + \cdots + \mathfrak{c}_n$$

is a measurable set and

$$\overline{\mathfrak{A}} \geq \widehat{\mathfrak{c}} > n(\alpha - \widehat{\mathfrak{E}}_{1n}).$$

The first and last members give

$$\widehat{\mathfrak{E}}_{1n} > \alpha - \frac{1}{n}\overline{\mathfrak{A}}.$$

Thus however small $\alpha > 0$ may be, there exists a μ such that

$$\widehat{\mathfrak{E}}_{1,\mu} \quad \left(1 - \frac{\epsilon}{2}\right)\alpha. \tag{4}$$

Let us now group the sets 2) in sets of μ. These sets give rise to a sequence of measurable sets

$$\mathfrak{E}_{1\mu} \quad , \quad \mathfrak{E}_{2\mu} \quad , \quad \mathfrak{E}_{3\mu} \cdots \tag{5}$$

such that the points of each set in 5) belong to at least two of the sets 1) and such that the measure of each is > the right side of 4).

We may now reason on the sets 5) as we did on those in 2). We would thus be led to a sequence of measurable sets

$$\mathfrak{E}_{1\nu} \quad , \quad \mathfrak{E}_{2\nu} \quad , \quad \mathfrak{E}_{3\nu} \cdots \tag{6}$$

such that the points of each set in 6) lie in at least two of the sets 5), and hence in at least 2^2 of the sets 1), and such that their measures are.

$$>\left(1-\frac{\epsilon}{2}\right)\left(1-\frac{\epsilon}{2^2}\right)\alpha>(1-\epsilon)\alpha.$$

In this way we may continue indefinitely. Let now $\mathfrak{B}_1$ be the union of all the points of $\mathfrak{A}$, common to at least two of the sets 1). Let $\mathfrak{B}_2$ be the union of the points of $\mathfrak{A}$ common to at least 2^2 of the sets 1), etc. In this way we get the sequence

$$\mathfrak{B}_1 \geq \mathfrak{B}_2 \geq \cdots$$

each of which contains a measurable set whose measure is $>(1-\epsilon)\alpha$.

We have now only to apply 25 and 364.

368. As corollaries of 367 we have:

1. *Let $\mathfrak{Q}_1, \mathfrak{Q}_2 \cdots$ be an infinite enumerable set of non-overlapping cubes whose union is limited. Let each $\widehat{\mathfrak{Q}}_n > \alpha > 0$. Then there exists a set of points $\mathfrak{d}$ whose cardinal number is $\mathfrak{c}$, lying in an infinity of the $\mathfrak{Q}_n$ and such that $\underline{\underline{\mathfrak{d}}} \geq \alpha$.*

2. *(Arzelà.) Let $y_1, y_2 \cdots \doteq \eta$. On each line y_n there exists an enumerable set of intervals of length δ_n. Should the number of intervals ν_n on the lines y_n be finite, let $\nu_n \doteq \infty$. In any case $\delta_n > \alpha > 0$, $n = 1, 2, \cdots$ and the projections of these intervals lie in $\mathfrak{A} = (a, b)$. Then there exists at least one point $x = \xi$ in $\mathfrak{A}$, such that the ordinate through ξ is cut by an infinity of these intervals.*

Associate Sets

369. 1. Let

$$\epsilon_1 > \epsilon_2 > \epsilon_3 \cdots \doteq 0. \tag{1}$$

Let $\mathfrak{E}_n$ be a standard ϵ_n-enclosure of $\mathfrak{A}_n$. If the cells of $\mathfrak{E}_{n+1}$ lie in $\mathfrak{E}_n$, we write

$$\mathfrak{E}_1 \geq \mathfrak{E}_2 \geq \cdots \tag{2}$$

and call 2) a *standard sequence of enclosures* belonging to 1).

Obviously such sequences exist. The set

$$\mathfrak{A}_e = Dv\,\{\mathfrak{E}_n\}$$

is called an *outer associated set* of $\mathfrak{A}$. Obviously

$$\mathfrak{A} \leq \mathfrak{A}_e.$$

2. *Each outer associated set* $\mathfrak{A}_e$ *is measurable, and*

$$\overline{\overline{\mathfrak{A}}} = \widehat{\mathfrak{A}}_e = \lim_{n=\infty} \widehat{\mathfrak{E}}_n. \tag{1}$$

For each $\mathfrak{E}_n$ is measurable; hence $\mathfrak{A}_e$ is measurable by 362, and

$$\begin{aligned} \widehat{\mathfrak{A}}_e &= \lim \widehat{\mathfrak{E}}_n \\ &= \lim(\overline{\overline{\mathfrak{A}}} + \epsilon_n'), \qquad 0 \leq \epsilon_n' < \epsilon_n \\ &= \overline{\overline{\mathfrak{A}}}, \qquad \text{as } \epsilon_n \doteq 0. \end{aligned}$$

370. 1. Let A be the complement of $\mathfrak{A}$ with respect to some cube $\mathfrak{Q}$ containing $\mathfrak{A}$. Let A_e be an outer associated set of A. Then

$$\mathfrak{A}_\iota = \mathfrak{Q} - A_e$$

is called *an inner associated set of* $\mathfrak{A}$. Obviously

$$\mathfrak{A}_\iota \leq \mathfrak{A}.$$

2. *The inner associated set* $\mathfrak{A}_\iota$ *is measurable, and*

$$\widehat{\mathfrak{A}}_\iota = \underline{\underline{\mathfrak{A}}}.$$

For A_e is measurable by 369, 2. Hence $\mathfrak{A}_\iota = \mathfrak{Q} - A_e$ is measurable. But

$$\widehat{A}_e = \overline{\overline{A}}$$

by 369, 2. Hence

$$\widehat{\mathfrak{A}}_\iota = \widehat{\mathfrak{Q}} - \widehat{A}_e = \widehat{\mathfrak{Q}} - \overline{\overline{A}} = \underline{\underline{\mathfrak{A}}}.$$

Separated Sets

371. Let $\mathfrak{A}$, $\mathfrak{B}$ be two limited point sets. If there exist measurable enclosures $\mathfrak{E}$, $\mathfrak{F}$ of $\mathfrak{A}$, $\mathfrak{B}$ such that $\mathfrak{D} = Dv(\mathfrak{E}, \mathfrak{F})$ is a null set, we say $\mathfrak{A}$, $\mathfrak{B}$ are *separated.*

If we superimpose $\mathfrak{F}$ on $\mathfrak{E}$, we get an enclosure of $\mathfrak{C} = (\mathfrak{A}, \mathfrak{B})$ such that those cells containing points of both $\mathfrak{A}$, $\mathfrak{B}$ form a null set, since these cells are precisely $\mathfrak{D}$. We shall call such an enclosure of $\mathfrak{C}$ a *null enclosure.*

Let $\mathfrak{A} = \{\mathfrak{A}_n\}$; we shall call this a *separated division* of $\mathfrak{A}$ into the subsets $\mathfrak{A}_n$, if each pair $\mathfrak{A}_m$, $\mathfrak{A}_n$ is separated. We shall also say the $\mathfrak{A}_n$ are separated.

372. *For $\mathfrak{A}$, $\mathfrak{B}$ to be separated, it is necessary and sufficient that*

$$\mathfrak{D} = Dv(\mathfrak{A}_e, \mathfrak{B}_e)$$

is a null set.

It is sufficient. For let

$$\mathfrak{C} = (\mathfrak{A}, \mathfrak{B}) \quad , \quad \mathfrak{A}_e = \mathfrak{D} + \mathfrak{a}, \qquad \mathfrak{B}_e = \mathfrak{D} + \mathfrak{b}.$$

Then

$$\mathfrak{E} = (\mathfrak{a}, \mathfrak{b}, \mathfrak{D})$$

is a measurable enclosure of $\mathfrak{C}$, consisting of three measurable cells. Of these only $\mathfrak{D}$ contains points of both $\mathfrak{A}$, $\mathfrak{B}$. But by hypothesis $\mathfrak{D}$ is a null set. Hence $\mathfrak{A}$, $\mathfrak{B}$ are separated.

It is necessary. For let $\mathfrak{M}$ be a null distinct enclosure of $\mathfrak{C}$, such that those of its cells $\mathfrak{N}$, containing points of $\mathfrak{A}$, $\mathfrak{B}$ form a null set. Let us superimpose $\mathfrak{M}$ on the enclosure $\mathfrak{E}$ above, getting an enclosure $\mathfrak{F}$ of $\mathfrak{A}$.

The cells of $\mathfrak{F}$ arising from $\mathfrak{a}$ contain no point of $\mathfrak{B}$; similarly the cells arising from $\mathfrak{b}$ contain no point of $\mathfrak{A}$. On the other hand, the cells arising from $\mathfrak{D}$, split up into three classes

$$\mathfrak{D}_a \quad , \quad \mathfrak{D}_b \quad , \quad \mathfrak{D}_{ab}.$$

The first contains no point of $\mathfrak{B}$, the second no point of $\mathfrak{A}$, the cells of the last contain both points of $\mathfrak{A}$, $\mathfrak{B}$. As $\mathfrak{D}_{a,b} \leq \mathfrak{N}$,

$$\widehat{\widehat{\mathfrak{D}}}_{a,b} = 0. \tag{1}$$

On the other hand,

$$\mathfrak{A}_e = \mathfrak{a} + \mathfrak{D} \geq \mathfrak{A};$$

hence

$$\mathfrak{a} + \mathfrak{D}_a + \mathfrak{D}_{ab} \geq \mathfrak{A}.$$

Thus

$$\widehat{\mathfrak{a}} + \widehat{\widehat{\mathfrak{D}}}_a \geq \overline{\overline{\mathfrak{A}}}, \tag{2}$$

by 1). Also

$$\widehat{\widehat{\mathfrak{A}}}_e = \widehat{\mathfrak{a}} + \widehat{\widehat{\mathfrak{D}}} = \overline{\overline{\mathfrak{A}}} \qquad \text{by 369, 2.}$$

This with 2) gives

$$\widehat{\mathfrak{a}} + \widehat{\widehat{\mathfrak{D}}}_a \geq \widehat{\mathfrak{a}} + \widehat{\widehat{\mathfrak{D}}}.$$

Hence

$$\widehat{\widehat{\mathfrak{D}}}_a = \widehat{\widehat{\mathfrak{D}}}. \tag{3}$$

But

$$\widehat{\widehat{\mathfrak{D}}} \geq \widehat{\widehat{\mathfrak{D}}}_a + \widehat{\widehat{\mathfrak{D}}}_b.$$

This with 3) gives $\widehat{\widehat{\mathfrak{D}}}_b = 0$.

In a similar manner we find that $\widehat{\widehat{\mathfrak{D}}}_a = 0$. Hence $\mathfrak{D}$ is a null set by 3).

373. 1. *If* $\mathfrak{A}$, $\mathfrak{B}$ *are separated, then* $\mathfrak{D} = Dv(\mathfrak{A}, \mathfrak{B})$ *is a null set.* For $\mathfrak{D}_e = Dv(\mathfrak{A}_e, \mathfrak{B}_e)$ is a null set by 372. But $\mathfrak{D} \leq \mathfrak{D}_e$.

2. Let $\mathfrak{A}$, $\mathfrak{B}$ be the Van Vleck sets in 366. We saw there that $\overline{\overline{\mathfrak{A}}} = \overline{\overline{\mathfrak{B}}} = 1$. Then by 369, 2, $\widehat{\widehat{\mathfrak{A}}}_e = \widehat{\widehat{\mathfrak{B}}}_e = 1$. The divisor of $\mathfrak{A}_e$, $\mathfrak{B}_e$ is not a null set. Hence by 372, $\mathfrak{A}$, $\mathfrak{B}$ are not separated. *Thus the condition that* $\mathfrak{D}$ *be a null set is necessary, but not sufficient.*

374. 1. *Let* $\{\mathfrak{A}_n\}$, $\{\mathfrak{B}_n\}$ *be separated divisions of* $\mathfrak{A}$. *Let* $\mathfrak{C}_{\iota\kappa} = Dv(\mathfrak{A}_\iota, \mathfrak{B}_\kappa)$. *Then* $\{\mathfrak{C}_{\iota\kappa}\}$ *is a separated division of* $\mathfrak{A}$ *also.*

We have to show there exists a null enclosure of any two of the sets $\mathfrak{C}_{\iota\kappa}$, $\mathfrak{C}_{mn}$. Now $\mathfrak{C}_{\iota\kappa}$ lies in $\mathfrak{A}_\iota$ and $\mathfrak{B}_\kappa$; also $\mathfrak{C}_{mn}$ lies in $\mathfrak{A}_m$, $\mathfrak{B}_n$. By hypothesis there exists a null enclosure $\mathfrak{E}$ of $\mathfrak{A}_\iota$, $\mathfrak{A}_m$; and a null enclosure $\mathfrak{F}$ of $\mathfrak{B}_\kappa$, $\mathfrak{B}_n$. Then $\mathfrak{G} = Dv(\mathfrak{E}, \mathfrak{F})$ is a null enclosure of $\mathfrak{A}_\iota$, $\mathfrak{A}_m$ and of $\mathfrak{B}_\kappa$, $\mathfrak{B}_n$. Thus those cells of $\mathfrak{G}$, call them $\mathfrak{G}_a$, containing points of both $\mathfrak{A}_\iota$, $\mathfrak{A}_m$ form a null set; and those of its cells $\mathfrak{G}_b$, containing points of both $\mathfrak{B}_\kappa$, $\mathfrak{B}_n$ also form a null set.

Let $G = \{g\}$ denote the cells of $\mathfrak{G}$ that contain points of both $\mathfrak{C}_{\iota\kappa}$, $\mathfrak{C}_{mn}$. Then a cell g contains points of $\mathfrak{A}_\iota\, \mathfrak{A}_m\, \mathfrak{B}_\kappa\, \mathfrak{B}_n$. Thus g lies in $\mathfrak{G}_a$ or $\mathfrak{G}_b$. Thus in either case G is a null set. Hence $\{\mathfrak{C}_{\iota\kappa}\}$ form a separated division of $\mathfrak{A}$.

2. Let D be a separated division of $\mathfrak{A}$ into the cells d_1, $d_2 \cdots$ Let E be another separated division of $\mathfrak{A}$ into the cells e_1, $e_2 \cdots$ We have seen that $F = \{f_{\iota\kappa}\}$ where $f_{\iota\kappa} = Dv(d_\iota, e_\kappa)$ is also a separated division of $\mathfrak{A}$. We shall say that F is obtained by superimposing E on D or D on E, and write $F = D + E = E + D$.

3. Let E be a separated division of the separated component $\mathfrak{B}$ of $\mathfrak{A}$, while D is a separated division of $\mathfrak{A}$. If d_ι is a cell of D, e_κ a cell of E, and $d_{\iota\kappa} = Dv(d_\iota, e_\kappa)$, then

$$d_\iota = (d_{\iota 1}, d_{\iota 2}, \cdots) + \delta_\iota.$$

Thus superposing E on D causes each cell d_ι to fall into separated cells $d_{\iota 1}$, $d_{\iota 2} \cdots \delta_\iota$. The union of all these cells, arising from different d_ι, gives a separated division of $\mathfrak{A}$ which we also denote by $D + E$.

375. *Let* $\{\mathfrak{A}_n\}$ *be a separated division of* $\mathfrak{A}$. *Let* $\mathfrak{B} < \mathfrak{A}$, *and let* $\mathfrak{B}_n$ *denote the points of* $\mathfrak{B}$ *in* $\mathfrak{A}_n$. *Then* $\{\mathfrak{B}_n\}$ *is a separated division of* $\mathfrak{B}$.

For let $\mathfrak{D}$ be a null enclosure of $\mathfrak{A}_m$, $\mathfrak{A}_n$. Let $\mathfrak{D}_{ab}$ denote the cells of $\mathfrak{D}$ containing points of both $\mathfrak{A}_m$, $\mathfrak{A}_n$. Let $\mathfrak{E}$ denote the cells of $\mathfrak{D}$ containing points of $\mathfrak{B}$; let $\mathfrak{E}_{a,b}$ denote the cells containing points of both $\mathfrak{B}_m$, $\mathfrak{B}_n$. Then

$$\mathfrak{E}_{ab} \leq \mathfrak{D}_{ab}.$$

As $\mathfrak{D}_{ab}$ is a null set, so is $\mathfrak{E}_{ab}$.

376. 1. *Let* $\mathfrak{A} = (\mathfrak{B}, \mathfrak{C})$ *be a separated division of* $\mathfrak{A}$. *Then*

$$\overline{\overline{\mathfrak{A}}} = \overline{\overline{\mathfrak{B}}} + \overline{\overline{\mathfrak{C}}}. \tag{1}$$

For let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. There exist ϵ_n-measurable enclosures of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$; call them respectively A_n, B_n, C_n. Then $\mathfrak{E}_n = A_n + B_n + C_n$ is an ϵ_n-enclosure of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ simultaneously.

Since $\mathfrak{B}$, $\mathfrak{C}$ are separated, there exist enclosures B, C of $\mathfrak{B}$, $\mathfrak{C}$ such that those cells of $D = B + C$ containing points of both $\mathfrak{B}$ and $\mathfrak{C}$ form a null set. Let us now superpose D on $\mathfrak{E}_n$ getting an ϵ_n-enclosure $E_n = \{e_{ns}\}$ of $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ simultaneously. Let e_{bn} denote the cells of E_n containing points of $\mathfrak{B}$ alone; e_{cn} those cells containing only points of $\mathfrak{C}$; and e_{bc} those cells containing points of both $\mathfrak{B}$, $\mathfrak{C}$. Then

$$\sum_s \widehat{e}_{ns} = \Sigma \widehat{e}_{bn} + \Sigma \widehat{e}_{cn} + \Sigma \widehat{e}_{bc}. \tag{2}$$

As $\Sigma e_{bc} = 0$, we see that as $n \doteq \infty$,

$$\Sigma \widehat{e}_{ns} \doteq \overline{\overline{\mathfrak{A}}} \quad , \quad \Sigma \widehat{e}_{bn} \doteq \overline{\overline{\mathfrak{B}}} \quad , \quad \Sigma \widehat{e}_{cn} \doteq \overline{\overline{\mathfrak{C}}}.$$

Hence passing to the limit $n = \infty$, in 2) we get 1).

2. *Let* $\mathfrak{A} = \{\mathfrak{B}_n\}$ *be a separated division of limited* $\mathfrak{A}$. *Then*

$$\overline{\overline{\mathfrak{A}}} = \Sigma \overline{\overline{\mathfrak{B}}}_n. \tag{1}$$

For in the first place, the series

$$B = \Sigma \overline{\overline{\mathfrak{B}}}_n \tag{2}$$

is convergent. In fact let $\mathfrak{A}_n = (\mathfrak{B}_1, \mathfrak{B}_2 \cdots \mathfrak{B}_n)$.

Then $\mathfrak{A}_n \leq \mathfrak{A}$, and hence $\overline{\overline{\mathfrak{A}}}_n \leq \overline{\overline{\mathfrak{A}}}$.

On the other hand, by 1

$$\overline{\overline{\mathfrak{A}}}_n = \overline{\overline{\mathfrak{B}}}_1 + \cdots + \overline{\overline{\mathfrak{B}}}_n = B_n,$$

the sum of the first n terms of the series 2). Thus

$$B_n \leq \overline{\overline{\mathfrak{A}}},$$

and hence B is convergent by 80, 4. Thus

$$B \leq \overline{\overline{\mathfrak{A}}}.$$

On the other hand, by 339,

$$B \geq \overline{\overline{\mathfrak{A}}}.$$

The last two relations give 1).

CHAPTER XII

LEBESGUE INTEGRALS

General Theory

377. In the foregoing chapters we have developed a theory of integration which rests on the notion of content. In this chapter we propose to develop a theory of integration due to Lebesgue, which rests on the notion of measure. The presentation here given differs considerably from that of Lebesgue. As the reader will see, the theory of Lebesgue integrals as here presented differs from that of the theory of ordinary integrals only in employing an infinite number of cells instead of a finite number.

378. In the following we shall suppose the field of integration $\mathfrak{A}$ to be limited, as also the integrand $\mathfrak{A}$ lies in $\mathfrak{R}_m$ and for brevity we set $f(x) = f(x_1 \cdots x_m)$. Let us effect a separated division of $\mathfrak{A}$ into cells $\delta_1, \delta_2 \cdots$. If each cell δ_ι lies in a cube of side d, we shall say D is a *separated division of norm d.*

As before, let

$$M_\iota = \text{Max} f \quad , \quad m_\iota = \text{Min} f \quad , \quad \omega_\iota = \text{Osc} f = M_\iota - m_\iota \quad \text{in } \delta_\iota .$$

Then

$$\overline{S}_D = \Sigma M_\iota \overline{\overline{\delta}}_\iota \quad , \quad \underline{S}_D = \Sigma m_\iota \overline{\overline{\delta}}_\iota ,$$

the summation extending over all the cells of $\mathfrak{A}$, are called the *upper and lower sums of f over* $\mathfrak{A}$ *with respect to D.*

The sum

$$\Omega_D f = \Sigma \omega_\iota \overline{\overline{\delta}}_\iota$$

is called the *oscillatory sum with respect to D.*

379. *If* $m = \text{Min} f$, $M = \text{Max} f$ *in* $\mathfrak{A}$, *then*

$$m \overline{\overline{\mathfrak{A}}} \le \underline{S}_D \le \overline{S}_D \le M \overline{\overline{\mathfrak{A}}} .$$

For

$$m \le m_\iota \le M_\iota \le M .$$

Hence $$\Sigma m\bar{\bar{\delta}}_\iota \le \Sigma m_\iota \bar{\bar{\delta}}_\iota \le \Sigma M_\iota \bar{\bar{\delta}}_\iota \le \Sigma M \bar{\bar{\delta}}_\iota .$$

Thus $$m\Sigma\bar{\bar{\delta}}_\iota \le \underline{S}_D \le \overline{S}_D \le M\Sigma\bar{\bar{\delta}}_\iota .$$

But $$\Sigma\bar{\bar{\delta}}_\iota = \bar{\bar{\mathfrak{A}}},$$

by 376, 2.

380. 1. Since f is limited in $\mathfrak{A}$,

$$\text{Max } \underline{S}_D \quad , \quad \text{Min } \overline{S}_D$$

with respect to the class of all separated divisions D of $\mathfrak{A}$, are finite. We call them respectively the *lower and upper Lebesgue integrals* of f over the field $\mathfrak{A}$, and write

$$\underline{\int}_{\mathfrak{A}} f = \text{Max } \underline{S}_D \quad ; \quad \overline{\int}_{\mathfrak{A}} f = \text{Min } \overline{S}_D .$$

In order to distinguish these new integrals from the old ones, we have slightly modified the old symbol $\int$ to resemble somewhat script L, or $\int$, in honor of the author of these integrals.

If $$\underline{\int}_{\mathfrak{A}} f = \overline{\int}_{\mathfrak{A}} f$$

we say f is *L-integrable over* $\mathfrak{A}$, and denote the common value by

$$\int_{\mathfrak{A}} f,$$

which we call the *L-integral.*

The integrals treated of in Vol. I we will call *R-integrals*, *i.e.* integrals in the sense of Riemann.

2. *Let f be limited over the null set $\mathfrak{A}$. Then f is L-integrable in $\mathfrak{A}$, and*

$$\int_{\mathfrak{A}} f = 0 .$$

This is obvious from 379.

381. *Let $\mathfrak{A}$ be metric or complete. Then*

$$\underline{\int}_{\mathfrak{A}} f \le \underline{\int}_{\mathfrak{A}} f \le \overline{\int}_{\mathfrak{A}} f . \qquad (1$$

For let $d_1, d_2 \cdots$ be an unmixed metric or complete division of $\mathfrak{A}$ of norm d. Let each cell d_ι be split up into the separated cells $d_{\iota 1}, d_{\iota 2} \cdots$

Then since d_ι is complete or metric,

$$\overline{d}_\iota = \overline{\overline{d}}_\iota = \Sigma \overline{\overline{d}}_{\iota\kappa}.$$

Hence using the customary notation,

$$m_\iota \overline{\overline{d}}_{\iota\kappa} \leq m_{\iota\kappa} \overline{\overline{d}}_{\iota\kappa} \leq M_{\iota\kappa} \overline{\overline{d}}_{\iota\kappa} \leq M_\iota \overline{\overline{d}}_{\iota\kappa}.$$

Thus summing over κ,

$$m_\iota \overline{d}_\iota \leq \sum_\kappa m_{\iota\kappa} \overline{\overline{d}}_{\iota\kappa} \leq \sum_\kappa M_{\iota\kappa} \overline{\overline{d}}_{\iota\kappa} \leq M_\iota \overline{d}_\iota.$$

Summing over ι gives

$$\Sigma m_\iota \overline{d}_\iota \leq \sum_{\iota\kappa} m_{\iota\kappa} \overline{\overline{d}}_{\iota\kappa} \leq \sum_{\iota\kappa} M_{\iota\kappa} \overline{\overline{d}}_{\iota\kappa} \leq \Sigma M_\iota \overline{d}_\iota.$$

Thus by definition,

$$\Sigma m_\iota \overline{d}_\iota \leq \underline{\overline{\int}}_{\mathfrak{A}} f \leq \Sigma M_\iota \overline{d}_\iota.$$

Letting now $d \doteq 0$, we get 1).

2. *Let $\mathfrak{A}$ be metric or complete. If f is R-integrable in $\mathfrak{A}$, it is L-integrable and*

$$\int_{\mathfrak{A}} f = \underline{\int}_{\mathfrak{A}} f. \tag{2}$$

3. In case that $\mathfrak{A}$ is not metric or complete, the relations 1), 2) may not hold.

Example 1. Let $\mathfrak{A}$ denote the rational points in the interval $(0, 1)$.

Let

$$f = 1, \text{ for } x = \frac{m}{n},\ n \text{ even}$$

$$= 2, \text{ when } n \text{ is odd.}$$

Then

$$\underline{\int}_{\mathfrak{A}} f = 1 \quad , \quad \overline{\int}_{\mathfrak{A}} f = 2;$$

while

$$\underline{\overline{\int}}_{\mathfrak{A}} f = 0,$$

since $\mathfrak{A}$ is a null set. Thus 1) does not hold.

Example 2. Let $f = 1$ at the rational points $\mathfrak{A}$ in $(0, 1)$. Then

$$\int_{\mathfrak{A}} f = 1 \quad , \quad \underline{\int}_{\mathfrak{A}} f = 0 \quad , \quad \text{and} \quad \underline{\int}_{\mathfrak{A}} f < \int_{\mathfrak{A}} f. \tag{3}$$

Let $g = -1$ in $\mathfrak{A}$. Then

$$\underline{\int}_{\mathfrak{A}} g = 0 \quad , \quad \int_{\mathfrak{A}} g = -1 \quad , \quad \text{and} \quad \int_{\mathfrak{A}} g < \underline{\int}_{\mathfrak{A}} g. \tag{4}$$

Thus in 3) the L-integral is less than the R-integral, while in 4) it is greater.

Example 3. Let $f = 1$ at the irrational points $\mathfrak{A}$ in $(0, 1)$. Then

$$\underline{\int}_{\mathfrak{A}} f = \int_{\mathfrak{A}} f,$$

although $\mathfrak{A}$ is neither metric nor complete.

382. *Let D, Δ be separated divisions of $\mathfrak{A}$. Let*

$$E = D + \Delta = \{e_\iota\}.$$

Then

$$\overline{S}_E \leq \overline{S}_D, \overline{S}_\Delta$$

$$\underline{S}_E \geq \underline{S}_D, \underline{S}_\Delta.$$

For any cell d_ι of D splits up into $d_{\iota 1}, d_{\iota 2} \cdots$ on superimposing Δ, and

$$\overline{\overline{d}}_\iota = \sum_\kappa \overline{\overline{d}}_{\iota\kappa}.$$

But

$$M_{\iota\kappa} \overline{\overline{d}}_{\iota\kappa} \leq M_\iota \overline{\overline{d}}_{\iota\kappa},$$

and

$$m_{\iota\kappa} \overline{\overline{d}}_{\iota\kappa} \geq m_\iota \overline{\overline{d}}_{\iota\kappa}.$$

Thus

$$\overline{S}_E \leq \overline{S}_D \quad , \quad \underline{S}_E \geq \underline{S}_D.$$

383. 1. *Extremal Sequences.* There exists a sequence of separated divisions

$$D_1 \quad , \quad D_2 \quad , \quad D_3 \cdots \tag{1}$$

each D_{n+1} being obtained from D_n by superposition, such that

$$\overline{S}_{D_1} \geq \overline{S}_{D_2} \geq \cdots \doteq \overline{\int}_{\mathfrak{A}} f, \tag{2}$$

$$\underline{S}_{D_1} \leq \underline{S}_{D_2} \leq \cdots \doteq \underline{\int}_{\mathfrak{A}} f. \tag{3}$$

For let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. For each ϵ_n, there exists a division E_n such that

$$0 < \overline{S}_{E_n} - \overline{\int_{\mathfrak{A}}} < \epsilon_n.$$

Let

$$E_2 + D_1 = D_2 \quad , \quad E_3 + D_2 = D_3, \cdots$$

and for uniformity set $E_1 = D_1$. Then by 382,

$$\overline{S}_{D_{n+1}} \leq \overline{S}_{D_n} \quad , \quad \overline{S}_{D_n} \leq \overline{S}_{E_n}.$$

Hence

$$0 \leq \overline{S}_{D_n} - \overline{\int_{\mathfrak{A}}} < \epsilon_n.$$

Letting $n \doteq \infty$ we get 2).

Thus there exists a sequence $\{D_n'\}$ of the type 1) for 2), and a sequence $\{D_n''\}$ of the same type for 3). Let now $D_n = D_n' + D_n''$. Obviously 2), 3) hold simultaneously for the sequence $\{D_n\}$.

2. The sequence 1) is called an *extremal sequence.*

3. *Let* $\{D_n\}$ *be an extremal sequence, and* E *any separated division of* $\mathfrak{A}$. *Let* $E_n = D_n + E$. *Then* $E_1, E_2 \cdots$ *is an extremal sequence also.*

384. *Let f be L-integrable in* $\mathfrak{A}$. *Then for any extremal sequence* $\{D_n\}$,

$$\int_{\mathfrak{A}} f = \lim_{n=\infty} \Sigma f(\xi_\iota)\,\overline{\overline{d}}_\iota, \tag{1}$$

where d_ι *are the cells of* D_n, *and* ξ_ι *any point of* $\mathfrak{A}$ *in* d_ι.

For

$$m_\iota \leq f(\xi_\iota) \leq M_\iota.$$

Hence

$$\underline{S}_{D_n} \leq \Sigma f(\xi_\iota)\,\overline{\overline{d}}_\iota \leq \overline{S}_{D_n}.$$

Passing to the limit we get 1).

385. 1. *Let* $m = \mathrm{Min}\, f$, $M = \mathrm{Max}\, f$ *in* $\mathfrak{A}$. *Then*

$$m\overline{\overline{\mathfrak{A}}} \leq \underline{\overline{\int_{\mathfrak{A}}}} f \leq M\overline{\overline{\mathfrak{A}}}.$$

This follows at once from 379 and 383, 1.

2. *Let* $F = \text{Max}\,|f|$ *in* $\mathfrak{A}$, *then*

$$\left|\int_{\mathfrak{A}} f\right| \leq F\overline{\overline{\mathfrak{A}}}.$$

This follows from 1.

386. *In order that f be L-integrable in $\mathfrak{A}$, it is necessary that, for each extremal sequence* $\{D_n\}$,

$$\lim_{n=\infty} \Omega_{D_n} f = 0\,;$$

and it is sufficient if there exists a sequence of superimposed separated divisions $\{E_n\}$, *such that*

$$\lim_{n=\infty} \Omega_{E_n} f = 0.$$

It is *necessary*. For

$$\underline{\int_{\mathfrak{A}}} = \lim \underline{S}_{D_n} \quad , \quad \overline{\int_{\mathfrak{A}}} = \lim \overline{S}_{D_n}.$$

As f is L-integrable,

$$0 = \overline{\int_{\mathfrak{A}}} - \underline{\int_{\mathfrak{A}}} = \lim\,(\overline{S}_{D_n} - \underline{S}_{D_n}) = \lim \Omega_{D_n} f.$$

It is *sufficient*. For

$$\underline{S}_{E_n} \leq \overline{\underline{\int_{\mathfrak{A}}}} \leq \overline{S}_{E_n}.$$

Both $\{\underline{S}_{E_n}\}$, $\{\overline{S}_{E_n}\}$ are limited monotone sequences. Their limits therefore exist. Hence

$$0 = \lim \Omega_{E_n} = \lim \overline{S}_{E_n} - \lim \underline{S}_{E_n}.$$

Thus

$$\underline{\int_{\mathfrak{A}}} = \overline{\int_{\mathfrak{A}}}.$$

387. *In order that f be L-integrable, it is necessary and sufficient that for each $\epsilon > 0$, there exists a separated division D of $\mathfrak{A}$, for which*

$$\Omega_D f < \epsilon. \tag{1}$$

It is *necessary*. For by 386, there exists an extremal sequence $\{D_n\}$, such that

$$0 \leq \Omega_{D_n} f < \epsilon \quad , \quad \text{for any } n \geq \text{some } m.$$

Thus we may take D_m for D.

It is *sufficient*. For let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. Let $\{D_n\}$ be an extremal sequence for which

$$0 \leq \Omega_{D_n} f < \epsilon_n.$$

Let $\Delta_1 = D_1$, $\Delta_2 = \Delta_1 + D_2$, $\Delta_3 = \Delta_2 + D_3 \cdots$ Then $\{\Delta_n\}$ is a set of superimposed separated divisions, and obviously

$$\Omega_{\Delta_n} f < \epsilon_n \doteq 0.$$

Hence f is L-integrable by 386.

388. *In order that f be L-integrable, it is necessary and sufficient that, for each pair of positive numbers ω, σ there exists a separated division D of $\mathfrak{A}$, such that if $\eta_1, \eta_2, \cdots$ are those cells in which* Osc $f > \omega$, *then*

$$\Sigma \overline{\overline{\eta}}_\iota < \sigma. \tag{1}$$

It is *necessary*. For by 387 there exists a separated division $D = \{\delta_\iota\}$ for which

$$\Omega_D f = \Sigma \omega_\iota \overline{\overline{\delta}}_\iota < \omega\sigma. \tag{2}$$

If $\theta_1, \theta_2 \cdots$ denote the cells of D in which Osc$f \leq \omega$,

$$\Omega_D f = \Sigma \omega_\iota \overline{\overline{\eta}}_\iota + \Sigma \omega_\iota \overline{\overline{\theta}}_\iota \geq \omega \Sigma \overline{\overline{\eta}}_\iota. \tag{3}$$

This in 2) gives 1).

It is *sufficient*. For taking $\epsilon > 0$ small at pleasure, let us then take

$$\sigma = \frac{\epsilon}{2\,\Omega} \quad , \quad \omega = \frac{\epsilon}{2\,\overline{\overline{\mathfrak{A}}}}, \tag{4}$$

where $\Omega =$ Osc f in $\mathfrak{A}$.

From 1), 3), and 4) we have, since $\omega_\iota \leq \Omega$,

$$\Omega_D f \leq \Sigma \Omega \overline{\overline{\eta}}_\iota + \Sigma \omega_\iota \overline{\overline{\theta}}_\iota \leq \sigma\Omega + \Sigma \omega \overline{\overline{\theta}}_\iota < \sigma\Omega + \omega\overline{\overline{\mathfrak{A}}} = \epsilon.$$

We now apply 387.

389. 1. *If f is L-integrable in $\mathfrak{A}$, it is in $\mathfrak{B} < \mathfrak{A}$.*

For let $\{D_n\}$ be an extremal sequence of f relative to $\mathfrak{A}$. Then by 386,

$$\Omega_{D_n} f \doteq 0. \tag{1}$$

But the sequence $\{D_n\}$ defines a sequence of superposed separated divisions of $\mathfrak{B}$, which we denote by $\{E_n\}$. Obviously

$$\Omega_{E_n} f \leq \Omega_{D_n} f.$$

Hence by 1),

$$\Omega_{E_n} f \doteq 0,$$

and f is L-integrable in $\mathfrak{B}$ by 386.

2. *If f is L-integrable in $\mathfrak{A}$, so is $|f|$.*

The proof is analogous to I, 507, using an extremal sequence for f.

390. 1. *Let $\{\mathfrak{A}_n\}$ be a separated division of $\mathfrak{A}$ into a finite or infinite number of subsets. Let f be limited in $\mathfrak{A}$. Then*

$$\overline{\underline{\int}}_{\mathfrak{A}} f = \overline{\underline{\int}}_{\mathfrak{A}_1} f + \overline{\underline{\int}}_{\mathfrak{A}_2} f + \cdots \tag{1}$$

For *let us* 1° *suppose* that the subsets $\mathfrak{A}_1 \cdots \mathfrak{A}_r$ are finite in number. Let $\{D_n\}$ be an extremal sequence of f relative to $\mathfrak{A}$, and $\{D_{mn}\}$ an extremal sequence relative to $\mathfrak{A}_m$. Let

$$E_n = D_n + D_{1n} + \cdots + D_{rn}.$$

Then $\{E_n\}$ is an extremal sequence of f relative to $\mathfrak{A}$, and also relative to each $\mathfrak{A}_m$.

Now

$$\overline{\underline{S}}_{\mathfrak{A}, E_n} = \overline{\underline{S}}_{\mathfrak{A}_1, E_n} + \cdots + \overline{\underline{S}}_{\mathfrak{A}_r, E_n}.$$

Letting $n \doteq \infty$, we get 1), for this case.

Let now r be infinite. We have

$$\overline{\overline{\mathfrak{A}}} = \sum_1^{\infty} \overline{\overline{\mathfrak{A}}}_m. \tag{2}$$

Let

$$\mathfrak{B}_n = (\mathfrak{A}_1 \cdots \mathfrak{A}_n) \quad , \quad \mathfrak{C}_n = \mathfrak{A} - \mathfrak{B}_n.$$

Then $\mathfrak{B}_n$, $\mathfrak{C}_n$ form a separated division of $\mathfrak{A}$, and

$$\overline{\overline{\mathfrak{A}}} = \overline{\overline{\mathfrak{B}}}_n + \overline{\overline{\mathfrak{C}}}_n.$$

If ν is taken large enough, 2) shows that

$$\overline{\overline{\mathfrak{C}}}_n < \frac{\epsilon}{M} \quad , \quad n \geq \nu \quad , \quad M = \text{Max}\,|f| \quad \text{in } \mathfrak{A}.$$

Thus by case 1°,

$$\overline{\int_{\mathfrak{A}}} f = \overline{\int_{\mathfrak{B}_n}} f + \overline{\int_{\mathfrak{C}_n}} f$$

$$= \overline{\int_{\mathfrak{A}_1}} + \cdots + \overline{\int_{\mathfrak{A}_n}} + \epsilon', \tag{3}$$

where by 385, 2

$$|\epsilon'| \leq M\overline{\overline{\mathfrak{C}}}_n < \epsilon \quad , \quad n \geq \nu.$$

Thus 1) follows from 3) in this case.

2. *Let $\{\mathfrak{A}_n\}$ be a separated division of $\mathfrak{A}$. Then*

$$\int_{\mathfrak{A}} f = \Sigma \int_{\mathfrak{A}_n} f,$$

if f is L integrable in $\mathfrak{A}$, or if it is in each $\mathfrak{A}_n$, and limited in $\mathfrak{A}$.

391. 1. *Let $f = g$ in $\mathfrak{A}$ except at the points of a null set $\mathfrak{N}$. Then*

$$\overline{\int_{\mathfrak{A}}} f = \overline{\int_{\mathfrak{A}}} g. \tag{1}$$

For let

$$\mathfrak{A} = \mathfrak{B} + \mathfrak{N}. \quad \text{Then}$$

$$\overline{\int_{\mathfrak{A}}} f = \overline{\int_{\mathfrak{B}}} f + \overline{\int_{\mathfrak{N}}} f = \overline{\int_{\mathfrak{B}}} f. \tag{2}$$

Similarly

$$\overline{\int_{\mathfrak{A}}} g = \overline{\int_{\mathfrak{B}}} g. \tag{3}$$

But $f = g$ in $\mathfrak{B}$. Thus 2), 3) give 1).

392. 1. *If $c > 0$;* $$\overline{\int} cf = c \overline{\int} f.$$

If $c < 0$; $$\underline{\int} cf = c \overline{\int} f, \qquad \overline{\int} cf = c \underline{\int} f.$$

The proof is similar to 3, 3, using extremal sequences.

2. *If f is L-integrable in $\mathfrak{A}$, so is cf, and*

$$\int_{\mathfrak{A}} cf = c \int_{\mathfrak{A}} f,$$

where c is a constant.

393. 1. *Let* $F(x) = f_1(x) + \cdots + f_n(x)$, *each* f_m *being limited in* $\mathfrak{A}$. *Then*

$$\sum_1^n \underline{\int}_{\mathfrak{A}} f_m \leq \underline{\overline{\int}}_{\mathfrak{A}} F \leq \sum_1^n \overline{\int}_{\mathfrak{A}} f_m. \tag{1}$$

For let $\{D_n\}$ be an extremal sequence common to $F, f_1, \cdots f_n$. In each cell

$$d_{n1} \quad , \quad d_{n2} \cdots$$

of D_n we have

$$\Sigma \operatorname{Min} f_m \leq \operatorname{Min} F \leq \operatorname{Max} F \leq \Sigma \operatorname{Max} f_m.$$

Multiplying by $\overline{\overline{d}}_{ns}$, summing over s and then letting $n \doteq \infty$, gives 1).

2. *If* $f_1(x), \cdots f_n(x)$ *are each L-integrable in* $\mathfrak{A}$, *so is*

$$F = c_1 f_1 + \cdots + c_n f_n,$$

and

$$\int_{\mathfrak{A}} F = c_1 \int_{\mathfrak{A}} f_1 + \cdots + c_n \int_{\mathfrak{A}} f_n.$$

394. 1.

$$\underline{\int}_{\mathfrak{A}} (f+g) \leq \underline{\int}_{\mathfrak{A}} f + \overline{\int}_{\mathfrak{A}} g \leq \overline{\int}_{\mathfrak{A}} (f+g).$$

For using the notation of 393,

$$\operatorname{Min}(f+g) \leq \operatorname{Min} f + \operatorname{Max} g \leq \operatorname{Max}(f+g)$$

in each cell d_{ns} of D_n.

2. *If* g *is L-integrable in* $\mathfrak{A}$,

$$\overline{\int}_{\mathfrak{A}} (f+g) = \overline{\int}_{\mathfrak{A}} f + \int_{\mathfrak{A}} g.$$

Reasoning similar to 3, 4, using extremal sequences.

3.

$$\underline{\int}_{\mathfrak{A}} (f-g) \leq \underline{\int}_{\mathfrak{A}} f - \underline{\int}_{\mathfrak{A}} g.$$

$$\overline{\int}_{\mathfrak{A}} f - \overline{\int}_{\mathfrak{A}} g \leq \overline{\int}_{\mathfrak{A}} (f-g).$$

For
$$\underline{\int}_{\mathfrak{A}}(f-g) \leq \underline{\int}_{\mathfrak{A}} f + \overline{\int}_{\mathfrak{A}}(-g) \leq \underline{\int}_{\mathfrak{A}} f - \underline{\int}_{\mathfrak{A}} g;$$
etc.

4. *If f, g are L-integrable in $\mathfrak{A}$, so is $f-g$, and*
$$\int_{\mathfrak{A}}(f-g) = \int_{\mathfrak{A}} f - \int_{\mathfrak{A}} g.$$

395. *If f, g are L-integrable in $\mathfrak{A}$, so is $f \cdot g$.*

Also their quotient f/g is L-integrable provided it is limited in $\mathfrak{A}$.

The proof of the first part of the theorem is analogous to I, 505, using extremal sequences common to both f and g. The proof of the second half is obvious and is left to the reader.

396. 1. *Let f, g be limited in $\mathfrak{A}$, and $f \leq g$, except possibly in a null set $\mathfrak{N}$. Then*
$$\overline{\underline{\int}}_{\mathfrak{A}} f \leq \overline{\underline{\int}}_{\mathfrak{A}} g. \tag{1}$$

Let us *suppose first* that $f \leq g$ everywhere in $\mathfrak{A}$.

Let $\{D_n\}$ be an extremal sequence common to both f and g. Then
$$\overline{\underline{S}}_{D_n} f \leq \overline{\underline{S}}_{D_n} g.$$

Letting $n \doteq \infty$, we get 1).

We consider now the *general case.* Let $\mathfrak{A} = \mathfrak{B} + \mathfrak{N}$. Then
$$\overline{\underline{\int}}_{\mathfrak{A}} f = \overline{\underline{\int}}_{\mathfrak{B}} f \quad , \quad \overline{\underline{\int}}_{\mathfrak{A}} g = \overline{\underline{\int}}_{\mathfrak{B}} g,$$
since
$$\overline{\underline{\int}}_{\mathfrak{N}} f = \overline{\underline{\int}}_{\mathfrak{N}} g = 0.$$

But in $\mathfrak{B}$, $f \leq g$ without exception. We may therefore use the result of case 1°.

2. *Let $f \geq 0$ in $\mathfrak{A}$. Then*
$$\operatorname{Min} g \cdot \underline{\int}_{\mathfrak{A}} f \leq \overline{\underline{\int}}_{\mathfrak{A}} f \cdot g \leq \operatorname{Max} g \cdot \overline{\underline{\int}}_{\mathfrak{A}} f.$$

For
$$f \cdot \operatorname{Min} g \leq fg \leq f \operatorname{Max} g.$$

397. *The relations of* 4 *also hold for L-integrals, viz.:*

$$\left|\overline{\underline{\int}}_{\mathfrak{A}} f\right| \leq \overline{\int}_{\mathfrak{A}} |f|. \tag{1}$$

$$\overline{\underline{\int}}_{\mathfrak{A}} f \leq \left|\overline{\int}_{\mathfrak{A}} f\right|. \tag{2}$$

$$-\underline{\int}_{\mathfrak{A}} |f| \leq \overline{\int}_{\mathfrak{A}} f \leq \overline{\int}_{\mathfrak{A}} |f|. \tag{3}$$

$$-\overline{\int}_{\mathfrak{A}} |f| \leq \underline{\int}_{\mathfrak{A}} f \leq \underline{\int}_{\mathfrak{A}} |f|. \tag{4}$$

The proof is analogous to that employed for the R-integrals, using extremal sequences.

398. *Let* $\mathfrak{A} = (\mathfrak{B}_u, \mathfrak{C}_u)$ *be a separated division for each* $u \doteq 0$. *Let* $\overline{\overline{\mathfrak{C}}}_u \doteq 0$. *Then*

$$\lim_{u=0} \overline{\underline{\int}}_{\mathfrak{B}_u} f = \overline{\underline{\int}}_{\mathfrak{A}} f.$$

For by 390, 1,

$$\overline{\underline{\int}}_{\mathfrak{A}} = \overline{\underline{\int}}_{\mathfrak{B}_u} + \overline{\underline{\int}}_{\mathfrak{C}_u}.$$

But by 385, 2, the last integral $\doteq 0$, since $\overline{\overline{\mathfrak{C}}}_u \doteq 0$, and since f is limited.

399. *Let f be limited and continuous in $\mathfrak{A}$, except possibly at the points of a null set $\mathfrak{N}$. Then f is L-integrable in $\mathfrak{A}$.*

Let us *first* take $\mathfrak{N} = 0$. Then f is continuous in $\mathfrak{A}$. Let $\mathfrak{A}$ lie in a standard cube $\mathfrak{Q}$. If $\text{Osc} f$ is not $< \epsilon$ in $\mathfrak{A}$, let us divide $\mathfrak{Q}$ into 2^n cubes. If in one of these cubes

$$\text{Osc} f < \epsilon, \tag{1}$$

let us call it a *black cube*. A cube in which 1) does not hold we will call *white*. Each white cube we now divide in 2^n cubes. These we call black or white according as 1) holds for them or does not. In this way we continue until we reach a stage where all cubes are black, or if not we continue indefinitely. In the latter case, we get an infinite enumerable set of cubes

$$\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3 \cdots \tag{2}$$

Each point a of $\mathfrak{A}$ lies in at least one cube 2). For since f is continuous at $x = a$,

$$|f(x) - f(a)| < \epsilon/2 \quad , \quad x \text{ in } V_\delta(a).$$

Thus when the process of division has been carried so far that the diagonals of the corresponding cubes are $< \delta$, the inequality 1) holds for a cube containing a. This cube is a black cube.

Thus, in either case, each point of $\mathfrak{A}$ lies in a black cube.

Now the cubes 2) effect a separated division D of $\mathfrak{A}$, and in each of its cells 1) holds. Hence f is L-integrable in $\mathfrak{A}$.

Let us now suppose $\mathfrak{N} > 0$. We set

$$\mathfrak{A} = \mathfrak{C} + \mathfrak{N}.$$

Then f is L-integrable in $\mathfrak{C}$ by case 1°. It is L-integrable in $\mathfrak{N}$ by 380, 2. Then it is L-integrable in $\mathfrak{A}$ by 390, 1.

2. If f is L-integrable in $\mathfrak{A}$, we cannot say that the points of discontinuity of f form a null set.

Example. Let $f = 1$ at the irrational points $\mathfrak{J}$, in $\mathfrak{A} = (0, 1)$;

$= 0$ at the other points $\mathfrak{R}$, in $\mathfrak{A}$.

Then each point of $\mathfrak{A}$ is a point of discontinuity. But here

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{R}} + \int_{\mathfrak{J}} = \int_{\mathfrak{J}} = 1,$$

since $\mathfrak{R}$ is a null set. Thus f is L-integrable.

400. *If $f(x_1 \cdots x_m)$ has limited variation in $\mathfrak{A}$, it is L-integrable.*

For let D be a cubical division of space of norm d. Then by I, 709, there exists a fixed number V, such that

$$\Sigma \omega_\iota d^{m-1} \leq V$$

for any D. Let ω, σ be any pair of positive numbers. We take d such that

$$d < \frac{\sigma\omega}{V}. \tag{1}$$

Let d_ι' denote those cells in which $\operatorname{Osc} f \geq \omega$, and let the number of these cells be ν. Let η_ι denote the points of $\mathfrak{A}$ in d_ι'. Then

$$\nu\omega d^{m-1} \leq \Sigma \omega_\iota d^{m-1} \leq V.$$

Hence
$$\nu \leq \frac{V}{\omega d^{m-1}}. \tag{2}$$

Thus
$$\Sigma \bar{\bar{\eta}}_\iota \leq \nu d^m \leq \frac{V d^m}{\omega d^{m-1}} \quad , \quad \text{by 2)},$$
$$\leq \frac{Vd}{\omega} < \sigma \quad , \quad \text{by 1)}.$$

Hence f is L-integrable by 388.

401. *Let*
$$\phi = f, \text{ in } \mathfrak{A} < \mathfrak{B};$$
$$= 0, \text{ in } A = \mathfrak{B} - \mathfrak{A}.$$

Then
$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \phi, \tag{1}$$

if 1°, ϕ *is L-integrable in* $\mathfrak{B}$; *or* 2°, f *is L-integrable in* $\mathfrak{A}$, *and* $\mathfrak{A}$, A *are separated parts of* $\mathfrak{B}$.

On the 1° *hypothesis* let $\{\mathfrak{E}_s\}$ be an extremal sequence of ϕ. Let the cells of $\mathfrak{E}_s$ be $e_1, e_2 \cdots$ They effect a separated division of $\mathfrak{A}$ into cells $d_1, d_2 \cdots$ Let m_ι, M_ι be the extremes of f in d_ι and n_ι, N_ι the extremes of ϕ in e_ι. Then for those cells containing at least a point of $\mathfrak{A}$,
$$n_\iota \bar{\bar{e}}_\iota \leq m_\iota \bar{\bar{d}}_\iota \leq M_\iota \bar{\bar{d}}_\iota \leq N_\iota \bar{\bar{e}}_\iota, \tag{2}$$
is obviously true when $e_\iota = d_\iota$. Let $d_\iota < e_\iota$. If $m_\iota \leq 0$,
$$n_\iota \bar{\bar{e}}_\iota \leq m_\iota \bar{\bar{d}}_\iota, \qquad \text{since } m_\iota = n_\iota. \tag{3}$$

If $m_\iota > 0$, $\qquad n_\iota = 0$, and 3) holds.

If $M_\iota \leq 0$, $\qquad M_\iota \bar{\bar{d}}_\iota \leq N_\iota \bar{\bar{e}}_\iota, \qquad \text{since } N_\iota = 0.$ (4

If $M_\iota > 0$, 4) still holds, since $M_\iota = N_\iota$.

Thus 2) holds in all these cases. Summing 2) gives
$$\sum_{\mathfrak{B}} n_\iota \bar{\bar{e}}_\iota \leq \underline{\int}_{\mathfrak{A}}^{-} f \leq \sum_{\mathfrak{B}} N_\iota e_\iota$$
for the division $\mathfrak{E}_s$, since in a cell e of $\mathfrak{E}$, containing no point of $\mathfrak{A}$, $\phi = 0$. Letting $s \doteq \infty$, we get 1), since the end members
$$\doteq \int_{\mathfrak{B}} \phi.$$

On the 2° *hypothesis*,

$$\int_{\mathfrak{B}} \phi = \int_{\mathfrak{A}} \phi + \int_{A} \phi = \int_{\mathfrak{A}} \phi = \int_{\mathfrak{A}} f,$$

since ϕ being $= 0$ in A, is L-integrable, and we can apply 390.

402. 1. If

$$\int_{\mathfrak{A}} f = 0,$$

we call f a *null function in* $\mathfrak{A}$.

2. *If* $f \geq 0$ *is a null function in* $\mathfrak{A}$, *the points* $\mathfrak{P}$ *where* $f > 0$ *form a null set.*

For let $\mathfrak{A} = \mathfrak{Z} + \mathfrak{P}$, so that $f = 0$ in $\mathfrak{Z}$.

By 401,

$$0 = \int_{\mathfrak{A}} f = \int_{\mathfrak{P}} f. \qquad (1$$

Let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. Let $\mathfrak{P}_n$ denote the points of $\mathfrak{P}$ where $f \geq \epsilon_n$. Then

$$\int_{\mathfrak{P}} \geq \int_{\mathfrak{P}_n} = 0, \qquad \text{by 1).}$$

Each $\mathfrak{P}_n$ is a null set. For

$$\int_{\mathfrak{P}_n} \geq \epsilon_n \overline{\overline{\mathfrak{P}}}_n = 0.$$

Hence $\overline{\overline{\mathfrak{P}}}_n = 0$.

Then $$\mathfrak{P} = \{\mathfrak{P}_n\} = Q_1 + Q_2 + \cdots$$

where $$Q_1 = \mathfrak{P}_1, \qquad Q_2 = \mathfrak{P}_2 - \mathfrak{P}_1, \qquad Q_3 = \mathfrak{P}_3 - Q_3 \cdots$$

As each Q_n is a null set, $\mathfrak{P}$ is a null set.

Integrand Sets

403. Let $\mathfrak{A}$ be a limited point set lying in an m-way space $\mathfrak{R}_m$. Let $f(x_1 \cdots x_m)$ be a limited function defined over $\mathfrak{A}$. Any point of $\mathfrak{A}$ may be represented by

$$a = (a_1 \cdots a_m).$$

The point $$x = (a_1 \cdots a_m x_{m+1})$$

lies in an $m+1$ way space $\mathfrak{R}_{m+1}$. The set of points $\{x\}$ in which x_{m+1} ranges from $-\infty$ to $+\infty$ is called an *ordinate through a*. If x_{m+1} is restricted by
$$0 \le x_{m+1} \le l,$$
we shall call the ordinate a *positive ordinate of length l*; if it is restricted by
$$-l \le x_{m+1} \le 0,$$
it is a *negative* ordinate. The set of ordinates through all the points a of $\mathfrak{A}$, each having a length $= f(a)$, and taken positively or negatively, as $f(a)$ is $\gtrless 0$, form a point set $\mathfrak{J}$ in $\mathfrak{R}_{m+1}$ which we call an *integrand set*. The points of $\mathfrak{J}$ for which x_{m+1} has a fixed value $x_{m+1} = c$ form a *section of* $\mathfrak{J}$, and is denoted by $\mathfrak{J}(c)$ or by $\mathfrak{J}_c$.

404. *Let $\mathfrak{A} = \{a\}$ be a limited point set in $\mathfrak{R}_m$. Through each point a, let us erect a positive ordinate of constant length l, getting a set $\mathfrak{O}$, in $\mathfrak{R}_{m+1}$. Then*
$$\overline{\overline{\mathfrak{O}}} = l\overline{\overline{\mathfrak{A}}}. \qquad (1$$

For let $\mathfrak{E}_1 > \mathfrak{E}_2 > \cdots$ form a standard sequence of enclosures of $\mathfrak{O}$, such that
$$\widehat{\mathfrak{E}}_n \doteq \overline{\overline{\mathfrak{O}}}. \qquad (2$$

Let us project each section of $\mathfrak{E}_n$ corresponding to a given value of x_{m+1} on $\mathfrak{R}_m$, and let $\mathfrak{A}_n$ be their divisor. Then $\mathfrak{A}_n \ge \mathfrak{A}$. Thus
$$\overline{\overline{\mathfrak{O}}} \le \overline{\overline{\mathfrak{A}}} l \le \overline{\overline{\mathfrak{A}}}_n l \le \widehat{\mathfrak{E}}_n.$$

Letting $n \doteq \infty$, and using 2), we get
$$\overline{\overline{\mathfrak{O}}} = \overline{\overline{\mathfrak{A}}} \cdot l.$$

To prove the rest of 1), let O be the complement of $\mathfrak{O}$ with respect to some standard cube $\mathfrak{Q}$ in $\mathfrak{R}_{m+1}$, of base Q in $\mathfrak{R}_m$.

Then, as just shown,
$$\overline{\overline{O}} = l\overline{\overline{A}} \quad , \quad \text{where } A = Q - \mathfrak{A}.$$

Hence
$$\underline{\underline{\mathfrak{O}}} = \widehat{\mathfrak{Q}} - \overline{\overline{O}} = \widehat{Q}l - l\overline{\overline{A}}$$
$$= l\{\widehat{Q} - (\widehat{Q} - \underline{\underline{\mathfrak{A}}})\}$$
$$= l\underline{\underline{\mathfrak{A}}}.$$

405. *Let $f \geq 0$ be L-integrable in $\mathfrak{A}$. Then*

$$\int_{\mathfrak{A}} f = \overline{\overline{\mathfrak{J}}}, \tag{1}$$

where $\mathfrak{J}$ is the integrand set corresponding to f.

For let $\{\delta_\iota\}$ be a separated division D of $\mathfrak{A}$. On each cell δ_ι erect a cylinder $\mathfrak{C}_\iota$ of height $M_\iota = \text{Max}\, f$ in δ_ι. Then by 404,

$$\overline{\overline{\mathfrak{C}}}_\iota = \overline{\overline{\delta}}_\iota M_\iota.$$

Let $\mathfrak{C} = \{\mathfrak{C}_\iota\}$; the $\mathfrak{C}_\iota$ are separated. Hence, $\epsilon > 0$ being small at pleasure,

$$\overline{\overline{\mathfrak{J}}} \leq \overline{\overline{\mathfrak{C}}} = \Sigma \overline{\overline{\mathfrak{C}}}_\iota = \Sigma \overline{\overline{\delta}}_\iota M_\iota < \overline{\int_{\mathfrak{A}}} f + \epsilon,$$

for a properly chosen D. Thus

$$\overline{\overline{\mathfrak{J}}} \leq \overline{\int_{\mathfrak{A}}} f. \tag{2}$$

Similarly we find

$$\underline{\int_{\mathfrak{A}}} f \leq \overline{\overline{\mathfrak{J}}}. \tag{3}$$

From 2), 3) follows 1).

406. *Let $f \geq 0$ be L-integrable over the measurable field $\mathfrak{A}$. Then the corresponding integrand set $\mathfrak{J}$ is measurable, and*

$$\widehat{\widehat{\mathfrak{J}}} = \int_{\mathfrak{A}} f. \tag{1}$$

For by 2) in 405,

$$\overline{\overline{\mathfrak{J}}} \leq \overline{\int_{\mathfrak{A}}} f.$$

Using the notation of 405, let $\mathfrak{c}_n$ be a cylinder erected on δ_n of height $m_n = \text{Min}\, f$ in δ_n. Let $\mathfrak{c} = \{\mathfrak{c}_n\}$. Then $\mathfrak{c} \leq \mathfrak{J}$, and hence

$$\underline{\underline{\mathfrak{c}}} \leq \underline{\underline{\mathfrak{J}}}. \tag{2}$$

But $\mathfrak{A}$ being measurable, each $\mathfrak{c}_n$ is measurable, by 404. Hence $\mathfrak{c}$ is by 359. Thus 2) gives

$$\widehat{\widehat{\mathfrak{c}}} \leq \underline{\underline{\mathfrak{J}}}. \tag{3}$$

Now for a properly chosen D,

$$-\epsilon + \underline{\int_{\mathfrak{A}}} f \leq \Sigma m_\iota \widehat{\widehat{\delta}}_\iota = \widehat{\widehat{\mathfrak{c}}}.$$

Hence

$$\underline{\int_{\mathfrak{A}}} \leq \bar{\mathfrak{c}}, \tag{4}$$

as ϵ is arbitrarily small. From 2), 3), 4)

$$\underline{\int_{\mathfrak{A}}} f \leq \underline{\underline{\mathfrak{J}}} \leq \bar{\bar{\mathfrak{J}}} \leq \overline{\int_{\mathfrak{A}}} f,$$

from which follows 1).

Measurable Functions

407. Let $f(x_1 \cdots x_m)$ be limited in the limited measurable set $\mathfrak{A}$. Let $\mathfrak{A}_{\lambda\mu}$ denote the points of $\mathfrak{A}$ at which

$$\lambda \leq f < \mu.$$

If each $\mathfrak{A}_{\lambda\mu}$ is measurable, we say *f is measurable in $\mathfrak{A}$*.

We should bear in mind that when f is measurable in $\mathfrak{A}$, necessarily $\mathfrak{A}$ itself is measurable, by hypothesis.

408. 1. *If f is measurable in $\mathfrak{A}$, the points $\mathfrak{C}$ of $\mathfrak{A}$, at which $f = C$, form a measurable set.*

For let $\mathfrak{A}_n$ denote the points where

$$-\epsilon_n + C \leq f < C + \epsilon_n,$$

where

$$\epsilon_1 > \epsilon_2 > \cdots \doteq 0.$$

Then by hypothesis, $\mathfrak{A}_n$ is measurable. But $\mathfrak{C} = Dv\{\mathfrak{A}_n\}$. Hence $\mathfrak{C}$ is measurable by 361.

2. *If f is measurable in $\mathfrak{A}$, the set of points where*

$$\lambda \leq f \leq \mu$$

is measurable, and conversely.

Follows from 1, and 407.

3. *If the points $\mathfrak{A}_\lambda$ in $\mathfrak{A}$ where $f \geq \lambda$ form a measurable set for each λ, f is measurable in $\mathfrak{A}$.*

For $\mathfrak{A}_{\lambda\mu}$ having the same meaning as in 407,

$$\mathfrak{A}_{\lambda\mu} = \mathfrak{A}_\lambda - \mathfrak{A}_\mu.$$

Each set on the right being measurable, so is $\mathfrak{A}_{\lambda\mu}$.

409. 1. *If f is measurable in $\mathfrak{A}$, it is L-integrable.*

For setting $m = \text{Min } f$, $M = \text{Max } f$ in $\mathfrak{A}$, let us effect a division D of the interval $\mathfrak{J} = (m, M)$ of norm d, by interpolating a finite number of points

$$m_1 < m_2 < m_3 < \cdots$$

Let us call the resulting segments, as well as their lengths,

$$d_1, d_2, d_3 \cdots$$

Let $\mathfrak{A}_\iota$ denote the points of $\mathfrak{A}$ in which

$$m_{\iota-1} \leq f < m_\iota \quad , \quad \iota = 1, 2, \cdots ; \ m_0 = m.$$

We now form the sums

$$\underline{s}_D = \Sigma m_{\iota-1} \widehat{\widehat{\mathfrak{A}}}_\iota \quad , \quad \overline{s}_D = \Sigma m_\iota \widehat{\widehat{\mathfrak{A}}}_\iota .$$

Obviously

$$\underline{s}_D \leq \overline{\underline{\int}}_{\mathfrak{A}} f \leq \overline{s}_D . \tag{1}$$

But

$$\begin{aligned} \overline{s}_D - \underline{s}_D &= m_1 \widehat{\widehat{\mathfrak{A}}}_1 + m_2 \widehat{\widehat{\mathfrak{A}}}_2 + \cdots - \{ m \widehat{\widehat{\mathfrak{A}}}_1 + m_1 \widehat{\widehat{\mathfrak{A}}}_2 + \cdots \} \\ &= \widehat{\widehat{\mathfrak{A}}}_1 (m_1 - m) + \widehat{\widehat{\mathfrak{A}}}_2 (m_2 - m_1) + \cdots \\ &\leq d \{ \widehat{\widehat{\mathfrak{A}}}_1 + \widehat{\widehat{\mathfrak{A}}}_2 + \cdots \} \\ &\leq d \widehat{\widehat{\mathfrak{A}}} \\ &\doteq 0 \quad , \quad \text{as } d \doteq 0. \end{aligned} \tag{2}$$

We may now apply 387.

2. *If f is measurable in $\mathfrak{A}$*

$$\underline{\int}_{\mathfrak{A}} f = \lim \Sigma m_{\iota-1} \widehat{\widehat{\mathfrak{A}}}_\iota = \lim \Sigma m_\iota \widehat{\widehat{\mathfrak{A}}}_\iota , \tag{3}$$

using the notation in 1.

This follows from 1), 2) in 1.

3. The relation 3) is taken by Lebesgue as definition of his integrals. His theory is restricted to measurable fields and to measurable functions. For Lebesgue's own development of his theory the reader is referred to his paper, *Intégrale, Longueur, Aire, Annali di Mat.*, Ser. 3, vol. 7 (1902); and to his book, *Leçons sur l'Intégration.* Paris, 1904. He may also consult the excellent account of it in *Hobson's* book, *The Theory of Functions of a Real Variable.* Cambridge, England, 1907.

Semi-Divisors and Quasi-Divisors

410. 1. The convergence of infinite series leads to the two following classes of point sets.

Let
$$F = \Sigma f_\iota(x_1 \cdots x_m) = \sum_1^n f_\iota + \sum_{n+1}^\infty f_\iota = F_n + \overline{F}_n, \tag{1}$$
each f_ι being defined in $\mathfrak{A}$.

Let us take $\epsilon > 0$ small at pleasure, and then fix it.

Let us denote by $\mathfrak{A}_n$ the points of $\mathfrak{A}$ at which
$$-\epsilon \leq \overline{F}_n(x) \leq \epsilon. \tag{2}$$
Of course $\mathfrak{A}_n$ may not exist. We are thus led in general to the sets
$$\mathfrak{A}_1 \quad , \quad \mathfrak{A}_2 \quad , \quad \mathfrak{A}_3 \cdots \tag{3}$$

The complementary set $A_n = \mathfrak{A} - \mathfrak{A}_n$ will denote the points where
$$|\, \overline{F}_n(x) \,| > \epsilon. \tag{4}$$

If now F is convergent at x, there exists a ν such that this point lies in
$$\mathfrak{A}_\nu \quad , \quad \mathfrak{A}_{\nu+1} \quad , \quad \mathfrak{A}_{\nu+2} \cdots \tag{5}$$

The totality of the points of convergence forms a set which has this property: corresponding to each of its points x, there exists a ν such that x lies in the set 5). A set having this property is called the *semi-divisor* of the sets 3), and is denoted by
$$\text{Sdv } \{\mathfrak{A}_n\}.$$

Suppose now, on the other hand, that 1) does not converge at the point x in $\mathfrak{A}$. Then there exists an infinite set of indices
$$n_1 < n_2 < \cdots \doteq \infty,$$
such that
$$|\, \overline{F}_{n_s}(x) \,| > \epsilon.$$

Thus, the point x lies in an infinity of the sets
$$A_1 \quad , \quad A_2 \quad , \quad A_3 \cdots \tag{6}$$

The totality of points such that each lies in an infinity of the sets 6) is called the *quasi-divisor* of 6) and is denoted by
$$\text{Qdv } \{A_n\}.$$

Obviously,
$$\text{Sdv } \{\mathfrak{A}_n\} + \text{Qdv } \{A_n\} = \mathfrak{A}. \tag{7}$$

We may generalize these remarks at once. Since $F(x)$ is nothing but

$$\lim F_n(x),$$

we can apply these notions to the case that the functions $f_\iota(x_1 \cdots x_m)$ are defined in $\mathfrak{A}$, and that

$$\lim f_\iota = \phi.$$

2. We may go still farther and proceed in the following abstract manner.

The divisor $\mathfrak{D}$ of the point sets

$$\mathfrak{A}_1 \quad , \quad \mathfrak{A}_2 \cdots \tag{1}$$

is the set of points lying in all the sets 1).

The totality of points each of which lies in an infinity of the sets 1) *is called the quasi-divisor* and is denoted by

$$\mathrm{Qdv}\,\{\mathfrak{A}_n\}. \tag{2}$$

The totality of points a, to each of which correspond an index m_a, such that a lies in

$$\mathfrak{A}_{m_a} \quad , \quad \mathfrak{A}_{m_a+1}, \cdots$$

forms a set called the semi-divisor of 1), and is denoted by

$$\mathrm{Sdv}\,\{\mathfrak{A}_n\}. \tag{3}$$

If we denote 2), 3) by $\mathfrak{Q}$ and $\mathfrak{S}$ respectively, we have, obviously,

$$\mathfrak{D} \leq \mathfrak{S} \leq \mathfrak{Q}. \tag{4}$$

3. In the special case that $\mathfrak{A}_1 > \mathfrak{A}_2 > \cdots$ we have

$$\mathfrak{Q} = \mathfrak{S} = \mathfrak{D}. \tag{5}$$

For denoting the complementary sets by the corresponding Roman letters, we have

$$D = A_1 + Dv(\mathfrak{A}_1, A_2) + Dv(\mathfrak{A}_2, A_3) + \cdots$$

But Q has precisely the same expression.

Thus $\mathfrak{Q} = \mathfrak{D}$, and hence by 4), $\mathfrak{S} = \mathfrak{D}$.

4. *Let* $\mathfrak{A}_n + A_n = \mathfrak{B}$, $n = 1, 2, \cdots$ *Then*

$$\mathrm{Qdv}\,\{\mathfrak{A}_n\} + \mathrm{Sdv}\,\{A_n\} = \mathfrak{B}.$$

For each point b of $\mathfrak{B}$ lies

either 1° only in a finite number of $\mathfrak{A}_n$, or in none at all,

or 2° in an infinite number of $\mathfrak{A}_n$.

In the 1° case, b does not lie in $\mathfrak{A}_s, \mathfrak{A}_{s+1} \cdots$; hence it lies in $A_s, A_{s+1} \cdots$ In the 2° case b lies obviously in $\mathrm{Qdv}\,\{\mathfrak{A}_n\}$.

5. *If* $\mathfrak{A}_1, \mathfrak{A}_2 \cdots$ *are measurable, and their union is limited,*

$$\mathfrak{Q} = \mathrm{Qdv}\,\{\mathfrak{A}_n\} \quad , \quad \mathfrak{S} = \mathrm{Sdv}\,\{\mathfrak{A}_n\}$$

are measurable.

For let $\mathfrak{D}_n = Dv\,(\mathfrak{A}_n, \mathfrak{A}_{n+1} \cdots)$. Then $\mathfrak{S} = \{\mathfrak{D}_n\}$.

But $\mathfrak{S}$ is measurable, as each $\mathfrak{D}_n$ is. Thus $\mathrm{Sdv}\,\{A_n\}$ is measurable, and hence $\mathfrak{Q}$ is by 4.

6. *Let* $\mathfrak{Q} = \mathrm{Qdv}\,\{\mathfrak{A}_n\}$, *each* $\mathfrak{A}_n$ *being measurable, and their union limited. If there are an infinity of the* $\mathfrak{A}_n$, *say*

$$\mathfrak{A}_{\iota_1}, \mathfrak{A}_{\iota_2} \cdots \quad ; \quad \iota_1 < \iota_2 < \cdots$$

whose measure is $\geq \alpha$, *then*

$$\widehat{\mathfrak{Q}} \geq \alpha. \tag{6}$$

For let $\mathfrak{B}_n = (\mathfrak{A}_{\iota_n}, \mathfrak{A}_{\iota_{n+1}} \cdots)$, then $\widehat{\mathfrak{B}}_n \geq \alpha$.

Let $\mathfrak{B} = Dv\{\mathfrak{B}_n\}$. As $\mathfrak{B}_n \geq \mathfrak{B}_{n+1}$,

$$\widehat{\mathfrak{B}} = \lim \widehat{\mathfrak{B}}_n \geq \alpha \tag{7}$$

by 362. As $\mathfrak{Q} \geq \mathfrak{B}$ we have 6) at once, from 7).

Limit Functions

411. *Let*

$$\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m),$$

as x ranges over $\mathfrak{A}$, *τ finite or infinite. Let f be measurable in* $\mathfrak{A}$ *and numerically* $\leq M$, *for each t near τ. Then ϕ is measurable in* $\mathfrak{A}$ *also.*

To prove this we show that the points $\mathfrak{B}$ of $\mathfrak{A}$ where

$$\lambda \leq \phi \leq \mu \tag{1}$$

form a measurable set for each λ, μ. For simplicity let τ be finite. Let $t_1, t_2 \cdots \doteq \tau$; also let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. Let $\mathfrak{C}_{n,s}$ denote the points of $\mathfrak{A}$ where

$$\lambda - \epsilon_n \leq f(x, t_s) \leq \mu + \epsilon_n. \tag{2}$$

Then for each point x of $\mathfrak{B}$, there is an s_0 such that 2) holds for any t_s, if $s \geq s_0$. Let $\mathfrak{C}_n = \underset{s}{\mathrm{Sdv}} \{\mathfrak{C}_{ns}\}$. Then $\mathfrak{B} \leq \mathfrak{C}_n$. But the $\mathfrak{C}_{ns}$ being measurable, $\mathfrak{C}_n$ is by 410, 5. Finally $\mathfrak{B} = Dv\{\mathfrak{C}_n\}$, and hence $\mathfrak{B}$ is measurable.

412. *Let* $$\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m),$$

for x in $\mathfrak{A}$, and τ finite or infinite. Let $t', t'' \cdots \doteq \tau$. Let each $f_s = f(x, t^{(s)})$ be measurable, and numerically $\leq M$. Let $\phi = f_s + g_s$. Let $\mathfrak{G}_s$ denote the points where

$$|g_s| > \epsilon.$$

Then for each $\epsilon > 0$,
$$\lim_{s=\infty} \widehat{\mathfrak{G}}_s = 0. \tag{1}$$

For by 411, ϕ is measurable, hence g_s is measurable in $\mathfrak{A}$, hence $\mathfrak{G}_s$ is measurable.

Suppose now that 1) does not hold. Then

$$\overline{\lim_{s=\infty}} \widehat{\mathfrak{G}}_s = l > 0.$$

Then there are an infinity of the $\mathfrak{G}_s$, as $\mathfrak{G}_{s_1}, \mathfrak{G}_{s_2} \cdots$ whose measures are $\geq \lambda > 0$. Then by 410, 6, the measure of

$$\mathfrak{G} = \mathrm{Qdv}\{\mathfrak{G}_s\}$$

is $\geq \lambda$. But this is not so, since $f_s \doteq \phi$, at each point of $\mathfrak{A}$.

413. 1. *Let* $$\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m),$$

for x in $\mathfrak{A}$, and τ finite or infinite.

Let
$$t', t'' \cdots \doteq \tau. \tag{1}$$

If each $f_s = f(x, t^{(s)})$ is measurable, and numerically $\leq M$ in $\mathfrak{A}$ for each sequence 1), then

$$\int_{\mathfrak{A}} \phi = \lim_{t=\tau} \int_{\mathfrak{A}} f(x, t). \tag{2}$$

For set

$$\phi = f_s + g_s,$$

and let

$$|g_s| \leq N \quad , \quad s = 1, 2 \cdots$$

Then as in 412, ϕ and g_s are measurable in $\mathfrak{A}$. Then by 409, they are L-integrable, and

$$\int_{\mathfrak{A}} \phi = \int_{\mathfrak{A}} f_s + \int_{\mathfrak{A}} g_s. \tag{3}$$

Let $\mathfrak{B}_s$ denote the points of $\mathfrak{A}$, at which

$$|g_s| \geq \epsilon;$$

and let $\mathfrak{B}_s + B_s = \mathfrak{A}$. Then $\mathfrak{B}_s$, B_s are measurable, since g_s is. Thus by 390,

$$\int_{\mathfrak{A}} g_s = \int_{\mathfrak{B}_s} g_s + \int_{B_s} g_s.$$

Hence

$$\left| \int_{\mathfrak{A}} g_s \right| \leq N\widehat{\mathfrak{B}}_s + \epsilon \widehat{B}_s \leq N\widehat{\mathfrak{B}}_s + \epsilon\widehat{\mathfrak{A}}.$$

By 412, $\widehat{\mathfrak{B}}_s \doteq 0$. Thus

$$\lim_{s=\infty} \int_{\mathfrak{A}} g_s = 0.$$

Hence passing to the limit in 3), we get 2), for the sequence 1). Since we can do this for every sequence of points t which $\doteq \tau$, the relation 2) holds.

2. *Let*

$$F = \Sigma f_{\iota_1 \cdots \iota_n}(x_1 \cdots x_m)$$

converge in $\mathfrak{A}$. *If each term* f_ι *is measurable, and each* $|F_\lambda| \leq M$, *then* F *is* L*-integrable, and*

$$\int_{\mathfrak{A}} F = \Sigma \int_{\mathfrak{A}} f_\iota.$$

Iterated Integrals

414. In Vol. I, 732, seq. we have seen that the relation,

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \overline{\underline{\int_{\mathfrak{C}}}} f,$$

holds when f is R-integrable in the *metric* field $\mathfrak{A}$. This result was extended to *iterable* fields in 14 of the present volume. We

wish now to generalize still further to the case that f is L-integrable in the *measurable* field $\mathfrak{A}$. The method employed is due to *Dr. W. A. Wilson*,* and is essentially simpler than that employed by Lebesgue.

1. Let $x = (z_1 \cdots z_s)$ denote a point in s-way space $\mathfrak{R}_s$, $s = m + n$. If we denote the first m coördinates by $x_1 \cdots x_m$, and the remaining coördinates by $y_1 \cdots y_n$, we have

$$z = (x_1 \cdots x_m \, y_1 \cdots y_n).$$

The points

$$x = (x_1 \cdots x_m \, 0 \, 0 \cdots 0)$$

range over an m-way space $\mathfrak{R}_m$, when z ranges over $\mathfrak{R}_s$. We call *x the projection of z on $\mathfrak{R}_m$*.

Let z range over a point set $\mathfrak{A}$ lying in $\mathfrak{R}_s$, then x will range over a set $\mathfrak{B}$ in $\mathfrak{R}_m$, called *the projection of $\mathfrak{A}$ on $\mathfrak{R}_m$*. The points of $\mathfrak{A}$ whose projection is x is called *the section of $\mathfrak{A}$ corresponding to x*. We may denote it by

$$\mathfrak{A}(x), \text{ or more shortly by } \mathfrak{C}.$$

We write

$$\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$$

to denote that $\mathfrak{A}$ is conceived of as formed of the sections $\mathfrak{C}$, corresponding to the different points of its projection $\mathfrak{B}$.

2. Let $\mathfrak{Q}$ denote a standard cube containing $\mathfrak{A}$, let $\mathfrak{q}$ denote its projection on $\mathfrak{R}_m$. Then $\mathfrak{B} \leq \mathfrak{q}$. Suppose each section $\mathfrak{A}(x)$ is measurable. It will be convenient to let $\widehat{\mathfrak{A}}(x)$ denote a function of x defined over $\mathfrak{q}$ such that

$$\begin{aligned} \widehat{\mathfrak{A}}(x) = \text{Meas } \mathfrak{A}(x) = \widehat{\mathfrak{C}} \qquad & \text{when } x \text{ lies in } \mathfrak{B}, \\ = 0 \qquad & \text{when } x \text{ lies in } \mathfrak{q} - \mathfrak{B}. \end{aligned}$$

This function therefore is equal to the measure of the section of $\mathfrak{A}$ corresponding to the point x, when such a section exists; and when not, the function $= 0$.

When each section $\mathfrak{A}(x)$ is not measurable, we can introduce the functions

$$\overline{\overline{\mathfrak{A}}}(x) \quad , \quad \underline{\underline{\mathfrak{A}}}(x).$$

* Dr. Wilson's results were obtained in August, 1909, and were presented by me in the course of an address which I had the honor to give at the Second Decennial Celebration of Clark University, September, 1909.

Here the first $= \overline{\overline{\mathfrak{C}}}$ when a section exists, otherwise it $= 0$, in $\mathfrak{q}$. A similar definition holds for the other function.

3. Let us note that *the sections*

$$\mathfrak{A}_e(x) \quad , \quad \mathfrak{A}_\iota(x),$$

where $\mathfrak{A}_e$, $\mathfrak{A}_\iota$ *are the outer and inner associated sets belonging to* $\mathfrak{A}$, *are always measurable.*

For $\mathfrak{A}_e = Dv\{\mathfrak{E}_n\}$, where each $\mathfrak{E}_n$ is a standard enclosure, each of whose cells $\mathfrak{e}_{nm}$ is rectangular. But the sections $\mathfrak{e}_{nm}(x)$ are also rectangular. Hence

$$\mathfrak{A}_e(x) = Dv\{\mathfrak{e}_{nm}(x)\},$$

being the divisor of measurable sets, is measurable.

415. *Let* $\mathfrak{A}_e$ *be an outer associated set of* $\mathfrak{A}$, *both lying in the standard cube* $\mathfrak{Q}$. *Then* $\widehat{\mathfrak{A}}_e(x)$ *is L-integrable in* $\mathfrak{q}$, *and*

$$\overline{\overline{\mathfrak{A}}} = \underline{\int_{\mathfrak{q}}} \widehat{\mathfrak{A}}_e(x). \tag{1}$$

For let $\{\mathfrak{E}_n\}$ be a sequence of standard enclosures of $\mathfrak{A}$, and $\mathfrak{E}_n = \{\mathfrak{e}_{nm}\}$. Then

$$\widehat{\mathfrak{E}}_n = \sum_m \widehat{\mathfrak{e}}_{nm} \tag{2}$$

and

$$\widehat{\mathfrak{E}}_n(x) = \sum_m \widehat{\mathfrak{e}}_{nm}(x). \tag{3}$$

Now $\mathfrak{e}_{nm}$ being a standard cell, $\widehat{\mathfrak{e}}_{nm}(x)$ has a constant value > 0 for all x contained in the projection of $\mathfrak{e}_{nm}$ on $\mathfrak{q}$. It is thus continuous in $\mathfrak{q}$ except for a discrete set. It thus has an R-integral, and

$$\widehat{\mathfrak{e}}_{nm} = \int_{\mathfrak{q}} \widehat{\mathfrak{e}}_{nm}(x).$$

This in 2) gives

$$\widehat{\mathfrak{E}}_n = \Sigma \int_{\mathfrak{q}} \widehat{\mathfrak{e}}_{nm}(x)$$

$$= \underline{\int_{\mathfrak{q}}} \Sigma \widehat{\mathfrak{e}}_{nm}(x), \qquad \text{by 413, 2,}$$

$$= \underline{\int_{\mathfrak{q}}} \widehat{\mathfrak{E}}_n(x), \tag{4}$$

by 3).

On the other hand, $\widehat{\widehat{\mathfrak{E}}}(x)$ is a measurable function by 411. Also

$$\overline{\overline{\mathfrak{A}}} = \widehat{\widehat{\mathfrak{A}}}_e = \lim \widehat{\widehat{\mathfrak{E}}}_n$$
$$= \lim \int_{\mathfrak{q}} \widehat{\widehat{\mathfrak{E}}}_n(x)$$
$$= \int_{\mathfrak{q}} \lim \widehat{\widehat{\mathfrak{E}}}_n(x), \qquad \text{by 413, 1.} \tag{5}$$

Now $$\widehat{\widehat{\mathfrak{A}}}_e(x) = \lim_{n=\infty} \widehat{\widehat{\mathfrak{E}}}_n(x).$$

Thus this in 5) gives 1).

416. *Let $\mathfrak{A}$ lie in the standard cube $\mathfrak{Q}$. Let $\mathfrak{A}_\iota$ be an inner associated set. Then $\widehat{\widehat{\mathfrak{A}}}_\iota(x)$ is L-integrable in $\mathfrak{q}$, and*

$$\underline{\underline{\mathfrak{A}}} = \int_{\mathfrak{q}} \widehat{\widehat{\mathfrak{A}}}_\iota(x).$$

For $$\mathfrak{Q} = \mathfrak{A}_\iota + A_e.$$

Thus $$\widehat{\widehat{\mathfrak{A}}}_\iota(x) = \widehat{\mathfrak{Q}}(x) - \widehat{A}_e(x).$$

Hence $\widehat{\widehat{\mathfrak{A}}}_\iota(x)$ is L-integrable in $\mathfrak{q}$, and

$$\int_{\mathfrak{q}} \widehat{\widehat{\mathfrak{A}}}_\iota(x) = \int_{\mathfrak{q}} \widehat{\mathfrak{Q}}(x) - \int_{\mathfrak{q}} \widehat{A}_e(x)$$
$$= \widehat{\mathfrak{Q}} - A_e \quad , \quad \text{by 415,}$$
$$= \widehat{\widehat{\mathfrak{A}}}_\iota = \underline{\underline{\mathfrak{A}}} \qquad \text{by 370, 2.}$$

417. *Let measurable $\mathfrak{A}$ lie in the standard cube $\mathfrak{Q}$. Then*

$$\widehat{\widehat{\mathfrak{A}}} = \int_{\mathfrak{q}} \overline{\overline{\mathfrak{A}}}(x). \tag{1}$$

For $$\mathfrak{A}_\iota(x) \le \mathfrak{A}(x) \le \mathfrak{A}_e(x).$$

Hence $$\underline{\underline{\mathfrak{A}}} = \int_{\mathfrak{q}} \widehat{\widehat{\mathfrak{A}}}_\iota(x) \le \underline{\overline{\int_{\mathfrak{q}}}} \overline{\overline{\mathfrak{A}}}(x) \le \overline{\int_{\mathfrak{q}}} \widehat{\widehat{\mathfrak{A}}}_e(x) = \overline{\overline{\mathfrak{A}}}, \tag{2}$$

using 396, 1, and 415, 416. From 2) we conclude 1) at once.

418. *Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ be measurable. Then $\underline{\underline{\overline{\overline{\mathfrak{C}}}}}$ are L-integrable in $\mathfrak{B}$, and*

$$\widehat{\widehat{\mathfrak{A}}} = \int_{\underline{\underline{\mathfrak{B}}}} \overline{\overline{\mathfrak{C}}}.$$

For by 417,
$$\widehat{\widehat{\mathfrak{A}}} = \int_{\underline{\underline{\mathfrak{Q}}}} \overline{\overline{\mathfrak{A}}}(x)$$
$$= \int_{\underline{\underline{\mathfrak{B}}}} \overline{\overline{\mathfrak{C}}},$$
by 401.

419. *If* $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ *is measurable, the points of* $\mathfrak{B}$ *at which* $\mathfrak{C}$ *is not measurable form a null set* $\mathfrak{N}$.

For by 418,
$$\widehat{\widehat{\mathfrak{A}}} = \int_{\mathfrak{B}} \overline{\overline{\mathfrak{C}}} = \int_{\underline{\underline{\mathfrak{B}}}} \mathfrak{C}.$$
Hence
$$0 = \int_{\mathfrak{B}} (\overline{\overline{\mathfrak{C}}} - \underline{\underline{\mathfrak{C}}}).$$
Thus
$$\phi = \overline{\overline{\mathfrak{C}}} - \underline{\underline{\mathfrak{C}}}$$
is a null function in $\mathfrak{B}$, and by 402, 2, points where $\phi > 0$ form a null set.

420. *Let* $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ *be measurable. Let* $\mathfrak{b}$ *denote the points of* $\mathfrak{B}$ *for which the corresponding sections* $\mathfrak{C}$ *are measurable. Then*
$$\widehat{\widehat{\mathfrak{A}}} = \int_{\mathfrak{b}} \widehat{\widehat{\mathfrak{C}}}.$$
For by 419,
$$\mathfrak{B} = \mathfrak{b} + \mathfrak{N},$$
and $\mathfrak{N}$ is a null set. Hence by 418,
$$\widehat{\widehat{\mathfrak{A}}} = \int_{\mathfrak{B}} \overline{\overline{\mathfrak{C}}} = \int_{\mathfrak{b}} \overline{\overline{\mathfrak{C}}} + \int_{\mathfrak{N}} \overline{\overline{\mathfrak{C}}}$$
$$= \int_{\mathfrak{b}} \widehat{\widehat{\mathfrak{C}}}.$$

421. *Let* $f \geq 0$ *in* $\mathfrak{A}$. *If the integrand set* $\mathfrak{J}$, *corresponding to* f *be measurable, then* f *is L-integrable in* $\mathfrak{A}$, *and*
$$\widehat{\widehat{\mathfrak{J}}} = \int_{\mathfrak{A}} f.$$

For the points of $\mathfrak{J}$ lying in an $m+1$ way space $\mathfrak{R}_{m+1}$ may be denoted by
$$x = (y_1 \cdots y_m, z),$$
where $y = (y_1 \cdots y_m)$ ranges over $\mathfrak{R}_m$, in which $\mathfrak{A}$ lies. Thus $\mathfrak{A}$ may be regarded as the projection of $\mathfrak{J}$ on $\mathfrak{R}_m$. To each point y

of $\mathfrak{A}$ corresponds a section $\mathfrak{J}(y)$, which for brevity may be denoted by $\mathfrak{K}$. Thus we may write

$$\mathfrak{J} = \mathfrak{A} \cdot \mathfrak{K}.$$

As $\mathfrak{K}$ is nothing but an ordinate through y of length $f(y)$, we have by 419,

$$\widehat{\mathfrak{J}} = \int_{\mathfrak{A}} \overline{\overline{\mathfrak{K}}} = \int_{\mathfrak{A}} f.$$

422. *Let f be L-integrable over the measurable field $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$. Let $\mathfrak{b}$ denote those points of $\mathfrak{B}$, for which f is L-integrable over the corresponding sections $\mathfrak{C}$. Then*

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{b}} \int_{\mathfrak{C}} f. \qquad (1$$

Moreover $\mathfrak{N} = \mathfrak{B} - \mathfrak{b}$ is a null set.

Let us 1° *suppose* $f \geq 0$. Then by 406, $\mathfrak{J}$ is measurable and

$$\widehat{\mathfrak{J}} = \int_{\mathfrak{A}} f. \qquad (2$$

Let β denote the points of $\mathfrak{B}$ for which $\mathfrak{J}(x)$ is measurable. Then by 420,

$$\widehat{\mathfrak{J}} = \int_{\beta} \widehat{\mathfrak{J}}(x). \qquad (3$$

By 419, the points

$$\mathfrak{P} = \mathfrak{B} - \beta \qquad (4$$

form a null set.

On the other hand, $\mathfrak{J}(x)$ is the integrand set of f, for $\mathfrak{A}(x) = \mathfrak{C}$. Hence by 421, for any x in β,

$$\widehat{\mathfrak{J}}(x) = \int_{\mathfrak{C}} f, \qquad (5$$

and

$$\beta \leq \mathfrak{b}. \qquad (6$$

From 2), 3), 5) we have

$$\int_{\mathfrak{A}} f = \int_{\beta} \int_{\mathfrak{C}} f. \qquad (7$$

From 6) we have

$$\mathfrak{N} = \mathfrak{B} - \mathfrak{b} \leq \mathfrak{B} - \beta = \mathfrak{P},$$

a null set by 4). Let us set

$$\mathfrak{b} = \beta + \mathfrak{n}.$$

Then $\mathfrak{n}$ lying in the null set $\mathfrak{P}$, is a null set. Hence

$$\int_{\beta}\int_{\mathfrak{C}} + \int_{\mathfrak{n}}\int_{\mathfrak{C}} = \int_{\mathfrak{b}}\int_{\mathfrak{C}}.$$

This with 7) gives 1).

Let f be now unrestricted as to sign. We take $C > 0$, such that the auxiliary function

$$g = f + C \geq 0, \qquad \text{in } \mathfrak{A}.$$

Then f, g are simultaneously L-integrable over any section $\mathfrak{C}$. Thus by case 1°

$$\int_{\mathfrak{A}}(f + C) = \int_{\mathfrak{b}}\int_{\mathfrak{C}}(f + C). \tag{8}$$

Now

$$\int_{\mathfrak{A}}(f + C) = \int_{\mathfrak{A}} f + \int_{\mathfrak{A}} C = \int_{\mathfrak{A}} f + C\widehat{\widehat{\mathfrak{A}}}, \tag{9}$$

$$\int_{\mathfrak{C}}(f + C) = \int_{\mathfrak{C}} f + C\overline{\overline{\mathfrak{C}}}. \tag{10}$$

By 418, $\overline{\overline{\mathfrak{C}}}$ is L-integrable in $\mathfrak{B}$, and hence in $\mathfrak{b}$. Thus

$$\int_{\mathfrak{b}}\int_{\mathfrak{C}}(f + C) = \int_{\mathfrak{b}}\int_{\mathfrak{C}} f + C\int_{\mathfrak{b}}\overline{\overline{\mathfrak{C}}}. \tag{11}$$

As $\mathfrak{b}$ differs from $\mathfrak{B}$ by a null set,

$$\underline{\int_{\mathfrak{b}}}\overline{\overline{\mathfrak{C}}} = \int_{\mathfrak{B}}\overline{\overline{\mathfrak{C}}} = \widehat{\widehat{\mathfrak{A}}}, \tag{12}$$

by 418. From 8), 9), 10), 11), 12) we have 1).

423. *If f is L-integrable over the measurable set $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$, then*

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}}\underline{\int_{\mathfrak{C}}}^{-} f. \tag{1}$$

For by 422,

$$\int_{\mathfrak{A}} = \int_{\mathfrak{b}}\int_{\mathfrak{C}}. \tag{2}$$

As $\mathfrak{B} - \mathfrak{b} = \mathfrak{N}$ is a null set,

$$\int_{\mathfrak{N}}\underline{\int_{\mathfrak{C}}}^{-} f = 0$$

may be added to the right side of 2) without altering its value. Thus

$$\int_{\mathfrak{A}} = \int_{\mathfrak{b}} \underline{\int_{\mathfrak{C}}} \overline{} + \int_{\mathfrak{N}} \underline{\int_{\mathfrak{C}}} = \int_{\mathfrak{B}} \underline{\int_{\mathfrak{C}}}.$$

424. 1. (*W. A. Wilson.*) *If $f(x_1 \cdots x_m)$ is L-integrable in measurable $\mathfrak{A}$, f is measurable in $\mathfrak{A}$.*

Let us *first suppose* that $f \geq 0$. We begin by showing that the set of points $\mathfrak{A}_\lambda$ of $\mathfrak{A}$ at which $f \geq \lambda$, is measurable. Then by 408, 3, f is measurable in $\mathfrak{A}$.

Now f being L-integrable in $\mathfrak{A}$, its integrand set $\mathfrak{J}$ is measurable by 406. Let $\mathfrak{J}_\lambda$ be the section of $\mathfrak{J}$ corresponding to $x_{m+1} = \lambda$. Then the projection of $\mathfrak{J}_\lambda$ on $\mathfrak{R}_m$ is $\mathfrak{A}_\lambda$. Since $\mathfrak{J}$ is measurable, the sections $\mathfrak{J}_\lambda$ are measurable, except at most over a null set L of values of λ, by 419. Thus there exists a sequence

$$\lambda_1 < \lambda_2 < \cdots \doteq \lambda$$

none of whose terms lies in L. Hence each $\mathfrak{J}_{\lambda_n}$ is measurable, and hence $\mathfrak{A}_{\lambda_n}$ is also.

As $\mathfrak{A}_{\lambda_{n+1}} \leq \mathfrak{A}_{\lambda_n}$, each point of $\mathfrak{A}_\lambda$ lies in

$$\mathfrak{D} = Dv\{\mathfrak{A}_{\lambda_n}\}, \tag{1}$$

so that

$$\mathfrak{A}_\lambda \leq \mathfrak{D}. \tag{2}$$

On the other hand, each point d of $\mathfrak{D}$ lies in $\mathfrak{A}_\lambda$. For if not, $f(d) < \lambda$.

There thus exists an s such that

$$f(d) < \lambda_s < \lambda. \tag{3}$$

But then d does not lie in $\mathfrak{A}_{\lambda_s}$, for otherwise $f(d) \geq \lambda_s$, which contradicts 3). But not lying in $\mathfrak{A}_{\lambda_s}$, d cannot lie in $\mathfrak{D}$, and this contradicts our hypothesis. Thus

$$\mathfrak{D} \leq \mathfrak{A}_\lambda. \tag{4}$$

From 2), 4) we have

$$\mathfrak{D} = \mathfrak{A}_\lambda.$$

But then from 1), $\mathfrak{A}_\lambda$ is measurable.

Let the sign of f be now unrestricted.

Since f is limited, we may choose the constant C, such that

$$g = f(x) + C \geq 0, \text{ in } \mathfrak{A}.$$

Then g is L-integrable, and hence, by case 1°, g is measurable. Hence f, differing only by a constant from g, is also measurable.

2. *Let $\mathfrak{A}$ be measurable. If f is L-integrable in $\mathfrak{A}$, it is measurable in $\mathfrak{A}$, and conversely.*

This follows from 1 and 409, 1.

3. From 2 and 409, 3, we have at once the theorem:

When the field of integration is measurable, an L-integrable function is integrable in Lebesgue's sense, and conversely; moreover, both have the same value.

Remark. In the theory which has been developed in the foregoing pages, the reader will note that neither the field of integration nor the integrand needs to be measurable. This is not so in Lebesgue's theory. In removing this restriction, we have been able to develop a theory entirely analogous to Riemann's theory of integration, and to extend this to a theory of upper and lower integration. We have thus a perfect counterpart of the theory developed in Chapter XIII of vol. I.

4. *Let $\mathfrak{A}$ be metric or complete. If $f(x_1 \cdots x_m)$ is limited and R-integrable, it is a measurable function in $\mathfrak{A}$.*

For by 381, 2, it is L-integrable. Also since $\mathfrak{A}$ is metric or complete, $\mathfrak{A}$ is measurable. We now apply 1.

IMPROPER L-INTEGRALS

Upper and Lower Integrals

425. 1. We propose now to consider the case that the integrand $f(x_1 \cdots x_m)$ is not limited in the limited field of integration $\mathfrak{A}$. In chapter II we have treated this case for R-integrals. To extend the definitions and theorems there given to L-integrals, we have in general only to replace metric or complete sets by measurable sets; discrete sets by null sets; unmixed sets by separated sets;

finite divisions by separated divisions; sequences of superposed cubical divisions by extremal sequences; etc.

As in 28 we may define an improper L-integral in any of the three ways there given, making such changes as just indicated. In the following we shall employ only the 3° Type of definition. To be explicit we define as follows:

Let $f(x_1 \cdots x_m)$ be defined for each point of the limited set $\mathfrak{A}$. Let $\mathfrak{A}_{\alpha\beta}$ denote the points of $\mathfrak{A}$ at which

$$-\alpha \leq f(x_1 \cdots x_m) \leq \beta \qquad \alpha, \beta \geq 0. \tag{1}$$

The limits

$$\lim_{\alpha,\,\beta=\infty} \underline{\int}_{\mathfrak{A}_{\alpha\beta}} f \quad , \quad \lim_{\alpha,\,\beta=\infty} \overline{\int}_{\mathfrak{A}_{\alpha\beta}} f \tag{2}$$

in case they exist, we call the *lower* and *upper* (improper) *L-integrals*, and denote them by

$$\underline{\int}_{\mathfrak{A}} f \quad , \quad \overline{\int}_{\mathfrak{A}} f.$$

In case the two limits 2) exist and are equal, we denote their common value by

$$\int_{\mathfrak{A}} f$$

and say f is (improperly) *L-integrable in* $\mathfrak{A}$, etc.

2. In order to use the demonstrations of Chapter II without too much trouble, we introduce the term *separated function.* A function f is such a function when the fields $\mathfrak{A}_{\alpha\beta}$ defined by 1) are separated parts of $\mathfrak{A}$.

We have defined measurable functions in 407 in the case that f is limited in $\mathfrak{A}$. We may extend it to unlimited functions by requiring that the fields $\mathfrak{A}_{\alpha\beta}$ are measurable however large α, β are taken.

This being so, we see that measurable functions are special cases of separated functions.

In case the field $\mathfrak{A}$ of integration is measurable, $\mathfrak{A}_{\alpha\beta}$ is a measurable part of $\mathfrak{A}$, if it is a separated part. From this follows the important result:

If f is a separated function in the measurable field $\mathfrak{A}$, it is L-integrable in each $\mathfrak{A}_{\alpha\beta}$.

From this follows also the theorem:

Let f be a separated function in the measurable field $\mathfrak{A}$. If either the lower or upper integral of f over $\mathfrak{A}$ is convergent, f is L-integrable in $\mathfrak{A}$, and

$$\int_{\mathfrak{A}} f = \lim_{\alpha,\beta=\infty} \int_{\mathfrak{A}_{\alpha\beta}} f.$$

426. To illustrate how the theorems on improper R-integrals give rise to analogous theorems on improper L-integrals, which may be demonstrated along the same lines as used in Chapter II, let us consider the analogue of 38, 2, viz.:

If f is a separated function such that $\overline{\int}_{\mathfrak{A}} f$ converges, so do $\overline{\int}_{\mathfrak{P}} f$.

Let $\{E_n\}$ be an extremal sequence common to both

$$\overline{\int}_{\mathfrak{A}_{\alpha\beta'}} , \quad \overline{\int}_{\mathfrak{A}_{\alpha\beta}} \qquad \beta' > \beta .$$

Let e denote the cells of E_n containing a point of $\mathfrak{P}_\beta$; e' those cells containing a point of $\mathfrak{P}_{\beta'}$; δ those cells containing a point of $\mathfrak{A}_{\alpha\beta}$ but none of $\mathfrak{P}_{\beta'}$. Then

$$\overline{\int}_{\mathfrak{A}_{\alpha\beta'}} = \lim_{n=\infty} \{\Sigma M_e' \cdot e + \Sigma M_{e'}' \cdot e' + \Sigma M_\delta' \cdot \delta\}.$$

In this manner we may continue using the proof of 38, and so establish our theorem.

427. As another illustration let us prove the theorem analogous to 46, viz.:

Let $\mathfrak{A}_1, \mathfrak{A}_2, \cdots \mathfrak{A}_n$ form a separated division of $\mathfrak{A}$. If f is a separated function in $\mathfrak{A}$, then

$$\underline{\overline{\int}}_{\mathfrak{A}} f = \underline{\overline{\int}}_{\mathfrak{A}_1} f + \cdots + \underline{\overline{\int}}_{\mathfrak{A}_n} f,$$

provided the integral on the left exists, or all the integrals on the right exist.

For let $\mathfrak{A}_{s,\alpha\beta}$ denote the points of $\mathfrak{A}_{\alpha\beta}$ in $\mathfrak{A}_s$. Then by 390, 1,

$$\underline{\overline{\int}}_{\mathfrak{A}_{\alpha\beta}} = \underline{\overline{\int}}_{\mathfrak{A}_{1,\alpha\beta}} + \cdots + \underline{\overline{\int}}_{\mathfrak{A}_{n,\alpha\beta}} .$$

In this way we continue with the reasoning of 46.

428. In this way we can proceed with the other theorems; in each case the requisite modification is quite obvious, by a consideration of the demonstration of the corresponding theorem in R-integrals given in Chapter II.

This is also true when we come to treat of *iterated* integrals along the lines of 70–78. We have seen, in 425, 2, that if $\mathfrak{A}$ is measurable, *upper* and *lower* integrals of separated functions do not exist as such; they reduce to L-integrals. We may still have a theory analogous to iterated R-integrals, by extending the notion of iterable fields, using the notion of upper measure. To this end we define:

A limited point set at $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ is *submeasurable* with respect to $\mathfrak{B}$, when

$$\overline{\overline{\mathfrak{A}}} = \underline{\overline{\int}}_{\mathfrak{B}} \overline{\overline{\mathfrak{C}}}.$$

We do not care to urge this point at present, but prefer to pass on at once to the much more interesting case of L-integrals over measurable fields.

L-Integrals

429. These we may define for our purpose as follows:

Let $f(x_1 \cdots x_m)$ be defined over the limited measurable set $\mathfrak{A}$. As usual let $\mathfrak{A}_{\alpha\beta}$ denote the points of $\mathfrak{A}$ at which

$$-\alpha \leq f \leq \beta, \qquad \alpha, \beta \geq 0.$$

Let each $\mathfrak{A}_{\alpha\beta}$ be measurable, and let f have a proper L-integral in each $\mathfrak{A}_{\alpha\beta}$. Then the improper integral of f over $\mathfrak{A}$ is

$$\int_{\mathfrak{A}} f = \lim_{\alpha, \beta = \infty} \int_{\mathfrak{A}_{\alpha\beta}} f, \tag{1}$$

when this limit exists. We shall also say that the integral on the left of 1) is *convergent.*

On this hypothesis, the reader will note at once that the demonstrations of Chapter II admit ready adaptation; in fact some of the theorems require no demonstration, as they follow easily from results already obtained.

430. Let us group together for reference the following theorems, analogous to those on improper R-integrals.

1. *If f is (improperly) L-integrable in $\mathfrak{A}$, it is in any measurable part of $\mathfrak{A}$.*

2. *If g, h denote as usual the non-negative functions associated with f, then*

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}} g - \int_{\mathfrak{A}} h. \tag{1}$$

3. *If $\int_{\mathfrak{A}} f$ is convergent, so is $\int_{\mathfrak{A}} |f|$, and conversely.*

4. *When convergent,*

$$\left|\int_{\mathfrak{A}} f\right| \leq \int_{\mathfrak{A}} f. \tag{2}$$

5. *If $\int_{\mathfrak{A}} f$ is convergent, then*

$$\epsilon > 0, \qquad \sigma > 0, \qquad \left|\int_{\mathfrak{B}} f\right| \leq \epsilon,$$

for any measurable $\mathfrak{B} < \mathfrak{A}$, such that $\widehat{\mathfrak{B}} < \sigma$.

6. *Let $\mathfrak{A} = (\mathfrak{A}_1, \mathfrak{A}_2 \cdots \mathfrak{A}_n)$ be a separated division of $\mathfrak{A}$, each $\mathfrak{A}_\iota$ being measurable. Then*

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}_1} f + \cdots + \int_{\mathfrak{A}_n} f, \tag{3}$$

provided the integral on the left exists, or all the integrals on the right exist.

7. *Let $\mathfrak{A} = \{\mathfrak{A}_n\}$ be a separated division of $\mathfrak{A}$, into an enumerable infinite set of measurable sets $\mathfrak{A}_n$. Then*

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{A}_1} f + \int_{\mathfrak{A}_2} f + \cdots \tag{4}$$

provided the integral on the left exists.

8. *If $f \leq g$ in $\mathfrak{A}$, except possibly at a null set, then*

$$\int_{\mathfrak{A}} f \leq \int_{\mathfrak{A}} g, \tag{5}$$

when convergent.

431. 1. To show how simple the proofs run in the present case, let us consider, in the first place, the theorem analogous to 38, 2, viz.:

If $\int_{\mathfrak{A}} f$ *converges, so do* $\int_{\mathfrak{P}} f$ *and* $\int_{\mathfrak{N}} f$.

The rather difficult proof of 38, 2 can be replaced by the following simpler one. Since

$$\mathfrak{A}_{\alpha\beta} = \mathfrak{P}_\beta + \mathfrak{N}_\alpha \tag{1}$$

is a *separated* division of $\mathfrak{A}_{\alpha\beta}$, we have

$$\int_{\mathfrak{A}_{\alpha\beta}} = \int_{\mathfrak{P}_\beta} + \int_{\mathfrak{N}_\alpha},$$

$$\int_{\mathfrak{A}_{\alpha\beta}} = \int_{\mathfrak{P}_\beta} + \int_{\mathfrak{N}_\alpha}.$$

Hence
$$\left| \int_{\mathfrak{A}_{\alpha\beta}} - \int_{\mathfrak{A}_{\alpha\beta'}} \right| = \left| \int_{\mathfrak{P}_\beta} - \int_{\mathfrak{P}_{\beta'}} \right|.$$

But the left side is $< \epsilon$, for a sufficiently large α, and $\beta, \beta' >$ some β_0. This shows that $\int_{\mathfrak{P}}$ is convergent. Similarly we show the other integral converges.

2. This form of proof could not be used in 38, 2, since 1) in general is not an unmixed division of $\mathfrak{A}_{\alpha\beta}$.

3. In a similar manner we may establish the theorem analogous to 39, viz.:

If $\int_{\mathfrak{P}} f$ *and* $\int_{\mathfrak{N}} f$ *converge, so does* $\int_{\mathfrak{A}} f$.

4. Let us look at the demonstration of the theorem analogous to 43, 1, viz.:

$$\int_{\mathfrak{A}} g = \int_{\mathfrak{P}} f \quad ; \quad \int_{\mathfrak{A}} h = -\int_{\mathfrak{N}} f,$$

provided the integral on either side of these equations converges.

Let us prove the first relation. Let $\mathfrak{B}_\beta$ denote the points of $\mathfrak{A}$ at which $f \leq \beta$. Then

$$\mathfrak{B}_\beta = \mathfrak{N} + \mathfrak{P}_\beta$$

is a separated division of $\mathfrak{B}_\beta$, and hence

$$\int_{\mathfrak{B}_\beta} g = \int_{\mathfrak{N}_\beta} g + \int_{\mathfrak{P}_\beta} g = \int_{\mathfrak{P}_\beta} g = \int_{\mathfrak{P}_\beta} f, \quad \text{etc.}$$

5. It is now obvious that the analogue of 44, 1 is the relation 1) in 430.

6. The analogue of 46 is the relation 3) in 430. Its demonstration is precisely similar to that in 46.

7. We now establish 430, 7. Let

$$\mathfrak{B}_m = (\mathfrak{A}_1, \mathfrak{A}_2 \cdots \mathfrak{A}_m).$$

Then
$$\mathfrak{A} = \mathfrak{B}_m + B_m$$

is a separated division of $\mathfrak{A}$, and we may take m so large that $\widehat{B}_m < \sigma$, an arbitrarily small positive number. Hence by 430, 5, we may take m so large that

$$\left| \int_{B_m} f \right| < \epsilon.$$

Thus
$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}_m} f + \int_{B_m} f$$
$$= \int_{\mathfrak{A}_1} f + \cdots + \int_{\mathfrak{A}_m} f + \epsilon' \quad , \quad |\epsilon'| < \epsilon.$$

From this our theorem follows at once.

Iterated Integrals

432. 1. Let us see how the reasoning of Chapter II may be extended to this case. We will of course suppose that the field of integration $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ is measurable. Then by 419, the points of $\mathfrak{B}$ for which the sections are not measurable form a null set. Since the integral of any function over a null set is zero, we may therefore in our reasoning suppose that every $\mathfrak{C}$ is measurable.

Since $\mathfrak{A}$ is measurable, there exists a sequence of complete components $A_m = B_m C_m$ in $\mathfrak{A}$, such that the measure of $A = \{A_m\}$ is $\widehat{\mathfrak{A}}$.

Since A_m is complete, its projection B_m is complete, by I, 717, 4. The points of B_m for which the corresponding sections C_m are not measurable form a null set ν_m. Hence the union $\{\nu_m\}$ is a null set. Thus we may suppose, without loss of generality in our demonstrations, that $\mathfrak{A}$ is such that every section in each A_m is measurable.

Now from

$$0 = \widehat{\widehat{\mathfrak{A}}} - \widehat{A} = \int_{\mathfrak{B}} \widehat{\widehat{\mathfrak{C}}} - \int_B \widehat{\widehat{C}} = \int_{\mathfrak{B}} (\widehat{\widehat{\mathfrak{C}}} - \widehat{C}),$$

we see that those points of $\mathfrak{B}$ where $\widehat{\widehat{\mathfrak{C}}} > \widehat{C}$ form a null set. We may therefore suppose that $\widehat{\widehat{\mathfrak{C}}} = C$ everywhere. Then $\mathfrak{C} - C$ is a null set at each point; we may thus adjoin them to C. Thus we may suppose that $\mathfrak{C} = C$ at each point of $\mathfrak{B}$, and that $\mathfrak{B} = B$ is the union of an enumerable set of complete sets B_m.

As we shall suppose that

$$\int_{\mathfrak{A}} f$$

is convergent, let

$$\alpha_1 < \alpha_2 < \cdots \doteq \infty,$$

$$\beta_1 < \beta_2 < \cdots \doteq \infty.$$

Let us look at the sets $\mathfrak{A}_{\alpha_n}$, $\mathfrak{B}_{\beta_n}$, which we shall denote by $\mathfrak{A}_n$. These are measurable by 429. Moreover, the reasoning of 72, 2 shows that without loss of generality we may suppose that $\mathfrak{A}$ is such that $\mathfrak{B}_n = \mathfrak{B}$. We may also suppose that each $\mathfrak{C}_n$ is measurable, as above.

2. Let us finally consider the integrals

$$\int_{\mathfrak{C}} f. \tag{1}$$

These may not exist at every point of $\mathfrak{B}$, because f does not admit a proper or an improper integral at this point. It will suffice for our purpose to suppose that 1) does not exist at a null set in $\mathfrak{B}$. Then without loss of generality we may suppose in our demonstrations that 1) converges at each point of $\mathfrak{B}$.

On these assumptions let us see how the theorems 73, 74, 75, and 76 are to be modified, in order that the proofs there given may be adapted to the present case.

433. 1 The first of these may be replaced by this:

Let $B_{\sigma, n}$ denote the points of $\mathfrak{B}$ at which $\hat{\mathfrak{c}}_n > \sigma$. Then

$$\lim_{n=\infty} \overline{\overline{B}}_{\sigma, n} = 0.$$

For by 419,

$$\hat{\mathfrak{A}} = \int_{\mathfrak{B}} \hat{\mathfrak{C}},$$

as by hypothesis the sections $\mathfrak{C}$ are measurable. Moreover, by hypothesis

$$\mathfrak{C} = \mathfrak{C}_n + \mathfrak{c}_n$$

is a separated division of $\mathfrak{C}$, each set on the right being measurable. Thus the proof in 73 applies at once.

2. The theorem of 74 becomes:

Let the integrals

$$\int_{\mathfrak{C}} f \quad , \quad f \geq 0$$

be limited in the complete set $\mathfrak{B}$. Let $\mathfrak{E}_n$ denote the points of $\mathfrak{B}$ at which

$$\int_{\mathfrak{c}_n} f \leq \epsilon.$$

Then

$$\lim_{n=\infty} \overline{\overline{\mathfrak{E}}}_n = \hat{\mathfrak{B}}.$$

The proof is analogous to that in 74. Instead of a cubical division of the space $\mathfrak{R}_p$, we use a standard enclosure. The sets $\mathfrak{B}_n$ are now measurable, and thus

$$\mathfrak{b} = Dv\{\mathfrak{B}_n\}$$

is measurable. Thus $\overline{\overline{\mathfrak{b}}}_n \doteq \hat{\mathfrak{b}}$. The rest of the proof is as in 74.

3. The theorem of 75 becomes:

Let the integral

$$\int_{\mathfrak{C}} f \quad , \quad f \geq 0$$

be limited in complete $\mathfrak{B}$. Then

$$\lim_{n=} \underline{\int_{\mathfrak{B}}} \int_{\mathfrak{c}_n} f = 0.$$

The proof is entirely similar to that in 75, except that we use extremal sequences, instead of cubical divisions.

4. As a corollary of 3 we have

Let the integral

$$\int_{\mathfrak{C}} f \quad , \quad f \geq 0$$

be limited and L-integrable in $\mathfrak{B}$. *Let* $\mathfrak{B} = \{B_m\}$ *the union of an enumerable set of complete sets. Then*

$$\lim_{n=\infty} \int_{\mathfrak{B}} \int_{\mathfrak{c}_n} f = 0.$$

For if $\mathfrak{B}_m = (B_1, B_2 \cdots B_m)$, and $\mathfrak{B} = \mathfrak{B}_m + \mathfrak{D}_m$, we have

$$\int_{\mathfrak{B}} \int_{\mathfrak{c}_n} = \int_{\mathfrak{B}_m} \int_{\mathfrak{c}_n} + \int_{\mathfrak{D}_m} \int_{\mathfrak{c}_n} \cdot$$

But for m sufficiently large, $\widehat{\mathfrak{D}}_m$ is small at pleasure. Hence

$$\int_{\mathfrak{D}_m} \int_{\mathfrak{c}_r.} \leq \int_{\mathfrak{D}_m} \int_{\mathfrak{C}} < \epsilon.$$

We have now only to apply 3.

434. 1. We are now in position to prove the analogue of 76, viz.:

Let $\mathfrak{A} = \mathfrak{B} \cdot \mathfrak{C}$ *be measurable. Let* $\int_{\mathfrak{A}} f$ *be convergent. Let the integrals* $\int_{\mathfrak{C}} f$ *converge in* $\mathfrak{B}$, *except possibly at a null set. Then*

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f, \tag{1}$$

provided the integral on the right is convergent.

We follow along the line of proof in 76, and begin by taking $f \geq 0$ in $\mathfrak{A}$. By 423, we have

$$\int_{\mathfrak{A}_n} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} f;$$

hence

$$\int_{\mathfrak{A}} f = \lim_{n=\infty} \int_{\mathfrak{B}} \int_{\mathfrak{C}_n} f. \tag{2}$$

Now $\epsilon > 0$ being small at pleasure,

$$-\epsilon + \int_{\mathfrak{B}}\int_{\mathfrak{C}} f < \int_{\mathfrak{B}_G}\int_{\mathfrak{C}} f \quad , \quad \text{for } G > \text{some } G_0,$$

$$\leq \int_{\mathfrak{B}_G}\left\{\int_{\mathfrak{C}_n} + \int_{c_n}\right\}$$

$$\leq \int_{\mathfrak{B}}\int_{\mathfrak{C}_n} + \int_{\mathfrak{B}}\int_{c_n}.$$

Since we have seen that we may regard $\mathfrak{B}$ as the union of an enumerable set of complete sets, we see that the last term on the right $\doteq 0$, as $n \doteq \infty$, by 433, 4. Thus

$$\int_{\mathfrak{B}}\int_{\mathfrak{C}} \leq \lim \int_{\mathfrak{B}}\int_{\mathfrak{C}_n} = \int_{\mathfrak{A}}, \tag{3}$$

by 2). On the other hand,

$$\int_{\mathfrak{B}}\int_{\mathfrak{C}_n} \leq \int_{\mathfrak{B}}\int_{\mathfrak{C}}.$$

Hence
$$\int_{\mathfrak{A}} = \lim \int_{\mathfrak{B}}\int_{\mathfrak{C}_n} \leq \int_{\mathfrak{B}}\int_{\mathfrak{C}}. \tag{4}$$

From 3) and 4) we have 1), when $f \geq 0$.

The general case is now obviously true. For

$$\mathfrak{A} = \mathfrak{P} + \mathfrak{N},$$

where $f \geq 0$ in $\mathfrak{P}$, and < 0 in $\mathfrak{N}$. Here $\mathfrak{P}$ and $\mathfrak{N}$ are measurable. We have therefore only to use 1) for each of these fields and add the results.

2. The theorem 1 states that if

$$\int_{\mathfrak{A}} f \quad , \quad \int_{\mathfrak{B}}\int_{\mathfrak{C}} f,$$

both converge, they are equal. *Hobson** in a remarkable paper on Lebesgue Integrals has shown that it is only necessary to assume the convergence of the first integral; the convergence of the second follows then as a necessary consequence.

* *Proceedings of the London Mathematical Society*, Ser. 2, vol. 8 (1909), p. 31.

435. We close this chapter by proving a theorem due to *Lebesgue*, which is of fundamental importance in the theory of Fourier's Series.

Let $f(x)$ be properly or improperly L-integrable in the interval $\mathfrak{A} = (a < b)$. Then

$$\lim_{\delta=0} J_f = \lim_{\delta=0} \int_a^\beta |f(x+\delta) - f(x)|\, dx$$

$$= \lim_{\delta=0} \int_a^\beta |\Delta f|\, dx = 0, \qquad a < \beta < \beta + \delta \leq b. \tag{1}$$

For in the first place,

$$J_f \leq \int_a^\beta |f(x+\delta)|\, dx + \int_a^\beta |f|\, dx \leq 2 \int_a^b |f|\, dx. \tag{2}$$

Next we note that

$$|f(x+\delta) - f(x)| - |g(x+\delta) - g(x)|$$
$$\leq |(f(x+\delta) - g(x+\delta) - (f(x) - g(x))|.$$

Hence

$$\int_a^\beta |\Delta f|\, dx - \int_a^\beta |\Delta g|\, dx \leq \int_a^\beta |\Delta(f-g)|\, dx,$$

or

$$J_f - J_g \leq J_{f-g}. \tag{3}$$

From 2), 3) we have

$$J_f < J_g + 2 \int_a^b |f - g|\, dx. \tag{4}$$

Let now

$$g = f \quad \text{for } |f| \leq G,$$
$$= 0 \quad \text{for } |f| > G.$$

Then by 4),

$$J_f \leq J_g + 2 \int_a^b |f - g|\, dx$$
$$\leq J_g + \epsilon',$$

where ϵ' is small at pleasure, for G sufficiently large. Thus the theorem is established, if we prove it for a limited function, $|g(x)| < G$.

Let us therefore effect a division of the interval $\Gamma = (-G, G)$, of norm d, by interpolating the points

$$-G < c_1 < c_2 < \cdots < G,$$

causing Γ to fall into the intervals

$$\gamma_1, \gamma_2, \gamma_3 \cdots$$

Let $h_m = c_m$ for those values of x for which $g(x)$ falls in the interval γ_m, and $= 0$ elsewhere in $\mathfrak{A}$. Then

$$J_g \leq \Sigma J_{h_\iota} + 2\int_a^\beta (g - \Sigma h_\iota)dx$$

$$\leq \Sigma J_{h_\iota} + 2\, d\widehat{\mathfrak{A}}$$

$$< \Sigma J_{h_\iota} + \epsilon', \qquad \epsilon' \text{ small at pleasure,}$$

for d sufficiently small.

Thus we have reduced the demonstration of our theorem to a function $h(x)$ which takes on but two values in $\mathfrak{A}$, say 0 and γ.

Let $\mathfrak{E}$ be a $\sigma/4$ enclosure of the points where $h = \gamma$, while $\mathfrak{F}$ may denote a finite number of intervals of $\mathfrak{E}$ such that $\widehat{\mathfrak{F}} - \widehat{\mathfrak{E}} < \sigma/4$.

Let $\phi = \gamma$ in $\mathfrak{E}$, and elsewhere $= 0$; let $\psi = \gamma$ in $\mathfrak{F}$, and elsewhere $= 0$. Thus using 4),

$$J_h \leq J_\phi + \int_a^\beta |h - \phi|$$

$$\leq J_\phi + \frac{\sigma}{2}\gamma,$$

since $h = \phi$ in (α, β), except at points of measure $< \sigma/4$. Similarly

$$J_\phi \leqq J_\psi + \frac{\sigma}{2}\gamma.$$

Thus
$$J_h \leq J_\psi + \sigma\gamma < J_\psi + \epsilon,$$

for σ sufficiently small.

Thus the demonstration is reduced to proving it for a ψ which is continuous, except at a finite number of points. But for such a function, it is obviously true.

CHAPTER XIII

FOURIER'S SERIES

Preliminary Remarks

436. 1. Let us suppose that the limited function $f(x)$ can be developed into a series of the type

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots \quad (1$$

which is valid in the interval $\mathfrak{A} = (-\pi, \pi)$. If it is also known that this series can be integrated termwise, the coefficients a_n, b_n can be found at once as follows. By hypothesis

$$\int_{-\pi}^{\pi} f dx = a_0 \int_{-\pi}^{\pi} dx + a_1 \int_{-\pi}^{\pi} \cos x dx + \cdots$$
$$+ b_1 \int_{-\pi}^{\pi} \sin x dx + \cdots$$

As the terms on the right all vanish except the first, we have

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f dx. \quad (2'$$

Let us now multiply 1) by $\cos nx$ and integrate.

$$\int_{-\pi}^{\pi} f(x) \cos nx dx = a_0 \int_{-\pi}^{\pi} \cos nx dx + a_1 \int_{-\pi}^{\pi} \cos x \cos nx dx + \cdots$$
$$+ b_1 \int_{-\pi}^{\pi} \sin x \cos nx + \cdots$$

Now
$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = 0 \quad , \quad m \neq n,$$
$$\int_{-\pi}^{\pi} \cos^2 nx dx = \pi,$$
$$\int_{-\pi}^{\pi} \sin mx \cos nx = 0.$$

Thus all the terms on the right of the last series vanish except the one containing a_n. Hence

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx. \tag{2''}$$

Finally multiplying 1) by $\sin nx$, integrating, and using the relations

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = 0 \quad , \quad m \neq n,$$

$$\int_{-\pi}^{\pi} \sin^2 nx dx = \pi,$$

we get

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx dx. \tag{2'''}$$

Thus under our present hypothesis,

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)du + \frac{1}{\pi}\sum_1^\infty \cos nx \int_{-\pi}^{\pi} f(u) \cos nu du$$
$$+ \frac{1}{\pi}\sum_1^\infty \sin nx \int_{-\pi}^{\pi} f(u) \sin nu du. \tag{3}$$

The series on the right is known as *Fourier's series;* the coefficients 2) are called *Fourier's coefficients* or *constants*. When the relation 3) holds for a set of points $\mathfrak{B}$, we say $f(x)$ can be developed in a Fourier's series in $\mathfrak{B}$, or Fourier's development is valid in $\mathfrak{B}$.

2. Fourier thought that every continuous function in $\mathfrak{A}$ could be developed into a trigonometric series of the type 3). The demonstration he gave is not rigorous. Later *Dirichlet* showed that such a development is possible, provided the continuous function has only a finite number of oscillations in $\mathfrak{A}$. The function still regarded as limited may also have a finite number of *discontinuities of the first kind*, *i.e.* where

$$f(a+0) \quad , \quad f(a-0) \tag{4}$$

exist, but one at least is $\neq f(a)$.

At such a point a, Fourier's series converges to

$$\tfrac{1}{2}\{f(a+0)+f(a-0)\}.$$

Jordan has extended Dirichlet's results to functions having limited variation in $\mathfrak{A}$. Thus Fourier's development is valid in certain cases when f has an infinite number of oscillations or points of discontinuity. Fourier's development is also valid in certain cases when f is not limited in $\mathfrak{A}$, as we shall see in the following sections.

We have supposed that $f(x)$ is given in the interval $\mathfrak{A} = (-\pi, \pi)$. This restriction was made only for convenience. For if $f(x)$ is given in the interval $\mathfrak{J} = (a < b)$, we have only to change the variable by means of the relation

$$u = \frac{\pi(2x - a - b)}{b - a}.$$

Then when x ranges over $\mathfrak{J}$, u will range over $\mathfrak{A}$.

Suppose f is an *even function* in $\mathfrak{A}$; its development in Fourier's series will contain only cosine terms. For

$$f(x) = \sum_0^\infty (a_n \cos nx + b_n \sin nx),$$

$$f(-x) = \sum_0^\infty (a_n \cos nx - b_n \sin nx).$$

Adding and remembering that $f(x) = f(-x)$ in $\mathfrak{A}$, we get

$$f(x) = \tfrac{1}{2}\sum_0^\infty a_n \cos nx, \qquad f \text{ even}.$$

Similarly if f is *odd*, its development in Fourier's series will contain only sine terms;

$$f(x) = \tfrac{1}{2}\sum_1^\infty b_n \sin nx, \qquad f \text{ odd}.$$

Let us note that if $f(x)$ is given only in $\mathfrak{B} = (0, \pi)$, and has limited variation in $\mathfrak{B}$, we may develop f either as a sine or a cosine series in $\mathfrak{B}$. For let

$$\begin{aligned} g(x) &= f(x) \quad , \quad x \text{ in } \mathfrak{B} \\ &= f(-x) \quad , \quad x \text{ in } (-\pi, 0). \end{aligned}$$

Then g is an even function in $\mathfrak{A}$ and has limited variation. Using Jordan's result, we see g can be developed in a cosine series valid in $\mathfrak{A}$. Hence f can be developed in a cosine series valid in $\mathfrak{B}$.

In a similar manner, let

$$h(x) = f(x) \quad , \quad x \text{ in } \mathfrak{B}$$
$$= -f(-x) \quad , \quad -\pi \leq x < 0.$$

Then h is an odd function in $\mathfrak{A}$, and Fourier's development contains only sine terms.

Unless $f(0) = 0$, the Fourier series will not converge to $f(0)$ but to 0, on account of the discontinuity at $x = 0$. The same is true for $x = \pi$.

If f can be developed in Fourier's series valid in $\mathfrak{A} = (-\pi, \pi)$, the series 3) will converge for all x, since its terms admit the period 2π. Thus 3) will represent $f(x)$ in $\mathfrak{A}$, but will not represent it unless f also admits the period 2π. The series 3) defines a periodic function admitting 2π as a period.

EXAMPLES

437. We give now some examples. They may be verified by the reader under the assumption made in 436. Their justification will be given later

Example 1. $\quad f(x) = x \quad , \quad \text{for } -\pi \leq x \leq \pi.$

Then

$$x = 2\left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right\}.$$

If we set $x = \frac{\pi}{2}$, we get *Leibnitz's formula*,

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Example 2. $\quad f(x) = x \quad , \quad 0 \leq x \leq \pi$

$$= -x \quad , \quad -\pi \leq x \leq 0.$$

Then

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right\}.$$

If we set $x = 0$, we get

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Example 3.

$$\begin{aligned} f(x) &= 1 \quad , \quad 0 < x < \pi \\ &= 0 \quad , \quad x = 0, \pm\pi \\ &= -1 \quad , \quad -\pi < x < 0. \end{aligned}$$

Then

$$f(x) = \frac{4}{\pi}\left\{\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right\}.$$

Example 4.

$$\begin{aligned} f(x) &= x \quad , \quad 0 \le x \le \frac{\pi}{2} \\ &= \pi - x \quad , \quad \frac{\pi}{2} \le x \le \pi. \end{aligned}$$

By defining f as an odd function, it can be developed in a sine series, *valid in* $(0, \pi)$. We find

$$f(x) = \frac{4}{\pi}\left\{\frac{\sin x}{1} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right\}.$$

Example 5.

$$\begin{aligned} f(x) &= 1 \quad , \quad 0 \le x \le \frac{\pi}{2} \\ &= -1 \quad , \quad \frac{\pi}{2} < x < \pi. \end{aligned}$$

By defining f as an even function, we get a development in cosines,

$$f(x) = \frac{4}{\pi}\left\{\frac{\cos x}{1} - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \cdots\right\},$$

valid in $(0, \pi)$.

Example 6.

$$f(x) = \tfrac{1}{2}(\pi - x) \quad , \quad 0 < x \le \pi.$$

By defining f as an odd function we get a development in sines,

$$f(x) = \sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \cdots$$

valid in $(-\pi, \pi)$.

Example 7.

$$\begin{aligned} \text{Let } f(x) &= \frac{\pi}{3} \quad , \quad 0 < x < \frac{\pi}{3} \\ &= 0 \quad , \quad \frac{\pi}{3} < x < \frac{2\pi}{3} \\ &= -\frac{\pi}{3} \quad , \quad \frac{2\pi}{3} < x < \pi. \end{aligned}$$

Developing f as a sine series, we get

$$f(x) = \sin 2x + \frac{\sin 4x}{2} + \frac{\sin 8x}{4} + \cdots$$

valid in $(0, \pi)$.

Example 8. $\qquad f(x) = e^x \quad , \quad \text{in } (-\pi, \pi).$

We find

$$f(x) = \frac{2 \sinh \pi}{\pi} \left\{ \frac{1}{2} - \frac{1}{1+1^2} \cos x + \frac{1}{1+2^2} \cos 2x - \frac{1}{1+3^2} \cos 3x + \cdots \right.$$
$$\left. + \frac{1}{1+1^2} \sin x - \frac{2}{1+2^2} \sin 2x + \frac{3}{1+3^2} \sin 3x - \cdots \right\}$$

valid for $-\pi < x < \pi$.

Example 9. We find

$$\cos \mu x = \frac{2\mu}{\pi} \sin \pi\mu \left\{ \frac{1}{2\mu^2} - \frac{\cos x}{\mu^2 - 1} + \frac{\cos 2x}{\mu^2 - 2^2} - \frac{\cos 3x}{\mu^2 - 3^2} + \cdots \right\}$$

valid for $\qquad -\pi \leq x \leq \pi \quad , \quad \mu^2 \neq 1, 2^2, 3^2, \cdots$

Let us set $x = \pi$, and replace μ by x; we get

$$\frac{\pi}{2x} \cot \pi x = \frac{1}{2x^2} + \frac{1}{x^2 - 1^2} + \frac{1}{x^2 - 2^2} + \frac{1}{x^2 - 3^2} + \cdots$$

a decomposition of $\cot \pi x$ into partial fractions, a result already found in 216.

Example 10. We find

$$\sin x = \frac{2}{\pi} \left\{ 1 - \frac{2 \cos 2x}{1 \cdot 3} - \frac{2 \cos 4x}{3 \cdot 5} - \frac{2 \cos 6x}{5 \cdot 7} \cdots \right\},$$

valid for $0 \leq x \leq \pi$.

Summation of Fourier's Series

438. In order to justify the development of $f(x)$ in Fourier's series F, we will actually sum the F series and show that it converges to $f(x)$ in certain cases. To this end let us suppose that $f(x)$ is given in the interval $\mathfrak{A} = (-\pi, \pi)$, and let us extend f by giving it the period 2π. Moreover, at the points of discontinuity of the first kind, let us suppose

$$f(x) = \tfrac{1}{2}\{f(x+0) + f(x-0)\}.$$

Then the function

$$\phi(u) = f(x + 2u) + f(x - 2v) - 2f(x)$$

is continuous at $u = 0$, and has the value 0, at points of continuity, and at points of discontinuity of 1° kind of f. Finally let us suppose that f is (properly or improperly) L-integrable in $\mathfrak{A}$; this last condition being necessary, in order to make the Fourier coefficients a_n, b_n have a sense.

Let

$$F = F(x) = \tfrac{1}{2} a_0 + a_1 \cos x + a_2 \cos 2x + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2x + \cdots \tag{1}$$
$$= \tfrac{1}{2} a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx),$$

where we will now write

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \cos nx dx, \tag{2'}$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \sin nx dx. \tag{2''}$$

Since $f(x)$ is periodic, the coefficients a_n, b_n have the same value however c is chosen. If we make $c = -\pi$, these integrals reduce to those given in 436.

We may write

$$F = \frac{1}{\pi}\int_c^{c+2\pi} f(t)dt \left\{\tfrac{1}{2} + \sum_1^\infty (\cos nx \cos nt + \sin nx \sin nt)\right\}$$

$$= \frac{1}{\pi}\int_c^{c+2\pi} \left\{\tfrac{1}{2} + \sum_1^\infty \cos n(t - x)\right\} . f(t)dt. \tag{3}$$

Thus

$$F_n = \frac{1}{\pi}\int_c^{c+2\pi} P_n \cdot f(t)dt,$$

where

$$P_n = \tfrac{1}{2} + \sum_1^n \cos m(t - x). \tag{4}$$

Provided

$$\sin \tfrac{1}{2}(t - x) \neq 0, \tag{5}$$

we may write

$$P_n = \frac{1}{2 \sin \frac{1}{2}(t - x)}\left\{\sin \tfrac{1}{2}(t - x) + \sum_1^m 2 \sin \tfrac{1}{2}(t - x) \cos m(t - x)\right.$$

$$= \frac{1}{2 \sin \frac{1}{2}(t - x)}\left[\sin \tfrac{1}{2}(t - x)\right.$$
$$\left. + \sum_1^n \left\{\sin \frac{2m + 1}{2}(t - x) - \sin \frac{2m - 1}{2}(t - x)\right\}\right].$$

Thus

$$P_n = \frac{\sin \frac{1}{2}(2n+1)(t-x)}{2 \sin \frac{1}{2}(t-x)}, \tag{6}$$

if 5) holds. Let us see what happens when 5) does not hold. In this case $\frac{1}{2}(t-x)$ is a multiple of π. As both t and x lie in $(c, c+2\pi)$, this is only possible for three singular values:

$$t = x \quad ; \quad t = c, \quad x = c + 2\pi \quad ; \quad t = c + 2\pi, \quad x = c.$$

For these singular values 4) gives

$$P_n = \frac{2n+1}{2}. \tag{7}$$

As P_n is a continuous function of t, x, the expression on the right of 6) must converge to the value 7) as x, t converge to these singular values. We will therefore assign to the expression on the right of 6) the value 7), for the above singular values. Then in all cases

$$F_n = \frac{1}{\pi}\int_c^{c+2\pi} \frac{\sin \frac{1}{2}(2n+1)(t-x)}{2 \sin \frac{1}{2}(t-x)} f(t)dt.$$

Let us set

$$2n+1 = \nu \quad , \quad t - x = u.$$

Then

$$F_n = \frac{1}{\pi}\int_{\frac{1}{2}(c-x)}^{\frac{1}{2}(c-x)+\pi} f(x+2u)\frac{\sin \nu u}{\sin u}du.$$

Let us choose c so that

$$c - x = -\pi,$$

then

$$\pi F_n = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \int_{-\frac{\pi}{2}}^{0} + \int_0^{\frac{\pi}{2}}.$$

Replacing u by $-u$ in the first integral on the right, it becomes

$$\int_0^{\frac{\pi}{2}} f(x-2u)\frac{\sin \nu u}{\sin u}du.$$

Thus we get

$$F_n = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \{f(x+2u) + f(x-2u)\} \frac{\sin \nu u}{\sin u}du. \tag{8}$$

Let us now introduce the term $-2f(x)$ under the sign of integration in order to replace the brace by $\phi(u)$. To this end let us

give x an arbitrary but fixed value and consider the Fourier's series for the function

$$g(t) = f(x), \text{ a constant.}$$

If we denote the Fourier series corresponding to the g function by

$$G = \tfrac{1}{2} g_0 + g_1 \cos t + g_2 \cos 2t + \cdots$$
$$+ h_1 \sin t + h_2 \sin 2t + \cdots$$

we have

$$g_0 = \frac{1}{\pi} \int_c^{c+2\pi} f(x) dt = 2 f(x),$$

$$g_n = \frac{f(x)}{\pi} \int_c^{c+2\pi} \cos ntdt = 0,$$

$$h_n = \frac{f(x)}{\pi} \int_c^{c+2\pi} \sin ntdt = 0.$$

Thus the sum of the first $n+1$ terms of the Fourier series belonging to $g(t)$ reduces to

$$G_n = f(x). \tag{9}$$

But this sum is also given by 8), if we replace

$$f(x + 2u) + f(x - 2u)$$

by

$$g(x + 2u) + g(x - 2u) = 2 f(x),$$

since g is a constant. We get thus

$$G_n = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} 2 f(t) \frac{\sin nu}{\sin u} du. \tag{10}$$

Let us therefore subtract $f(x)$ from both sides of 8), using 9), 10). We get

$$F_n(x) - f(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} \{ f(x + 2u) + f(x - 2u) - 2 f(x) \} \frac{\sin \nu u}{\sin u} du.$$

Setting

$$D_n(x) = \pi \{ F_n(x) - f(x) \}, \tag{11}$$

we have

$$D_n(x) = \int_0^{\frac{\pi}{2}} \phi(u) \frac{\sin \nu u}{\sin u} du. \tag{12}$$

We have thus the theorem:

For the Fourier Series to converge to $f(x)$ at the point x, it is necessary and sufficient that $D_n(x) \doteq 0$, as $n \doteq \infty$.

*Validity of Fourier's Development**

439. The integral on the right side of 438, 12), on which the validity of Fourier's development at the point x depends, is a special case of the integral

$$J_n = \int_{\mathfrak{B}} g(u) \sin nu du \quad , \quad \mathfrak{B} = (a < b). \tag{1}$$

In fact J_n goes over into D_n, if we set

$$g = \frac{\phi}{\sin u} \quad , \quad a = 0, \quad b = \frac{\pi}{2}.$$

To evaluate J_n let us break $\mathfrak{B}$ up into the intervals

$$\mathfrak{B}_0 = \left(a, a + \frac{\pi}{n}\right) \quad , \quad \mathfrak{B}_1 = \left(a + \frac{\pi}{n}, a + \frac{2\pi}{n}\right) \cdots \mathfrak{B}_r = \left(a + r\frac{\pi}{n}, b\right).$$

These intervals are equal except the last, which is shorter than the others unless $b - a$ is a multiple of π/n. We have thus

$$J_n = \int_{\mathfrak{B}_0} + \int_{\mathfrak{B}_1} + \cdots + \int_{\mathfrak{B}_r}.$$

If we set

$$v = u + \frac{\pi}{n},$$

we see that while v ranges over $\mathfrak{B}_{2s}$, u ranges over $\mathfrak{B}_{2s-1}$. This substitution enables us to replace the integrals over $\mathfrak{B}_{2s}$ by those over $\mathfrak{B}_{2s-1}$, since

$$\int_{\mathfrak{B}_{2s}} g(v) \sin nv dv = -\int_{\mathfrak{B}_{2s-1}} g\left(u + \frac{\pi}{n}\right) \sin nu du.$$

Hence grouping the integrals in pairs, we get

$$J_n = \int_{\mathfrak{B}_0} g(u) \sin nu du + \sum_s \int_{\mathfrak{B}_{2s-1}} \left\{g(u) - g\left(u + \frac{\pi}{n}\right)\right\} \sin nu du$$
$$+ \int_{\mathfrak{B}'} g(u) \sin nu du,$$

* The presentation given in 439–448 is due in the main to *Lebesgue.* Cf. his classic paper, *Mathematische Annalen*, vol. 61 (1905), p. 251. Also his *Leçons sur les Séries Trigonométriques*, Paris, 1906.

where $\mathfrak{B}'$ is $\mathfrak{B}_r$ or $\mathfrak{B}_{r-1} + \mathfrak{B}_r$, depending on the parity of r. Now

$$\left|\int_{\mathfrak{B}_0}\right| \leq \int_{\mathfrak{B}_0} |g|, \tag{2}$$

$$\left|\sum \int_{\mathfrak{B}_{2s-1}} \left\{g(u) - g\left(u + \frac{\pi}{n}\right)\right\} \sin nudu\right|$$

$$\leq \sum \int_{\mathfrak{B}_{2s-1}} \left|g\left(u + \frac{\pi}{n}\right) - g(u)\right| du$$

$$\leq \int_{a+\frac{\pi}{n}}^{b-\frac{\pi}{n}} \left|g\left(u + \frac{\pi}{n}\right) - g(u)\right| du. \tag{3}$$

$$\left|\int_{\mathfrak{B}'}\right| \leq \int_{\mathfrak{B}'} |g|. \tag{4}$$

Thus $J_n \doteq 0$, if the three integrals 2), 3), 4) $\doteq 0$. Moreover, if these three integrals are *uniformly evanescent* with respect to some point set $\mathfrak{C} \leq \mathfrak{B}$, J_n is also uniformly evanescent in $\mathfrak{C}$. In particular we note the theorem

$J_n \doteq 0$, if g is L-integrable in $\mathfrak{B}$.

We are now in a position to draw some important conclusions with respect to Fourier's series.

440. 1. *Let $f(x)$ be L-integrable in $(c, c + 2\pi)$. Then the Fourier constants a_n, $b_n \doteq 0$, as $n \doteq \infty$.*

For
$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \cos nxdx$$

is a special case of the J_n integral. As f is L-integrable, we need only apply the theorem at the close of the last article. Similar reasoning applies to b_n.

2. *For a given value of x in $\mathfrak{A} = (-\pi, \pi)$ let*

$$\psi(u) = \frac{\phi(u)}{\sin u}, \tag{1}$$

be L-integrable in $\mathfrak{B} = \left(0, \frac{\pi}{2}\right)$. Then Fourier's development is valid at the point x.

For by 438, Fourier's series $\doteq f(x)$ at the point x, if $D_n(x) \doteq 0$. But D_n is a special case of J_n for which the g function is integrable. We thus need only apply 439.

3. *For a given x in $\mathfrak{A} = (-\pi, \pi)$, let*

$$\chi(u) = \frac{\phi(u)}{u} \tag{2}$$

be L-integrable in $\mathfrak{B} = \left(0, \frac{\pi}{2}\right)$. Then Fourier's development is valid at the point x.

For let $\delta > 0$, then

$$\int_0^\delta |\psi| du = \int_0^\delta \left|\frac{\phi(u)}{\sin u}\right| \leq \int_0^\delta \frac{|\phi(u)|(1+\eta)}{u} du$$

$$\leq (1+\eta) \int_0^\delta |\chi(u)| du$$

$$\doteq 0 \quad , \quad \text{as } \delta \doteq 0 \quad , \quad \text{by hypothesis.}$$

4. *For a given x in $\mathfrak{A} = (-\pi, \pi)$, let*

$$\omega(u) = \frac{f(x+u) - f(x)}{u} \tag{3}$$

be L-integrable in $\mathfrak{A}$. Then Fourier's development is valid at the point x.

For

$$\chi(u) = \frac{f(x+2u) - f(x)}{u} + \frac{f(x-2u) - f(x)}{u}$$

$$= 2[\omega(2u) + \omega(-2u)].$$

Thus χ is L-integrable in $\left(0, \frac{\pi}{2}\right)$, as it is the difference of two integrable functions.

441. (*Lebesgue*). *For a given x in $\mathfrak{A} = (-\pi, \pi)$ let*

$$1^\circ \quad \lim_{n=\infty} n \int_0^{\frac{\pi}{n}} |\phi(u)| du = 0 ;$$

$$2^\circ \quad \lim_{\delta=0} \int_\delta^\eta |\psi(u+\delta) - \psi(u)| du = 0$$

for some η such that

$$0 < \delta < \eta \leq \frac{\pi}{2}.$$

Then Fourier's development is valid at the point x.

For as we have seen,

$$|D_n| \leq \int_0^{\frac{\pi}{\nu}} \left|\frac{\phi(u)}{\sin u}\sin \nu u\right| du + \int_{\frac{\pi}{\nu}}^{\frac{\pi}{2}} \left|\psi\left(u+\frac{\pi}{n}\right) - \psi(u)\right| du$$

$$+ \int_{\beta_n}^{\frac{\pi}{2}} \left|\frac{\phi(u)}{\sin u}\sin \nu u\right| du \leq D' + D'' + D''',$$

where β_n is a certain number which $\doteq \frac{\pi}{2}$, as $n \doteq \infty$.

Let us first consider D'. Since $0 \leq u \leq \frac{\pi}{\nu}$, we have $0 \leq \nu u \leq \pi$. Hence

$$\frac{\sin \nu u}{\sin u} = \frac{\nu u - \frac{\nu^3 u^3}{6} + \sigma \frac{\nu^4 u^4}{24}}{u - \frac{u^3}{6} + \tau \frac{u^4}{24}}, \qquad 0 < \sigma,\ \tau \leq 1$$

$$= \nu \frac{1 - \frac{\nu^2 u^2}{6}\left(1 - \frac{\sigma \nu u}{4}\right)}{1 - \frac{u^2}{6}\left(1 - \frac{\tau u}{4}\right)} = \nu \frac{1 - s\frac{u^2}{6}}{1 - t\frac{u^2}{6}}$$

$$\leq \nu, \text{ provided } s \geq t.$$

But this is indeed so. For

$$1 - \frac{\nu \sigma u}{4} \geq 1 - \frac{\pi}{4}.$$

Hence
$$s \geq \nu^2\left(1 - \frac{\pi}{4}\right) > 1 > t \quad , \quad \text{if } \nu \geq 5.$$

Thus
$$D' < \nu \int_0^{\frac{\pi}{\nu}} |\phi|\, du \doteq 0, \text{ by hypothesis.}$$

We now turn to D''. We have

$$D'' = \int_{\frac{\pi}{\nu}}^{\frac{\pi}{2}} = \int_{\frac{\pi}{\nu}}^{\eta} + \int_{\eta}^{\frac{\pi}{2}} \quad , \quad \frac{\pi}{\nu} < \eta < \frac{\pi}{2}.$$

Now f being L-integrable,

$$\left|\psi\left(u+\frac{\pi}{n}\right)-\psi(u)\right|$$

is L-integrable in $\left(\eta, \frac{\pi}{2}\right)$. Thus

$$\lim_{n=\infty}\int_\eta^{\frac{\pi}{2}} = 0.$$

But by condition 2°,
$$\lim_{\nu=\infty}\int_{\frac{\pi}{\nu}}^{\eta} = 0.$$

Thus
$$\lim_{\delta=0} D'' = 0.$$

Finally we consider D'''. But the integrand is an integrable function in $\left(\beta, \frac{\pi}{2}\right)$. Thus it $\doteq 0$ as $n \doteq \infty$.

442. 1. *The validity of Fourier's development at the point x depends only on the nature of f in a vicinity of x, of norm δ as small as we please.*

For the conditions of the theorem in 441 depend only on the value of f in such a vicinity.

2. Let us call a point x at which the function

$$\phi(u) = f(x+2u) + f(x-2u) - 2f(x)$$

is continuous at $u=0$, and has the value 0, a *regular point.*

In 438, we saw that if x is a point of discontinuity of the first kind for $f(x)$, then x is a regular point.

3. *Fourier's development is valid at a regular point x, provided for some η*

$$\lim_{\delta=0}\int_\delta^\eta |\psi(u+\delta)-\psi(u)|\,du = 0 \quad , \quad 0<\delta<\eta\leq\frac{\pi}{2}.$$

For at a regular point x, $\phi(u)$ is continuous at $u=0$, and $=0$ for $u=0$. Now

$$\lim_{h=0}\frac{1}{h}\int_0^h |\phi(u)|\,du = |\phi(0)| = 0.$$

Thus
$$n\int_0^{\frac{\pi}{n}} |\phi(u)|\,du = \pi\cdot\frac{1}{\frac{\pi}{n}}\cdot\int_0^{\frac{\pi}{n}}|\phi|\,du$$
$$\doteq \pi\,|\phi(0)| = 0.$$

Hence condition 1° of 441 is satisfied.

Limited Variation

443. 1. Before going farther we must introduce a few notions relative to the variation of a function $f(x)$ defined over an interval $\mathfrak{A}=(a<b)$. Let us effect a division D of $\mathfrak{A}$ into subintervals, by interpolating a finite number of points $a_1 < a_2 < \cdots$ The sum

$$V_D = \Sigma\,|f(a_\iota) - f(a_{\iota+1})| \qquad (1$$

is called *the variation of f in $\mathfrak{A}$ for the division D.* If

$$\text{Max } V_D \qquad (2$$

is finite with respect to the class of all finite divisions of $\mathfrak{A}$, we say *f has finite variation in $\mathfrak{A}$.* When 2) is finite, we denote its value by

$$\text{Var} f, \quad \text{or} \quad V_f, \quad \text{or} \quad V$$

and call it the *variation of f in $\mathfrak{A}$.*

We shall show in 5 that finite variation means the same thing as limited variation introduced in I, 509. We use the term finite variation in sections 1 to 4 only for clearness.

2. A most important property of functions having finite variation is brought out by the following geometric consideration.

Let us take two monotone increasing curves A, B such that one of them crosses the other a finite or infinite number of times. If $f(x)$, $g(x)$ are the continuous functions having these curves as graphs, it is obvious that

$$d(x) = f(x) - g(x)$$

is a continuous function which changes its sign, when the curves A, B cross each other. Thus we can construct functions in infinite variety, which oscillate infinitely often in a given interval, and which are the difference of two monotone increasing functions.

For simplicity we have taken the curves A, B continuous. A moment's reflection will show that this is not necessary.

Since $d(x)$ is the difference of two monotone increasing functions, its variation is obviously finite. Jordan has proved the following *fundamental theorem.*

3. *If $f(x)$ has finite variation in the interval $\mathfrak{A} = (a < b)$, there exists an infinity of limited monotone increasing functions $g(x)$, $h(x)$ such that*

$$f = g - h. \tag{1}$$

For let D be a finite division of $\mathfrak{A}$. Let

$$P_D = \text{sum of terms } \{f(a_{m+1}) - f(a_m)\} \text{ which are} \geq 0,$$
$$-N_D = \text{. . . . .} < 0.$$

Then

$$V_D = \Sigma\, |f(a_{m+1}) - f(a_m)| = P_D + N_D. \tag{2}$$

Also

$$\{f(a_1) - f(a)\} + \{f(a_2) - f(a_1)\} + \cdots + \{f(b) - f(a_n)\} = P_D - N_D.$$

On the left the sum is telescopic, hence

$$f(b) - f(a) = P_D - N_D. \tag{3}$$

From 2), 3) we have

$$V_D = 2\,P_D + f(a) - f(b) = 2\,N_D + f(b) - f(a). \tag{4}$$

Let now

$$\text{Max } P_D = P \quad , \quad \text{Max } N_D = N$$

with respect to the class of finite divisions D.

We call them the *positive* and *negative variation* of $f(x)$ in $\mathfrak{A}$. Then 4) shows that

$$V = 2\,P + f(a) - f(b) \quad , \quad V = 2\,N + f(b) - f(a). \tag{5}$$

Adding these, we get

$$V = P + N. \tag{6}$$

From 5) we have

$$f(b) - f(a) = P - N. \tag{7}$$

Instead of the interval $\mathfrak{A} = (a < b)$, let us take the interval $(a < x)$, where x lies in $\mathfrak{A}$. Replacing b by x in 7), we have

$$f(x) = f(a) + P(x) - N(x). \tag{8}$$

Obviously $P(x)$, $N(x)$ are monotone increasing functions. Let $\mu(x)$ be a monotone increasing function in $\mathfrak{A}$. If we set

$$\begin{aligned} g(x) &= f(a) + P(x) + \mu(x) \\ h(x) &= N(x) + \mu(x), \end{aligned} \tag{9}$$

we get 1) from 8) at once.

4. From 8) we have

$$\begin{aligned} |f(x)| &\leq |f(a)| + P(x) + N(x) \\ &\leq |f(a)| + V(x). \end{aligned} \tag{10}$$

5. We can now show that *when $f(x)$ has finite variation in the interval $\mathfrak{A} = (a < b)$ it has limited variation and conversely.*

For if *f has finite variation* in $\mathfrak{A}$ we can set

$$f(x) = \phi(x) - \psi(x),$$

where ϕ, ψ are monotone increasing in $\mathfrak{A}$. Then if $\mathfrak{A}$ is divided into the intervals $\delta_1, \delta_2 \cdots$ we have

$$\text{Osc} f \leq \text{Osc}\, \phi + \text{Osc}\, \psi \quad , \quad \text{in } \delta_\iota .$$

But

$$\text{Osc}\, \phi = \Delta\phi \quad , \quad \text{Osc}\, \psi = \Delta\psi \quad , \quad \text{in } \delta_\iota$$

since these functions are monotone. Hence summing over all the intervals δ_ι,

$$\begin{aligned} \Sigma\, \text{Osc} f &\leq \Sigma\Delta\phi + \Sigma\Delta\psi \\ &\leq \{\phi(b) - \phi(a)\} + \{\psi(b) - \psi(a)\} \\ &\leq \text{some } M, \text{ for any division.} \end{aligned}$$

Hence f has limited variation.

If *f has limited variation* in $\mathfrak{A}$,

$$|\Delta f| \leq \text{Osc} f \quad , \quad \text{in } \delta_\iota .$$

Hence

$$\Sigma\, |\Delta f| \leq \Sigma\, \text{Osc} f \leq \text{some } M.$$

Hence f has finite variation.

6. *If $f(x)$ has limited variation in the interval $\mathfrak{A}$, its points of continuity form a pantactic set in $\mathfrak{A}$.*

This follows from 5, and I, 508.

7. *Let $a < b < c$; then if f has finite variation in (a, c),*

$$V_{a,b}f + V_{b,c}f = V_{a,c}f, \tag{11}$$

where $V_{a,b}$ means the variation of f in the interval (a, b), etc.

For

$$V_{ac}f = \text{Max } V_D f$$

with respect to the class of all finite divisions D of (a, c). The divisions D fall into two classes:

1° those divisions E containing the point b,

2° the divisions F which do not.

Let Δ be a division obtained by interpolating one or more points in the interval. Obviously

$$V_\Delta f \geq V_D f.$$

Let now G be obtained from a division F by adding the point b. Then

$$V_G f \geq V_F f.$$

Hence

$$\underset{E}{\text{Max}}\ V_E \geq \underset{F}{\text{Max}}\ V_F.$$

Hence to find $V_{a,c}f$, we may consider only the class E. Let now E_1 be a division of (a, b), and E_2 a division of (b, c). Then $E_1 + E_2$ is a division of class E. Conversely each division of class E gives a division of (a, b), (b, c). Now

$$V_E f = V_{E_1} f + V_{E_2} f.$$

From this 11) follows at once.

444. We establish now a few simple relations concerning the variation of two functions in an interval $\mathfrak{A} = (a < b)$.

1.

$$V(f + c) = Vf. \tag{1}$$

For

$$\Sigma \,|\, (f_{\iota+1} + c) - (f_\iota + c) \,| = \Sigma \,| f_{\iota+1} - f_\iota |,$$

where for brevity we set

$$f_\iota = f(a_\iota).$$

2.

$$V(cf) = | c |\, Vf. \tag{2}$$

For

$$\Sigma \,|\, cf_{\iota+1} - cf_\iota \,| = | c | \,\Sigma \,| f_{\iota+1} - f_\iota |.$$

3. *Let f, g be monotone increasing functions in $\mathfrak{A}$. Then*

$$V(f+g)=Vf+Vg. \tag{3}$$

For
$$\Sigma\,|(f_{\iota+1}+g_{\iota+1})-(f_\iota+g_\iota)|=\Sigma\,|(f_{\iota+1}-f_\iota)+(g_{\iota+1}-g_\iota)|$$
$$=\Sigma\,|f_{\iota+1}-f_\iota|+\Sigma\,|g_{\iota+1}-g_\iota|.$$

4. *For any two functions f, g having limited variation,*

$$V(f+g)\leq Vf+Vg. \tag{4}$$

5. *Let f, f_1 have limited variation in $\mathfrak{A}=(a, b)$.*

Let
$$\alpha=|f(a)| \quad , \quad \alpha_1=|f_1(a)|.$$
Then
$$V(ff_1)\leq(\alpha+Vf)(\alpha_1+Vf_1). \tag{5}$$

For by 443, 8) we have

$$f=P-N+A \quad , \quad f_1=P_1-N_1+A_1,$$

where
$$A=f(a) \quad , \quad A_1=f_1(a).$$

Thus

$$ff_1=PP_1-PN_1+PA_1-NP_1+NN_1-NA_1+AP_1-AN_1+AA_1.$$

Hence by 2, 4,

$$Vff_1\leq VPP_1+VPN_1+VPA_1+\cdots$$
$$\leq V(PP_1+PN_1+\cdots) \quad , \quad \text{by } 3$$
$$\leq PP_1+PN_1+P\alpha_1+\cdots$$
$$\leq(P+N+\alpha)(P_1+N_1+\alpha_1).$$

But
$$Vf=P+N \quad , \quad \text{hence, etc.}$$

445. *Fourier's development is valid at the regular point x, if there exists a $0<\zeta\leq\frac{\pi}{2}$, such that in $(0, \zeta)$ the variation $V(u)$ of $\psi(u)$ in any (u, ζ) is limited, and such that $uV(u)\doteq 0$, $u\doteq 0$.*

By 442, we have only to show that

$$\Psi=\int_\delta^\eta|\psi(u+\delta)-\psi(u)|\,du \qquad 0<\delta<\eta\leq\frac{\pi}{2}$$

is evanescent with δ.

Let us *first* suppose that $\psi(u)$ is monotone in some $(0, \zeta)$, say monotone increasing. Similar reasoning will apply, if it is monotone decreasing. Then, taking $0 < \eta + \delta < \zeta$,

$$\Psi = \int_\delta^\eta \{\psi(u+\delta) - \psi(u)\}du = \int_\delta^\eta \psi(u+\delta)du - \int_\delta^\eta \psi(u)du.$$

In the second integral from the end, set $v = u + \delta$.

Then $$\int_\delta^\eta \psi(u+\delta)du = \int_{2\delta}^{\eta+\delta} \psi(v)dv.$$

Hence, $$\Psi = \int_{2\delta}^{\eta+\delta} \psi(u)du - \int_\delta^\eta \psi(u)du$$

$$= \int_{2\delta}^{\eta+\delta} - \int_\delta^{2\delta} - \int_{2\delta}^\eta .$$

Thus $$|\Psi(u)| \leq \int_\delta^{2\delta} |\psi| \, du + \int_\eta^{\eta+\delta} |\psi| \, du = \Psi_1 + \Psi_2 .$$

We will consider the integrals on the right separately. Let

$$\phi_m = \text{Max} \, |\phi|, \qquad \text{in } (\delta, 2\,\delta).$$

Then $$\Psi_1 \leq \int_\delta^{2\delta} \frac{|\phi|}{\sin u} du \leq \phi_m \int_\delta^{2\delta} \frac{du}{\sin u} .$$

Now $$\sin u = u - \sigma' u^2 \quad , \quad 0 < \sigma' < \tfrac{1}{2}.$$

Hence, $$\frac{1}{\sin u} = \frac{1}{u} + \sigma u \quad , \quad |\sigma| < \text{some } M.$$

Thus, $$\Psi_1 \leq \phi_m \left\{ \int_\delta^{2\delta} \frac{du}{u} + M \int_\delta^{2\delta} u du \right\}$$

$$< \phi_m \{ \log 2 + M' \delta^2 \}$$

$$\doteq 0 \quad , \quad \text{as } \delta \doteq 0 \quad , \quad \text{since } \phi(u) \doteq 0,$$

as x is a regular point.

We turn now to Ψ_2. In $(\eta, \eta + \delta)$, δ, η sufficiently small,

$$\sin u > u - \tfrac{1}{6} u^3 > \eta(1 - \eta^2).$$

Thus, if $\phi_\mu = \text{Max} \mid \phi' \mid$ in $(\eta, \eta + \delta)$,

$$\Psi_2 \leq \frac{\phi_\mu}{\eta(1-\eta)} \int_\eta^{\eta+\delta} du = \frac{\phi_\mu \delta}{\eta(1-\eta)} \doteq 0$$

with δ.

Thus, when ψ is monotone in some $(0, \zeta)$, Fourier's development is valid. But obviously when ψ is monotone, the condition that $uV(u) \doteq 0$ is satisfied. Our theorem is thus established in this case.

Let us now consider the case that the variation $V(u)$ of ψ is limited in (u, ζ).

From 443, 10), we have

$$| \psi(u) | \leq | \psi(\zeta) | + V(u).$$

As before we have

$$| \Psi | < \int_\delta^{2\delta} | \psi | du + \int_\eta^{\eta+\delta} | \psi | du = \Psi_1 + \Psi_2.$$

By hypothesis there exists for each $\epsilon > 0$, a $\delta_0 > 0$, such that

$$uV(u) \leq \epsilon \quad , \quad \text{for any } 0 < u \leq \delta_0.$$

Hence,

$$V(u) \leq \frac{\epsilon}{u} \quad , \quad 0 < u \leq \delta_0.$$

Thus,

$$\Psi_1 \leq \int_\delta^{2\delta} | \psi(\zeta) | du + \int_\delta^{2\delta} V(u) du$$

$$\leq | \psi(\zeta) | \delta + \epsilon \int_\delta^{2\delta} \frac{du}{u}$$

$$= | \psi(\zeta) | \delta + \epsilon \log 2.$$

Let us turn now to Ψ_2. Since $V(u)$ is the sum of two limited monotone decreasing functions P, N in (u, ζ), it is integrable.

Thus,

$$\Psi_2 \leq | \psi(\zeta) | \int_\eta^{\eta+\delta} du + \int_\eta^{\eta+\delta} V(u) du \leq \delta \{ | \psi(\zeta) | + V(\eta) \}$$

is evanescent with δ.

446. 1. *Fourier's development is valid at the regular point x, if $\phi(u)$ has limited variation in some interval* $(0 < \zeta)$, $\zeta < \frac{\pi}{2}$.

For let $0 < u < \gamma < \zeta$, then

$$V_{u\zeta}\psi = V_{u\gamma}\psi + V_{\gamma\zeta}\psi.$$

Now
$$\psi = \phi(u) \cdot \frac{1}{\sin u}.$$

Hence
$$V_{u\gamma}\psi \leq \{ V_{u\gamma}\phi + |\phi(\gamma)| \} \left\{ V_{u\gamma}\frac{1}{\sin u} + \frac{1}{\sin\gamma} \right\}.$$

But $\sin u$ being monotone,

$$V_{u\gamma}\frac{1}{\sin u} = \frac{1}{\sin u} - \frac{1}{\sin \gamma}.$$

Thus
$$V_{u\gamma}\psi \leq \frac{V_{u\gamma}\phi + |\phi(\gamma)|}{\sin u} = V_1.$$

Similarly,
$$V_{\gamma\zeta}\psi \leq \frac{V_{\gamma\zeta}\phi + |\phi(\zeta)|}{\sin \gamma} = V_2.$$

Now
$$0 < \frac{u}{\sin u} < M \quad , \quad \text{in } (0^*, \zeta).$$

The theorem now follows by 445. For we may take γ so small that
$$V_{0\gamma}\phi < \frac{\epsilon}{4M} \quad , \quad |\phi(\gamma)| < \frac{\epsilon}{4M}. \tag{1}$$

Thus for any $u < \gamma$,
$$uV_1 < \frac{\epsilon}{2}.$$

On the other hand, $\mathfrak{M}$ being sufficiently large, and γ chosen as in 1) and then fixed,
$$V_2 < \mathfrak{M}.$$

Thus
$$uV_2 < \frac{\epsilon}{2},$$

for $u <$ some δ'. Hence
$$uV_{u\zeta}\psi < \epsilon,$$

for $0 < u <$ some δ.

2. (*Jordan.*) *Fourier's development is valid at the regular point x, if $f(x)$ has limited variation in some domain of x.*

For
$$\phi(u) = \{f(x+2u) - f(u)\} + \{f(x-2u) - f(u)\}$$
has limited variation also.

3. *Fourier's development is valid at every point of* $\mathfrak{A} = (0, 2\pi)$, *if f is limited and has only a finite number of oscillations in* $\mathfrak{A}$.

Other Criteria

447. *Let* $$\mathrm{X} = \int_\delta^\eta | \chi(u+\delta) - \chi(u) | \, du \quad , \quad \chi(u) = \frac{\phi(u)}{u},$$
$$\Psi = \int_\delta^\eta | \psi(u+\delta) - \psi(u) | \, du \quad , \quad 0 < \delta < \eta \leq \frac{\pi}{2}.$$

If $\mathrm{X} \doteq 0$ *as* $\delta \doteq 0$, *so does* Ψ, *and conversely.*

For
$$\chi(u+\delta) - \chi(u) = \psi(u+\delta)\frac{\sin(u+\delta)}{u+\delta} - \psi(u)\frac{\sin u}{u}$$
$$= \{\psi(u+\delta) - \psi(u)\}\frac{\sin(u+\delta)}{u+\delta} + \rho,$$
where
$$\rho = \psi(u)\left\{\frac{\sin(u+\delta)}{u+\delta} - \frac{\sin u}{u}\right\}.$$

Obviously X and Ψ are simultaneously evanescent with δ, provided
$$R = \int_\delta^\eta | \rho | \doteq 0 \quad , \quad \text{as } \delta \doteq 0.$$

Let
$$Z(u) = \frac{\sin u}{u}.$$

Then
$$\rho = \psi(u)\{Z(u+\delta) - Z(u)\}$$
$$= \delta\psi(u)Z'(v) \quad , \quad u < v < u + \delta.$$

Now
$$Z'(v) = \frac{v\cos v - \sin v}{v^2} = \frac{v\left(1 - \frac{v^2}{2} + \cdots\right) - v\left(1 - \frac{v^2}{6} + \cdots\right)}{v^2}$$
$$= -\tfrac{1}{3}v + \tfrac{1}{30}v^3 + \cdots$$

Thus
$$| Z'(v) | < Mv < M \cdot 2u.$$

Hence

$$|\rho| < \frac{2u\delta\,|\phi|\,M}{\sin u} < 2\,\delta\,|\phi|\,\mathfrak{M}.$$

As

$$|\phi| \leq |f(x+2u)| + |f(x-2u)| + 2\,|f(x)|,$$

$$R < 2\,\delta\mathfrak{M}\int_\delta^\eta |\phi| \doteq 0 \quad , \quad \text{with } \delta.$$

448. (*Lipschitz-Dini.*) *At the regular point x, Fourier's development is valid, if for each $\epsilon > 0$, there exists a $\delta_0 > 0$, such that for each $0 < \delta < \delta_0$,*

$$|\phi(u+\delta) - \phi(u)| < \frac{\epsilon}{|\log\delta|}, \qquad \textit{for any } u \textit{ in } (\delta, \delta_0).$$

For

$$|\chi(u+\delta) - \chi(u)| = \left|\frac{\phi(u+\delta) - \phi(u)}{u+\delta} + \left\{\frac{1}{u+\delta} - \frac{1}{u}\right\}\phi(u)\right|$$

$$< \frac{|\phi(u+\delta) - \phi(u)|}{u} + \delta\frac{|\phi(u)|}{u^2}.$$

Now x being a regular point, there exists an η' such that

$$|\phi(u)| < \epsilon, \qquad \text{for } u \text{ in any } (\delta, \eta').$$

Thus taking

$$\eta > \delta_0, \eta',$$

$$\mathrm{X} = \int_\delta^\eta |\chi(u+\delta) - \chi(u)|\,du < \frac{\epsilon}{|\log\delta|}\int_\delta^\eta \frac{du}{u} + \epsilon\delta\int_\delta^\eta \frac{du}{u^2}$$

$$< \epsilon\frac{\log\eta - \log\delta}{|\log\delta|} + \epsilon\delta\left(\frac{1}{\delta} - \frac{1}{\eta}\right)$$

$$< 2\,\epsilon, \qquad \text{for any } \delta < \eta.$$

Thus

$$\mathrm{X} \doteq 0, \text{ as } \delta \doteq 0.$$

Uniqueness of Fourier's Development

449. Suppose $f(x)$ can be developed in Fourier's series

$$f(x) = \tfrac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx), \tag{1}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \quad , \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \tag{2}$$

valid in $\mathfrak{A} = (-\pi; \pi)$. We ask can $f(x)$ be developed in a similar series

$$f(x) = \tfrac{1}{2} a_0' + \sum_1^\infty (a_n' \cos nx + b_n' \sin nx), \tag{3}$$

also valid in $\mathfrak{A}$, where the coefficients are not Fourier's coefficients, at least not all of them.

Suppose this were true. Subtracting 1), 3) we get

$$0 = \tfrac{1}{2}(a_0 - a_0') + \Sigma\{(a_n - a_n')\cos nx + (b_n - b_n')\sin nx\} = 0,$$

or

$$c_0 + \Sigma\{c_n \cos nx + d_n \sin nx\} = 0, \qquad \text{in } \mathfrak{A}. \tag{4}$$

Thus it would be possible for a trigonometric series of the type 4) to vanish without all the coefficients c_m, d_m vanishing.

For a power series

$$p_0 + p_1 x + p_2 x^2 + \cdots \tag{5}$$

to vanish in an interval about the origin, however small, we know that all the coefficients p_m in 5) must $= 0$.

We propose to show now that a similar theorem holds for a trigonometric series. In fact we shall prove the *fundamental*

Theorem 1. Suppose it is known that the series 4) *converges to* 0 *for all the points of* $\mathfrak{A} = (-\pi, \pi)$, *except at a reducible set* $\mathfrak{R}$. *Then the coefficients* c_m, d_m *are all* 0, *and the series* 4) $\doteq 0$ *at all the points of* $\mathfrak{A}$.

From this we deduce at once as corollaries:

Theorem 2. Let $\mathfrak{R}$ *be a reducible set in* $\mathfrak{A}$. *Let the series*

$$\alpha_0 + \sum_1^\infty \{\alpha_n \cos nx + \beta_n \sin nx\} \tag{6}$$

converge in $\mathfrak{A}$, *except possibly at the points* $\mathfrak{R}$. *Then* 6) *defines a function* $F(x)$ *in* $\mathfrak{A} - \mathfrak{R}$.

If the series

$$\alpha_0' + \Sigma\{\alpha_n' \cos nx + \beta_n' \sin nx\}$$

converges to $F(x)$ *in* $\mathfrak{A} - \mathfrak{R}$, *its coefficients are respectively equal to those in* 6).

Theorem 3. If $f(x)$ *admits a development in Fourier's series for the set* $\mathfrak{A} - \mathfrak{R}$, *any other development of* $f(x)$ *of the type* 6), *valid in* $\mathfrak{A} - \mathfrak{R}$ *is necessarily Fourier's series, i.e. the coefficients* α_m, β_m *have the values given in* 2).

In order to establish the fundamental theorem, we shall make use of some results due to *Riemann*, *G. Cantor*, *Harnack* and *Schwarz* as extended by later writers. Before doing this let us prove the easy

Theorem 4. If $f(x)$ admits a development in Fourier's series which is uniformly convergent in $\mathfrak{A} = (-\pi, \pi)$, it admits no other development of the type 3), *which is also uniformly convergent in* $\mathfrak{A}$.

For then the corresponding series 4) is uniformly convergent in $\mathfrak{A}$, and may be integrated termwise. Thus making use of the method employed in 436, we see that all the coefficients in 4) vanish.

450. 1. Before attempting to prove the fundamental theorem which states that the coefficients a_n, b_n are 0, we will first show that the coefficients of any trigonometric series which converges in $\mathfrak{A}$, except possibly at a point set of a certain type, must be such that they $\doteq 0$, as $n \doteq \infty$. We have already seen, in 440, 1, that this is indeed so in the case of Fourier's series, whether it converges or not. It is not the case with every trigonometric series as the following example shows, viz.:

$$\sum_1^\infty \sin n!\, x. \tag{1}$$

When $x = \dfrac{m\pi}{r!}$ all the terms, beginning with the $r!^{\text{th}}$, vanish, and hence 1) is convergent at such points. Thus 1) is convergent at a pantactic set of points. In this series the coefficients a_n of the *cosine terms* are all 0, while the coefficients of the sine terms b_n, are 0 or 1. Thus b_n does not $\doteq 0$, as $n \doteq \infty$.

2. Before enunciating the theorem on the convergence of the coefficients of a trigonometric series to 0, we need the notion of *divergence of a series* due to *Harnack*.

Let
$$A = a_1 + a_2 + \cdots \tag{2}$$
be a series of real terms. Let g_n, G_n be the minimum and maximum of all the terms

$$A_{n+1} \quad , \quad A_{n+2} \quad , \quad \cdots$$

where as usual A_n is the sum of the first n terms of 2). Obviously

$$g_n \leq g_{n+1} \quad , \quad G_n \geq G_{n+1}.$$

Thus the two sequences $\{g_n\}$, $\{G_n\}$ are monotone, and if limited, their terms converge to fixed values. Let us say

$$g_n \doteq g \quad , \quad G_n \doteq G.$$

The difference

$$\mathfrak{d} = G - g$$

is called the *divergence* of the series 2).

3. *For the series* 2) *to converge it is necessary and sufficient that its divergence* $\mathfrak{d} = 0$.

For if A is convergent,

$$-\epsilon + A \leq A_{n+p} \leq A + \epsilon \quad , \quad p = 1, 2 \cdots$$

Thus

$$-\epsilon + A \leq g_n \leq G_n \leq A + \epsilon.$$

Thus the limits G, g exist, and

$$G - g \leq 2\epsilon \quad ; \quad \text{or } G = g,$$

as $\epsilon > 0$ is small at pleasure.

Suppose now $\mathfrak{d} = 0$. Then by hypothesis, G, g exist and are equal. There exists, therefore, an n, such that

$$g - \epsilon \leq g_n \leq G_n < G + \epsilon,$$

or

$$G_n - g_n \leq 2\epsilon.$$

Thus

$$|A_{n+p} - A_n| \leq 2\epsilon \quad , \quad p = 1, 2 \cdots$$

and A is convergent.

451. *Let the series*

$$\sum_0^\infty (a_n \cos nx + b_n \sin nx)$$

be such that for each $\delta > 0$, *there exists a subinterval of*

$$\mathfrak{A} = (-\pi, \pi)$$

at each point of which its divergence $\mathfrak{d} < \delta$. *Then* $a_n, b_n \doteq 0$, *as* $n \doteq \infty$.

For, as in 450, there exists for each x an m_x, such that

$$|a_n \cos nx + b_n \sin nx| < \frac{\delta}{8} \quad , \quad n > m_x \tag{1}$$

for any point x in some interval $\mathfrak{B}$ of $\mathfrak{A}$. Thus if b is an inner point of $\mathfrak{B}$, $x = b + \beta$ will lie in $\mathfrak{B}$, if β lies in some interval $B = (p, q)$. Now

$$a_n \cos n(b + \beta) + b_n \sin n(b + \beta)$$
$$= (a_n \cos nb + b_n \sin nb) \cos n\beta - (a_n \sin nb - b_n \cos nb) \sin n\beta.$$

$$a_n \cos n(b - \beta) + b_n \sin n(b - \beta)$$
$$= (a_n \cos nb + b_n \sin nb) \cos n\beta + (a_n \sin nb - b_n \cos nb) \sin n\beta.$$

Adding and subtracting these equations, and using 1) we have

$$|(a_n \cos nb + b_n \sin nb) \cos n\beta| < \frac{\delta}{4},$$

$$|(a_n \sin nb - b_n \cos nb) \sin n\beta| < \frac{\delta}{4},$$

for all $n > m_x$. Let us multiply the first of these inequalities by $\cos nb \sin n\beta$, and the second by $\sin nb \cos n\beta$, and add. We get

$$|a_n \sin n\beta_1| < \delta \quad , \quad \beta_1 = 2\beta \quad , \quad n > m_x. \tag{2}$$

Again if we multiply the first inequality by $\sin nb \sin n\beta$, and the second by $\cos nb \cos n\beta$, and subtract, we get

$$|b_n \sin n\beta_1| < \delta \quad , \quad n > m_x. \tag{3}$$

From 2), 3), we can infer that for any $\epsilon > 0$

$$|a_n| < \epsilon \quad , \quad |b_n| < \epsilon \quad , \quad n > \text{some } m, \tag{4}$$

or what is the same, that $a_n, b_n \doteq 0$.

For suppose that the first inequality of 4) did not hold. Then there exists a sequence

$$n_1 < n_2 < \cdots \doteq \infty \tag{5}$$

such that on setting

$$|a_{n_r}| = \delta + \delta'_{n_r} \quad , \quad \epsilon - \delta = \delta'$$

we will have

$$\delta_{n_r} \geq \delta'. \tag{6}$$

If this be so, we can show that there exists a sequence

$$\nu_1 < \nu_2 < \cdots \doteq \infty$$

in 5), such that for some β' in B,

$$|a_{\nu_r} \sin \nu_r \beta'| \geq \delta, \tag{7}$$

which contradicts 2). To this end we note that $\gamma_0 > 0$ may be chosen so small that for any r and any $|\gamma| \leq \gamma_0$,

$$|a_{\nu_r}| \cos\gamma \geq (\delta + \delta') \cos\gamma_0 > \delta. \tag{8}$$

Let us take the integer ν_1 so that

$$\nu_1 \geq \frac{\pi + 2\gamma_0}{q - p}. \tag{9}$$

Then
$$\frac{2}{\pi}(\nu_1(q-p) - 2\gamma_0) \geq 2.$$

Thus at least *one* odd integer lies in the interval determined by the two numbers

$$\frac{2}{\pi}(p\nu_1 + \gamma_0) \quad , \quad \frac{2}{\pi}(q\nu_1 - \gamma_0).$$

Let m_1 be such an integer. Then

$$\frac{2}{\pi}(p\nu_1 + \gamma_0) \leq m_1 \leq \frac{2}{\pi}(q\nu_1 - \gamma_0). \tag{10}$$

If we set

$$p_1 = \frac{1}{\nu_1}\left(m_1\frac{\pi}{2} - \gamma_0\right) \quad , \quad q_1 = \frac{1}{\nu_1}\left(m_1\frac{\pi}{2} + \gamma_0\right) \tag{11}$$

we see that the interval $B_1 = (p_1, q_1)$ lies in B. The length of B_1 is $2\gamma_0/\nu_1$. Then for any β in B_1,

$$\nu_1\beta = m_1\frac{\pi}{2} + \gamma_1 \quad , \quad |\gamma_1| \leq \gamma_0.$$

Thus by 8),
$$|a_{\nu_1} \sin \nu_1\beta| = |a_{\nu_1}| \cos\gamma_1 > \delta. \tag{12}$$

But we may reason on B_1 as we have on B. We determine ν_2 by 9), replacing p, q by p_1, q_1. We determine the odd integer m_2 by 10), replacing p, q, ν_1 by p_1, q_1, ν_2. The relation 11) determines the new interval $B_2 = (p_2, q_2)$, on replacing m_1, ν_1 by m_2, ν_2. The length of B_2 is $2\gamma_0/\nu_2$, and B_2 lies in B_1. For this relation of ν_2, and for any β in B_2 we have, similar to 12),

$$|a_{\nu_2} \sin \nu_2\beta| > \delta.$$

In this way we may continue indefinitely. The intervals $B_1 > B_2 > \cdots \doteq$ to a point β', and obviously for this β', the rela-

tion 7) holds for any x. In a similar manner we see that if b_n does not $\doteq 0$, the relation 3) cannot hold.

452. As corollaries of the last theorem we have:

1. *Let the series*

$$\sum_0^\infty (a_n \cos nx + b_n \sin nx) \tag{1}$$

be such that for each $\delta > 0$, the points in $\mathfrak{A} = (-\pi, \pi)$ at which the divergence of 1) *is $\geq \delta$, form an apantactic set in $\mathfrak{A}$. Then a_n, $b_n \doteq 0$, as $n \doteq \infty$.*

2. *Let the series* 1) *converge in $\mathfrak{A}$, except possibly at the points of a reducible set $\mathfrak{R}$. Then a_n, $b_n \doteq 0$.*

For $\mathfrak{R}$ being reducible [318, 6], there exists in $\mathfrak{A}$ an interval $\mathfrak{B}$ in which 1) converges at every point. We now apply 451.

453. *Let*
$$F(x) = \sum_0^\infty (a_n \cos nx + b_n \sin nx)$$

at the points of $\mathfrak{A} = (-\pi, \pi)$, where the series is convergent. At the other points of $\mathfrak{A}$, let $F(x)$ have an arbitrarily assigned value, lying between the two limits of indetermination g, G of the series. If F is R-integrable in $\mathfrak{A}$, the coefficients a_n, $b_n \doteq 0$.

For there exists a division of $\mathfrak{A}$, such that the sum of those intervals in which Osc $F \geq \omega$ is $< \sigma$. There is therefore an interval $\mathfrak{J}$ in which Osc $F < \omega$. If $\mathfrak{K}$ is an inner interval of $\mathfrak{J}$, the divergence of the above series is $< \omega$ at each point of $\mathfrak{K}$. We now apply 451.

454. *Riemann's Theorem.*

Let $F(x) = \frac{1}{2} a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx) = \Sigma A_n$ *converge at each point of $\mathfrak{A} = (-\pi, \pi)$, except possibly at the points of a reducible set $\mathfrak{R}$. The series obtained by integrating this series termwise, we denote by*

$$G(x) = \frac{1}{4} a_0 x^2 - \sum_1^\infty \frac{1}{n^2}(a_n \cos nx + b_n \sin nx) = \frac{1}{2} A_0 x^2 - \sum_1^\infty \frac{A_n}{n^2}.$$

Then G is continuous in $\mathfrak{A}$.

Let $$\Phi(u) = G(x + 2u) + G(x - 2u) - 2G(x). \tag{1}$$

Then at each point of $\mathfrak{B} = \mathfrak{A} - \mathfrak{R}$,

$$\lim_{u=0} \frac{\Phi(u)}{4u^2} = F(x); \tag{2}$$

and at each point of $\mathfrak{A}$,

$$\lim_{u=0} \frac{\Phi(u)}{u} = 0. \tag{3}$$

For, in the first place, since $\mathfrak{R}$ is a reducible set, $a_n, b_n \doteq 0$. The series G is therefore uniformly convergent in $\mathfrak{A}$, and is thus a continuous function.

Let us now compute Φ. We have

$$\begin{aligned} a_n \cos n(x + 2u) + a_n \cos n(x - 2u) - 2a_n \cos nx &\\ = 2a_n \cos nx\,(\cos 2nu - 1)&\\ = -4a_n \cos nx \sin^2 nu.& \end{aligned}$$

Also $$\begin{aligned} b_n \sin n(x + 2u) + b_n \sin n(x - 2u) - 2b_n \sin nx &\\ = 2b_n \sin nx(\cos 2nu - 1)&\\ = -4b_n \sin nx \sin^2 nu.& \end{aligned}$$

Thus $$\frac{\Phi(u)}{4u^2} = \sum_0^\infty A_n \left(\frac{\sin nu}{nu}\right)^2,$$

if we agree to give the coefficient of A_0 the value 1. Let us give x an arbitrary but fixed value in $\mathfrak{B}$. Then for each $\epsilon > 0$, there exists an m such that

$$A_0 + A_1 + \cdots + A_{n-1} = F(x) + \epsilon_n \quad , \quad |\epsilon_n| < \epsilon, \quad n \geq m.$$

Thus $$A_n = \epsilon_{n+1} - \epsilon_n.$$

Hence $$\begin{aligned} \frac{\Phi}{4u^2} &= F(x) + \epsilon_1 + \sum_1^\infty (\epsilon_{n+1} - \epsilon_n)\left(\frac{\sin nu}{nu}\right)^2 \\ &= F(x) + \sum_1^\infty \epsilon_n \left\{\left[\frac{\sin (n-1)u}{(n-1)u}\right]^2 - \left[\frac{\sin nu}{nu}\right]^2\right\} \qquad (4 \\ &= F(x) + S \end{aligned}$$

The index m being determined as above, let us take u such that

$$u < \frac{\pi}{m} \quad , \quad \text{so that} \quad m < \frac{\pi}{u};$$

and break S into three parts

$$S_1 = \sum_1^m \quad , \quad S_2 = \sum_{m+1}^{\kappa} \quad , \quad S_3 = \sum_{\kappa+1}^{\infty},$$

where κ is the greatest integer $< \pi/u$, and then consider each sum separately, as $u \doteq 0$.

Obviously $$\lim_{u=0} S_1 = 0.$$

As to the second sum, the number of its terms increases indefinitely as $u \doteq 0$.

For any u,

$$\begin{aligned} | S_2 | &< \epsilon \sum_{m+1}^{\kappa} \{ \cdots \} \\ &< \epsilon \left\{ \left[\frac{\sin mu}{mu}\right]^2 - \left[\frac{\sin \kappa u}{\kappa u}\right]^2 \right\} \\ &< \epsilon \left[\frac{\sin mu}{mu}\right]^2 < \epsilon, \end{aligned}$$

since each term in the brace is positive. In fact

$$\frac{\sin v}{v}$$

is a decreasing function of v as v ranges from 0 to π, and

$$nu \leq \kappa u \leq \pi \quad , \quad n = m,\, m+1,\, \cdots \kappa.$$

Finally we consider S_3. We may write the general term as follows:

$$\epsilon_n \left\{ \left[\frac{\sin (n-1)u}{(n-1)u}\right]^2 - \left[\frac{\sin (n-1)u}{nu}\right]^2 \right\}$$

$$+ \epsilon_n \left\{ \left[\frac{\sin (n-1)u}{nu}\right]^2 - \left[\frac{\sin nu}{nu}\right]^2 \right\}.$$

Now $$\frac{\sin^2 (n-1)u - \sin^2 nu}{n^2u^2} = \frac{-\sin (2n-1)u \sin u}{n^2u^2} \bigg| \leq \bigg| \frac{1}{n^2u}.$$

Thus

$$|S_3| \leq \frac{\epsilon}{u^2}\sum_{\kappa+1}^{\infty}\left\{\frac{1}{(n-1)^2}-\frac{1}{n^2}\right\}+\frac{\epsilon}{u}\sum_{\kappa+1}^{\infty}\frac{1}{n^2}$$

$$\leq \frac{\epsilon}{\kappa^2 u^2}+\frac{\epsilon}{\kappa u},$$

since

$$\sum_{\kappa+1}^{\infty}\frac{1}{n^2} < \int_{\kappa}^{\infty}\frac{dx}{x^2} = \frac{1}{\kappa}.$$

But

$$\kappa \geq \frac{\pi}{u} - 1 \quad , \quad \text{or } \kappa u \geq \pi - u.$$

Thus

$$|S_3| \leq \epsilon\left\{\frac{1}{(\pi-u)^2}+\frac{1}{\pi-u}\right\}.$$

Hence

$$S = S_1 + S_2 + S_3 \doteq 0, \text{ as } u \doteq 0,$$

which proves the limit 2), on using 4).

To prove the limit 3), we have

$$\frac{\Phi(u)}{4u} = \sum_0^{\infty} uA_n\left(\frac{\sin nu}{nu}\right)^2 = T.$$

Let us give u a definite value and break T into three sums.

$$T_1 = \sum_1^m,$$

where m is chosen so that

$$|A_n| < \epsilon \quad , \quad n > m;$$

$$T_2 = \sum_{m+1}^{\lambda},$$

where λ is the greatest integer such that

$$\lambda u \leq 1;$$

and

$$T_3 = \sum_{\lambda+1}^{\infty}.$$

Obviously for some M,

$$|T_1| \leq uM.$$

Also

$$|T_2| \leq \epsilon u \lambda \leq \epsilon,$$

since

$$\left(\frac{\sin nu}{nu}\right)^2 < 1.$$

As to the last sum,

$$|T_3| \leq \frac{\epsilon}{u}\sum_{\lambda+1}^{\infty}\frac{1}{n^2} \leq \epsilon\lambda \cdot \frac{1}{\lambda} \quad , \quad \text{since } \frac{1}{u} \leq \lambda,$$
$$< \epsilon.$$

Thus

$$T \doteq 0 \quad , \quad \text{as } u \doteq 0.$$

455. *Schwarz-Lüroth Theorem.*

In $\mathfrak{A} = (a < b)$ let the continuous function $f(x)$ be such that

$$S(x, u) = \frac{f(x+u) + f(x-u) - 2f(x)}{u^2} \doteq 0, \textit{ as } u \doteq 0, \tag{1}$$

except possibly at an enumerable set $\mathfrak{E}$ in $\mathfrak{A}$. At the points $\mathfrak{E}$, let

$$uS(x, u) \doteq 0 \qquad \textit{as } u \doteq 0. \tag{2}$$

Then f is a linear function in $\mathfrak{A}$.

Let us *first suppose* with Schwarz that $\mathfrak{E} = 0$. We introduce the auxiliary function,

$$g(x) = \eta L(x) - \tfrac{1}{2} c(x-a)(x-b),$$

where

$$L(x) = f(x) - f(a) - \frac{x-a}{b-a}\{f(b) - f(a)\},$$

$\eta = \pm 1$, and c is an arbitrary constant.

The function $g(x)$ is continuous in $\mathfrak{A}$, and $g(a) = g(b) = 0$. Moreover

$$\frac{g(x+u) + g(x-u) - 2g(x)}{u^2} \doteq c, \qquad \text{as } u \doteq 0.$$

Thus for all $0 < u <$ some δ,

$$G = g(x+u) + g(x-u) - 2g(x) > 0. \tag{3}$$

From this follows that $g(x) \leq 0$ in $\mathfrak{A}$. For if $g(x) > 0$, at any point in $\mathfrak{A}$, it takes on its maximum value at some point ξ within $\mathfrak{A}$. Thus

$$g(\xi + u) - g(\xi) \leq 0 \quad , \quad g(\xi - u) - g(\xi) \leq 0,$$

for $0 < u < \delta$, δ being sufficiently small. Adding these two inequalities gives $G \leq 0$, which contradicts 3). Thus $g \leq 0$ in $\mathfrak{A}$.

Let us now suppose $L \neq 0$ for some x in $\mathfrak{A}$. We take c so small that

$$\operatorname{sgn} g = \operatorname{sgn} \eta L = \eta \operatorname{sgn} L.$$

But η is at pleasure ± 1, hence the supposition that $L \neq 0$ is not admissible. Hence $L = 0$ in $\mathfrak{A}$, or

$$f(x) = f(a) - \frac{x-a}{b-a}\{f(b) - f(a)\} \tag{4}$$

is indeed a linear function.

Let us now suppose with Lüroth that $\mathfrak{C} > 0$. We introduce the auxiliary continuous function.

$$h(x) = L(x) + c(x-a)^2 \quad , \quad c > 0.$$

Thus

$$h(a) = 0 \quad , \quad h(b) = c(b-a)^2.$$

Suppose at some inner point ξ of $\mathfrak{A}$

$$L(\xi) > 0. \tag{5}$$

This leads to a contradiction, as we proceed to show. For then

$$h(\xi) - h(b) = L(\xi) + c\{(\xi - a)^2 - (b-a)^2\} > 0,$$

provided

$$C = \frac{L(\xi)}{(b-a)^2 - (\xi - a)^2} > c.$$

We shall take c so that this inequality is satisfied, *i.e.* c lies in the interval $\mathfrak{C} = (0^*, C^*)$. Thus

$$h(\xi) > h(b) > h(a).$$

Hence $h(x)$ takes on its maximum value at some inner point e of $\mathfrak{A}$. Hence for $\delta > 0$ sufficiently small,

$$h(e+u) - h(e) < 0 \quad , \quad h(e-u) - h(e) \leq 0 \quad , \quad 0 < u \leq \delta. \tag{6}$$

Hence

$$H(e, u) = \frac{h(e+u) + h(e-u) - 2\,h(e)}{u^2} \leq 0. \tag{7}$$

Now if e is a point of $\mathfrak{A} - \mathfrak{C}$,

$$\lim_{u=0} H(e, u) = 2\,c > 0.$$

But this contradicts 7), which requires that

$$\lim_{u=0} H(e, u) \leq 0.$$

Hence e is a point of $\mathfrak{E}$. Hence by 2),

$$\frac{h(e+u)-h(e)}{u}+\frac{h(e-u)-h(e)}{u} \doteq 0 \quad , \quad \text{as } u \doteq 0.$$

By 6), both terms have the same sign. Hence each term $\doteq 0$. Thus for $u > 0$

$$0 = \lim_{u=0} \frac{h(e \pm u)-h(e)}{\pm u} = \lim \frac{f(e \pm u)-f(e)}{\pm u} - \frac{f(b)-f(a)}{b-a} + 2\,c(e-a).$$

Hence

$$f'(e) = \frac{f(b)-f(a)}{b-a} + 2\,c(e-a). \tag{8}$$

Thus to each c in the interval $\mathfrak{C}$, corresponds an e in $\mathfrak{E}$, at which point the derivative of $f(x)$ exists and has the value given on the right of 8). On the other hand, two different c's, say c and c', in $\mathfrak{C}$ cannot correspond to the same e in $\mathfrak{E}$.

For then 8) shows that

$$c(e-a) = c'(e-a),$$

or as

$$e > a, \qquad c' = c.$$

Thus there is a uniform correspondence between $\mathfrak{C}$ whose cardinal number is $\mathfrak{c}$, and $\mathfrak{E}$ whose cardinal number is $\mathfrak{e}$, which is absurd. Thus the supposition 5) is impossible. In a similar manner, the assumption that $L < 0$ at some point in $\mathfrak{A}$, leads to a contradiction. Hence $L = 0$ in $\mathfrak{A}$, and 4) again holds, which proves the theorem.

456. *Cantor's Theorem. Let*

$$\tfrac{1}{2}\,a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx) \tag{1}$$

converge to 0 *in* $\mathfrak{A} = (-\pi, \pi)$, *except possibly at a reducible set* $\mathfrak{R}$, *where nothing is asserted regarding its convergence. Then it converges to* 0 *at every point in* $\mathfrak{A}$, *and all its coefficients*

$$a_0,\ a_1,\ a_2 \cdots b_1,\ b_2,\ b_3 \cdots = 0.$$

For by 452, 2, a_n, $b_n \doteq 0$. Then Riemann's function

$$f(x) = \tfrac{1}{4}\,a_0 x^2 - \sum \frac{1}{n^2}(a_n \cos nx + b_n \sin nx)$$

satisfies the conditions of the Schwarz-Lüroth theorem, 455, since $\mathfrak{R}$ is enumerable. Thus $f(x)$ is a linear function of x in $\mathfrak{A}$, and has the form $\alpha + \beta x$. Hence

$$\alpha + \beta x - \tfrac{1}{4} a_0 x^2 = -\sum_1^\infty \frac{1}{n^2}(a_n \cos nx + b_n \sin nx). \qquad (2$$

The right side admits the period 2π, and is therefore periodic. Its period ω must be 0. For if $\omega > 0$, the left side has this period, which is absurd. Hence $\omega = 0$, and the left side reduces to a constant, which gives $\beta = 0$, $a_0 = 0$. But in $\mathfrak{A} - \mathfrak{R}$, the right side of 1) has the sum 0. Hence $\alpha = 0$. Thus the right side of 2) vanishes in $\mathfrak{A}$. As it converges uniformly in $\mathfrak{A}$, we may determine its coefficients as in 436. This gives

$$a_n = 0 \quad , \quad b_n = 0 \quad , \quad n = 1, 2 \cdots$$

CHAPTER XIV

DISCONTINUOUS FUNCTIONS

Properties of Continuous Functions

457. 1. In Chapter VII of Volume I we have discussed some of the elementary properties of continuous and discontinuous functions. In the present chapter further developments will be given, paying particular attention to discontinuous functions. Here the results of Baire * are of foremost importance. Lebesgue † has shown how some of these may be obtained by simpler considerations, and we have accordingly adopted them.

2. Let us begin by observing that the definition of a continuous function given in I, 339, may be extended to sets having isolated points, if we use I, 339, 2 as definition.

p208

Let therefore $f(x_1 \cdots x_m)$ be defined over $\mathfrak{A}$, being either limited or unlimited. Let a be any point of $\mathfrak{A}$. If for each $\epsilon > 0$, there exists a $\delta > 0$, such that

$$|f(x) - f(a)| < \epsilon, \qquad \text{for any } x \text{ in } V_\delta(a),$$

we say *f is continuous at a.*

By the definition it follows at once that f is continuous at each isolated point of $\mathfrak{A}$. Moreover, when a is a proper limiting point of $\mathfrak{A}$, the definition here given coincides with that given in I, 339. If f is continuous at each point of $\mathfrak{A}$, we say it is *continuous in* $\mathfrak{A}$. The definition of discontinuity given in I, 347, shall still hold, except that we must regard isolated points as points of continuity.

* "*Sur les Functions de Variables réeles*," *Annali di Mat.*, Ser. 3, vol. 3 (1899).

Also his *Leçons sur les Functions Discontinues.* Paris, 1905.

† *Bulletin de la Société Mathématique de France*, vol. 32 (1904), p. 229.

3. The reader will observe that the theorems I, 350 to 354 inclusive, are valid not only for limited perfect domains, but also for limited complete sets.

458. 1. *If $f(x_1 \cdots x_m)$ is continuous in the limited set $\mathfrak{A}$, and its values are known at the points of $\mathfrak{B} < \mathfrak{A}$, then f is known at all points of $\mathfrak{B}'$ lying in $\mathfrak{A}$.*

For let $b_1, b_2, b_3 \cdots$ be points of $\mathfrak{B}$, whose limiting point b lies in $\mathfrak{A}$. Then

$$\lim_{n=\infty} f(b_n) = f(b).$$

2. *If f is known for a dense set $\mathfrak{B}$ in $\mathfrak{A}$, and is continuous in $\mathfrak{A}$, f is known throughout $\mathfrak{A}$.*

For

$$\mathfrak{B}' \geq \mathfrak{A}.$$

3. *If $f(x_1 \cdots x_m)$ is continuous in the complete set $\mathfrak{A}$, the points $\mathfrak{B}$ in $\mathfrak{A}$ where $f = c$, a constant, form a complete set. If $\mathfrak{A}$ is an interval, there is a first and a last point of $\mathfrak{B}$.*

For $f = c$ at $x = \alpha_1, \alpha_2 \cdots$ which $\doteq \alpha$; we have therefore

$$f(\alpha) = \lim_{n=\infty} f(\alpha_n) = c.$$

459. *The points of continuity $\mathfrak{C}$ of $f(x_1 \cdots x_m)$ in $\mathfrak{A}$ lie in a deleted enclosure $\mathfrak{K}$. If $\mathfrak{A}$ is complete, $\mathfrak{K} = \mathfrak{C}$.*

For let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. For each ϵ_n, and for each point of continuity c in $\mathfrak{A}$, there exists a cube $\mathfrak{Q}$ whose center is c, such that

$$\operatorname{Osc} f < \epsilon_n, \qquad \text{in } \mathfrak{Q}.$$

Thus the points of continuity of f lie in an enumerable non-overlapping set of complete metric cells, in each of which $\operatorname{Osc} f < \epsilon_n$. Let $\mathfrak{Q}_n$ be the inner points of this enclosure. Then each point of the *deleted enclosure*

$$\mathfrak{K} = Dv\,\{\mathfrak{Q}_n\}$$

which lies in $\mathfrak{A}$ is a point of continuity of f. For such a point c lies within each $\mathfrak{Q}_n$.

Hence

$$\operatorname{Osc} f < \epsilon, \qquad \text{in } V_\delta(c),$$

for $\delta > 0$ sufficiently small and n sufficiently great.

Oscillation

460. Let
$$\omega_\delta = \operatorname{Osc} f(x_1 \cdots x_m) \qquad \text{in } V_\delta(a).$$

This is a monotone decreasing function of δ. Hence if ω_δ is finite, for some $\delta > 0$,
$$\omega = \lim_{\delta=0} \omega_\delta$$
exists. We call ω *the oscillation of f at $x = a$*, and write

"The diff. btw max. and min. of a funct in an interval = oscillation"

$$\omega = \operatorname*{Osc}_{x=a} f.$$

v. 1 p 341

Should $\omega_\delta = +\infty$, however small $\delta > 0$ is taken, we say $\omega = +\infty$. When $\omega = 0$, f is continuous at $x = a$, if a is a point in the domain of definition of f. When $\omega > 0$, f is discontinuous at this point. It is a *measure of the discontinuity of f at $x = a$*; we write
$$\omega = \operatorname*{Disc}_{x=a} f(x_1 \cdots x_m).$$

461. 1. *Let*
$$d = \operatorname{Disc} f(x_1 \cdots x_m) \quad , \quad e = \operatorname{Disc} g(x_1 \cdots x_m),$$
at $x = a$. Then
$$|d - e| \le \operatorname*{Disc}_{x=a}(f \pm g) \le d + e.$$

For in $V_\delta(a)$,
$$|\operatorname{Osc} f - \operatorname{Osc} g| \le \operatorname{Osc}(f \pm g) \le \operatorname{Osc} f + \operatorname{Osc} g.$$

2. *If f is continuous at $x = a$, while* $\operatorname{Disc} g = d$, *then*
$$\operatorname*{Disc}_{x=a}(f + g) = d.$$

For f being continuous at a, $\qquad \operatorname{Disc} f = 0.$

Hence
$$\operatorname{Disc} g \le \operatorname{Disc}(f + g) \le \operatorname{Disc} g = d.$$

3. *If c is a constant,*
$$\operatorname{Disc}(cf) = |c| \operatorname{Disc} f \quad , \quad \text{at } x = a.$$

For
$$\operatorname{Osc}(cf) = |c| \operatorname{Osc} f \quad , \quad \text{in any } V_\delta(a).$$

4. When the limits
$$f(x - 0) \quad , \quad f(x + 0)$$

exist and at least one of them is different from $f(x)$, the point x is a discontinuity of the first kind, as we have already said. When at least one of the above limits does not exist, the point x is a *point of discontinuity of the second kind.*

462. 1. *The points of infinite discontinuity* $\mathfrak{J}$ *of* f, *defined over a limited set* $\mathfrak{A}$, *form a complete set.*

For let $\iota_1, \iota_2 \cdots$ be points of $\mathfrak{J}$, having k as limiting point. Then in any $V(k)$ there are an infinity of the points ι_n and hence in any $V(k)$, $\text{Osc} f = +\infty$. The point k does not of course need to lie in $\mathfrak{A}$.

2. We cannot say, however, that the points of discontinuity of a function form a complete set as is shown by the following

Example. In $\mathfrak{A} = (0, 1)$, let $f(x) = x$ when x is irrational, and $= 0$ when x is rational. Then each point of $\mathfrak{A}$ is a point of discontinuity except the point $x = 0$. Hence the points of discontinuity of f do not form a complete set.

3. *Let* f *be limited or unlimited in the limited complete set* $\mathfrak{A}$. *The points* $\mathfrak{K}$ *of* $\mathfrak{A}$ *at which* $\text{Osc} f \geq k$ *form a complete set.*

For let $a_1, a_2 \cdots$ be points of $\mathfrak{K}$ which $\doteq a$. However small $\delta > 0$ is taken, there are an infinity of the a_n lying in $V_\delta(a)$. But at any one of these points, $\text{Osc} f \geq k$. Hence $\text{Osc} f \geq k$ in $V_\delta(a)$, and thus a lies in $\mathfrak{K}$.

4. *Let* $f(x_1 \cdots x_m)$ *be limited and R-integrable in the limited set* $\mathfrak{A}$. *The points* $\mathfrak{K}$ *at which* $\text{Osc} f \geq k$ *form a discrete set.*

For let D be a rectangular division of space. Let us suppose $\overline{\mathfrak{K}}_D >$ some constant $c > 0$, however D is chosen. In each cell δ of D,

$$\text{Osc} f \geq k.$$

Hence the sum of the cells in which the oscillation is $\geq k$ cannot be made small at pleasure, since this sum is $\overline{\mathfrak{K}}_D$. But this contradicts I, 700, 5.

5. *Let* $f(x_1 \cdots x_m)$ *be limited in the complete set* $\mathfrak{A}$. *If the points* $\mathfrak{K}$ *in* $\mathfrak{A}$ *at which* $\text{Osc} f \geq k$ *form a discrete set, for each* k, *then* f *is R-integrable in* $\mathfrak{A}$.

For about each point of $\mathfrak{A} - \mathfrak{K}$ as center, we can describe a cube $\mathfrak{C}$ of varying size, such that $\mathrm{Osc} f < 2k$ in $\mathfrak{C}$. Let D be a cubical division of space of norm d. We may take d so small that $\overline{\mathfrak{K}}_D = \Sigma d_\iota$ is as small as we please. The points of $\mathfrak{A}$ lie now within the cubes $\mathfrak{C}$ and the set formed of the cubes d_ι. By Borel's theorem there are a finite number of cubes, say

$$\eta_1 \quad , \quad \eta_2 \cdots$$

such that all the points of $\mathfrak{A}$ lie within these η's. If we prolong the faces of these η's, we effect a rectangular division such that the sum of those cells in which the oscillation is $> 2k$ is as small as we choose, since this sum is obviously $\leq \overline{\mathfrak{K}}_D$. Hence by I, 700, 5, f is R-integrable.

6. *Let $f(x_1 \cdots x_m)$ be limited in $\mathfrak{A}$; let its points of discontinuity in $\mathfrak{A}$ be $\mathfrak{D}$. If f is R-integrable, $\mathfrak{D}$ is a null set. If $\mathfrak{A}$ is complete and $\mathfrak{D}$ is a null set, f is R-integrable.*

Let f be R-integrable. Then $\mathfrak{D}$ is a null set. For let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. Let $\mathfrak{D}_n$ denote the points at which $\mathrm{Osc} f \geq \epsilon_n$. Then $\mathfrak{D} = \{\mathfrak{D}_n\}$. But since f is R-integrable, each $\mathfrak{D}_n$ is discrete by 4. Hence $\mathfrak{D}$ is a null set.

Let $\mathfrak{A}$ be complete and $\mathfrak{D}$ a null set. Then each $\mathfrak{D}_n$ is complete by 3. Hence by 365, $\widehat{\mathfrak{D}}_n = \overline{\mathfrak{D}}_n$. As $\widehat{\mathfrak{D}} = 0$, we see $\mathfrak{D}_n$ is discrete. Hence by 5, f is R-integrable.

If $\mathfrak{A}$ is not complete, f does not need to be R-integrable when $\mathfrak{D}$ is a null set.

Example. Let $\mathfrak{A}_1 = \left\{\frac{m}{2^n}\right\} \quad , \quad n = 1, 2 \cdots ; \; m < 2^n.$

$$\mathfrak{A}_2 = \left\{\frac{r}{3^s}\right\} \quad , \quad s = 1, 2 \cdots ; \; r < 3^s.$$

Let $\mathfrak{A} = \mathfrak{A}_1 + \mathfrak{A}_2$.

Let
$$f(x) = \frac{1}{2^n} \quad , \quad \text{at } x = \frac{m}{2^n}$$
$$= 1 \qquad \text{in } \mathfrak{A}_2.$$

Then each point of $\mathfrak{A}$ is a point of discontinuity, and $\mathfrak{A} = \mathfrak{D}$. But $\mathfrak{A}_1$, $\mathfrak{A}_2$ are null sets, hence $\mathfrak{A}$ is a null set.

On the other hand,

$$\overline{\int_{\mathfrak{A}}} f = 1 \quad , \quad \underline{\int_{\mathfrak{A}}} f = 0,$$

and f is not R-integrable in $\mathfrak{A}$.

Pointwise and Total Discontinuity

463. Let $f(x_1 \cdots x_m)$ be defined over $\mathfrak{A}$. If each point of $\mathfrak{A}$ is a point of discontinuity, we say f is *totally discontinuous* in $\mathfrak{A}$.

We say f is *pointwise discontinuous* in $\mathfrak{A}$, if f is not continuous in $\mathfrak{A} = \{a\}$, but has in any $V(a)$ a point of continuity. If f is continuous or pointwise discontinuous, we may say it is *at most pointwise* discontinuous.

Example 1. A function $f(x_1 \cdots x_m)$ having only a finite number of points of discontinuity in $\mathfrak{A}$ is pointwise discontinuous in $\mathfrak{A}$.

Example 2. Let

$$\begin{aligned} f(x) &= 0 \quad , \quad \text{for irrational } x \text{ in } \mathfrak{A} = (0, 1) \\ &= \frac{1}{n} \quad , \quad \text{for } x = \frac{m}{n} \\ &= 1 \quad , \quad \text{for } x = 0, 1. \end{aligned}$$

Obviously f is continuous at each irrational x, and discontinuous at the other points of $\mathfrak{A}$. Hence f is pointwise discontinuous in $\mathfrak{A}$.

Example 3. Let $\mathfrak{D}$ be a Harnack set in the unit interval $\mathfrak{A} = (0, 1)$. In the associate set of intervals, end points included, let $f(x) = 1$. At the other points of $\mathfrak{A}$, let $f = 0$. As $\mathfrak{D}$ is apantactic in $\mathfrak{A}$, f is pointwise discontinuous.

Example 4. In Ex. 3, let $\mathfrak{D} = \mathfrak{E} + \mathfrak{F}$, where $\mathfrak{E}$ is the set of end points of the associate set of intervals. Let $f = 1/n$ at the end points of these intervals belonging to the n^{th} stage. Let $f = 0$ in $\mathfrak{F}$. Here f is defined only over $\mathfrak{D}$. The points $\mathfrak{F}$ are points of continuity in $\mathfrak{D}$. Hence f is pointwise discontinuous in $\mathfrak{D}$.

Example 5. Let $f(x)$ be Dirichlet's function, *i.e.* $f = 0$, for the irrational points $\mathfrak{J}$ in $\mathfrak{A} = (0, 1)$, and $= 1$ for the rational points.

As each point of $\mathfrak{A}$ is a point of discontinuity, f is totally discontinuous in $\mathfrak{A}$. Let us remove the rational points in $\mathfrak{A}$; the deleted domain is $\mathfrak{J}$. In this domain f is continuous. Thus on removing certain points, a discontinuous function becomes a continuous function in the remaining point set.

This is not always the case. For if in Ex. 4 we remove the points $\mathfrak{F}$, retaining only the points $\mathfrak{E}$, we get a function which is *totally* discontinuous in $\mathfrak{E}$, whereas before f was only pointwise discontinuous.

464. 1. *If $f(x_1 \cdots x_m)$ is totally discontinuous in the infinite complete set $\mathfrak{A}$, then the points $\mathfrak{d}_\omega$ where*

$$\mathrm{Disc} f \geq \omega \quad , \quad \omega > 0,$$

form an infinite set, if ω is taken sufficiently small.

For suppose $\mathfrak{d}_\omega$ were finite however small ω is taken. Let $\omega_1 > \omega_2 > \cdots \doteq 0$. Let $D_1, D_2, \cdots$ be a sequence of superposed cubical divisions of space of norms $d_n \doteq 0$. We shall only consider cells containing points of $\mathfrak{A}$. Then if d_1 is taken sufficiently small, D_1 contains a cell δ_1, containing an infinite number of points of $\mathfrak{A}$, but no point at which $\mathrm{Disc} f \geq \omega_1$. If d_2 is taken sufficiently small, D_2 contains a cell $\delta_2 < \delta_1$, containing no point at which $\mathrm{Disc} f \geq \omega_2$. In this way we get a sequence of cells,

$$\delta_1 > \delta_2 > \cdots$$

which $\doteq$ a point p. As $\mathfrak{A}$ is complete, p lies in $\mathfrak{A}$. But f is obviously continuous at p. Hence f is not totally discontinuous in $\mathfrak{A}$.

2. If $\mathfrak{A}$ is not complete, $\mathfrak{d}_\omega$ does not need to be infinite for any $\omega > 0$.

Example. Let $\mathfrak{A} = \left\{ \frac{m}{2^n} \right\}$, $n = 1, 2, \cdots$ and m odd and $< 2^n$. At $\frac{m}{2^n}$, let $f = \frac{1}{2^n}$. Then each point of $\mathfrak{A}$ is a point of discontinuity. But $\mathfrak{d}_\omega$ is finite, however small $\omega > 0$ is taken.

3. We cannot say f is not pointwise discontinuous in complete $\mathfrak{A}$, when $\mathfrak{d}_\omega$ is infinite.

Example. At the points $\left\{\frac{1}{n}\right\} = \mathfrak{N}$, let $f = 0$; at the other points of $\mathfrak{A} = (0, 1)$, let $f = 1$.

Obviously f is pointwise discontinuous in $\mathfrak{A}$. But $\mathfrak{d}_\omega$ is an infinite set for $\omega \leq 1$, as in this case it is formed of $\mathfrak{N}$, and the point 0.

Examples of Discontinuous Functions

465. In volume I, 330 *seq.* and 348 *seq.*, we have given examples of discontinuous functions. We shall now consider a few more.

Example 1. Riemann's Function.

Let (x) be the difference between x and the nearest integer; and when x has the form $n + \frac{1}{2}$, let $(x) = 0$. Obviously (x) has the period 1.

It can be represented by Fourier's series thus:

$$(x) = \frac{1}{\pi}\left\{\frac{\sin 2\pi x}{1} - \frac{\sin 2 \cdot 2\pi x}{2} + \frac{\sin 3 \cdot 2\pi x}{3} - \cdots\right\}. \qquad (1$$

Riemann's function is now

$$F(x) = \sum_1^\infty \frac{(nx)}{n^2}. \qquad (2$$

This series is obviously uniformly convergent in $\mathfrak{A} = (-\infty, \infty)$.

Since (x) has the period 1 and is continuous within $(-\frac{1}{2}, \frac{1}{2})$, we see that (nx) has the period $\frac{1}{n}$, and is continuous within $\left(-\frac{1}{2n}, \frac{1}{2n}\right)$. The points of discontinuity of (nx) are thus

$$\mathfrak{E}_n = \left\{\frac{1}{2n} + \frac{s}{n}\right\} \quad , \quad s = 0, \pm 1, \pm 2, \cdots$$

Let $\mathfrak{E} = \{\mathfrak{E}_n\}$. Then at any x not in $\mathfrak{E}$, each term of 2) is a continuous function of x. Hence $F(x)$ is continuous at this point.

On the other hand, F is discontinuous at any point e of $\mathfrak{E}$. For F being uniformly convergent,

$$R\lim_{x=e} F(x) = \Sigma R \lim_{x=e} \frac{(nx)}{n^2} \qquad (3$$

$$L\lim_{x=e} F(x) = \Sigma L \lim_{x=e} \frac{(nx)}{n^2}. \qquad (4$$

We show now that 3) has the value

$$F(e) - \frac{\pi^2}{16 n^2}, \qquad \text{for } e = \frac{2s+1}{2n}, \qquad e \text{ irreducible.} \tag{5}$$

and 4) the value

$$F(e) + \frac{\pi^2}{16 n^2}. \tag{6}$$

Hence

$$\operatorname*{Disc}_{x=e} F(x) = \frac{\pi^2}{8 n^2}. \tag{7}$$

To this end let us see when two of the numbers

$$\frac{1}{2m} + \frac{r}{m}, \quad \text{and} \quad \frac{1}{2n} + \frac{s}{n} \qquad m \neq n$$

are equal. If equal, we have

$$\frac{2r+1}{m} = \frac{2s+1}{n}. \tag{8}$$

Thus if we take $2s+1$ relatively prime to n, no two of the numbers in $\mathfrak{E}_n$ are equal. Let us do this for each n. Then no two of the numbers in $\mathfrak{E}$ are equal.

Let now $x = e = \dfrac{1}{2n} + \dfrac{s}{n}$. Then (mx) is continuous at this point, unless 8) holds; *i.e.* unless m is a multiple of n, say $m = ln$. In this case, 8) gives

$$2r + 1 = l(2s+1).$$

Thus l must be odd; $l = 1, 3, 5 \cdots$ In this case $(mx) = 0$ at e, while $R\lim\limits_{x=e} (mx) = -\frac{1}{2}$. When m is not an odd multiple of n, obviously

$$R\lim_{x=e} (mx) = (me).$$

Thus when $m = ln$, l odd,

$$R\lim_{x=e} \frac{(mx)}{m^2} = -\frac{1}{2}\frac{1}{l^2 n^2} = \frac{(mx)}{m^2} - \frac{1}{2}\frac{1}{n^2}\cdot\frac{1}{l^2}.$$

When m is not a multiple of n,

$$R\lim_{x=e} \frac{(mx)}{m^2} = \frac{(mx)}{m^2}.$$

Hence

$$R \lim_{x=e} F(x) = F(e) - \frac{1}{2n^2}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right\}$$

$$= F(e) - \frac{\pi^2}{16n^2}, \qquad \text{by 218.}$$ p. 264.

This establishes 5). Similarly we prove 6). Thus $F(x)$ is discontinuous at each point of $\mathfrak{E}$. As

$$|F(x)| \leq \sum \frac{1}{n^2}$$

F is limited. As the points $\mathfrak{E}$ form an enumerable set, F is R-integrable in any finite interval.

466. *Example 2.* Let $f(x) = 0$ at the points of a Cantor set $C = m \cdot a_1 a_2 \cdots$; $m = 0$, or a positive or negative integer, and the a's $= 0$ or 2. Let $f(x) = 1$ elsewhere. Since $f(x)$ admits the period 1, $f(3nx)$ admits the period $\frac{1}{3n}$. Let C_1 be the points of C which fall in $\mathfrak{A} = (0, 1)$. Let D_1 be the corresponding set of intervals. Let $C_2 = C_1 + \Gamma_1$, where Γ_1 is obtained by putting a C_1 set in each interval of D_1. Let D_2 be the intervals corresponding to C_2. Let $C_3 = C_2 + \Gamma_2$ where Γ_2 is obtained by putting a C_2 set in each interval of D_2, etc.

The zeros of $f(3nx)$ are obviously the points of C_n. Let

$$F = \sum \frac{1}{n^2} f(3nx) = \Sigma f_n(x).$$

The zeros of F are the points of $\mathfrak{C} = \{C_n\}$. Since each C_n is a null set, $\mathfrak{C}$ is also a null set. Let $A = \mathfrak{A} - \mathfrak{C}$. The points A, $\mathfrak{C}$ are each pantactic in $\mathfrak{A}$. Obviously F converges uniformly in $\mathfrak{A}$, since $0 \leq f(3nx) \leq 1$. Since $f_n(x)$ is continuous at each point a of A, F is continuous at a, and

$$F(a) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = H.$$

We show now that F is discontinuous at each point of $\mathfrak{E}$. For let e_m be an end point of one of the intervals of D_{m+1} but not of D_m. Then

$$f_1(e_m) = \frac{1}{1^2}\quad,\quad \cdots f_m(e_m) = \frac{1}{m^2},$$

$$f_{m+p}(e_m) = 0\quad,\quad p = 1, 2 \cdots$$

Hence $$F(e_m) = H_m = \frac{1}{1^2} + \cdots + \frac{1}{m^2}.$$

As the points A are pantactic in $\mathfrak{A}$, there exists a sequence in A which $\doteq e$. For this sequence $F \doteq H$. Hence

$$\operatorname*{Disc}_{x=e_m} F = H - H_m = \overline{H}_m.$$

Similarly, if η_m is not an end point of the intervals D_{m+1}, but a limiting point of such end points,

$$\operatorname*{Disc}_{x=\eta_m} = \overline{H}_m.$$

The function F is R-integrable in $\mathfrak{A}$ since its points of discontinuity $\mathfrak{E}$ form a null set.

467. *Let $\mathfrak{E} = \{e_{\iota_1 \cdots \iota_s}\}$ be an enumerable set of points lying in the limited or unlimited set $\mathfrak{A}$, which lies in $\mathfrak{R}_m$. For any x in $\mathfrak{A}$ and any e_ι in $\mathfrak{E}$, let $x - e_\iota$ lie in $\mathfrak{B}$. Let $g(x_1 \cdots x_m)$ be limited in $\mathfrak{B}$ and continuous, except at $x = 0$, where*

$$\operatorname{Disc} g(x) = \mathfrak{d}.$$

Let $C = \Sigma c_{\iota_1 \cdots \iota_s}$ converge absolutely. Then

$$F(x_1 \cdots x_m) = \Sigma c_\iota g(x - e_\iota)$$

is continuous in $A = \mathfrak{A} - \mathfrak{E}$, and at $x = e_\iota$,

$$\operatorname{Disc} F(x) = c_\iota \mathfrak{d}.$$

For when x ranges over $\mathfrak{A}$, $x - e_\iota$ remains in $\mathfrak{B}$, and g is limited in $\mathfrak{B}$. Hence F is uniformly and absolutely convergent in $\mathfrak{A}$.

Now each $g(x - e_\iota)$ is continuous in A; hence F is continuous in A by 147, 2.

On the other hand, F is discontinuous at $x = e_\kappa$. For

$$F(x) = c_\kappa g(x - e_\kappa) + H(x),$$

where H is the series F after removing the term on the right of the last equation. But H, as has just been shown, is continuous at $x = e_\kappa$.

468. *Example 1.* Let $\mathfrak{E} = \{e_n\}$ denote the rational numbers.

Let

$$g(x) = \sin\frac{\pi}{x} \quad , \quad x \neq 0$$
$$= 0 \quad , \quad x = 0.$$

Then

$$F(x) = \sum \frac{1}{n_\mu} g(x - e_n) \quad , \quad \mu > 1$$

is continuous, except at the points $\mathfrak{E}$. At $x = e_n$,

$$\text{Disc } F = \frac{2}{n^\mu}.$$

Example 2. Let $\mathfrak{E} = \{e_n\}$ denote the rational numbers.

Let

$$g(x) = \lim_{n=\infty} \frac{nx}{1 + nx} = 1 \quad , \quad x \neq 0$$
$$= 0 \quad , \quad x = 0,$$

which we considered in I, 331.

Then

$$F(x) = \sum \frac{1}{n!} g(x - e_n)$$

is continuous, except at the rational points, and at $x = e_m$,

$$\text{Disc } F(x) = \frac{1}{m!}.$$

469. In the foregoing $g(x)$ is limited. This restriction may be removed in many cases, as the reader will see from the following theorem, given as an example.

Let $E = \{e_{\iota_1 \cdots \iota_s}\}$ be an enumerable apantactic set in $\mathfrak{A}$. Let $\mathfrak{E} = (E, E')$. For any x in $\mathfrak{A}$, and any e_ι in E, let $x - e_\iota$ lie within a cube $\mathfrak{B}$. Let $g(x_1 \cdots x_m)$ be continuous in $\mathfrak{B}$ except at $x = 0$, where $g \doteq +\infty$, as $x \doteq 0$. Let $\Sigma c_{\iota_1 \cdots \iota_s}$ be a positive term convergent series.

Then
$$F(x_1 \cdots x_m) = \Sigma c_\iota g(x - e_\iota)$$
is continuous in $A = \mathfrak{A} - \mathfrak{E}$. *On the other hand, each point of* $\mathfrak{E}$ *is a point of infinite discontinuity.*

For any given point $x = a$ of A lies at a distance > 0 from $\mathfrak{E}$. Thus
$$\text{Min}\,(x - e_\iota) > 0,$$
as x ranges over some $V_\eta(a)$, and e_ι over E.

Hence
$$|\,g(x - e_\iota)\,| < \text{some } M,$$
for x in $V_\eta(a)$, and e_ι in E. Thus F is uniformly convergent at $x = a$. As each $g(x - e_\iota)$ is continuous at $x = a$, F is continuous at a.

Let next $x = e_\kappa$. Then there exists a sequence
$$x',\ x'' \cdots \doteq e_\kappa \tag{1}$$
whose points lie in A. Thus the term $g(x - e_\kappa) \doteq +\infty$ as x ranges over 1). Hence a fortiori $F = +\infty$. Thus each point of E is a point of infinite discontinuity.

Finally any limit point of E is a point of infinite discontinuity, by 462, 1.

470. *Example.* Let $g(x) = \dfrac{1}{x}$, $a_n = -\dfrac{1}{a^n}$, $a > 1$.
$$c_n = \frac{1}{a^n n!}.$$
Then
$$F(x) = \Sigma c_n g(x - a_n)$$
$$= \sum \frac{1}{n!} \frac{1}{1 + a^n x}$$
is a continuous function, except at the points
$$0,\ -\frac{1}{a},\ -\frac{1}{a^2},\ -\frac{1}{a^3} \cdots$$
which are points of infinite discontinuity.

471. Let us show how to construct functions by limiting processes, whose points of discontinuity are any given complete limited apantactic set $\mathfrak{E}$ in an m-way space $\mathfrak{R}_m$.

1. Let us for simplicity take $m = 2$, and call the coördinates of a point x, y.

Let Q denote the square whose center is the origin, and one of whose vertices is the point $(1, 0)$.

The edge of Q is given by the points x, y satisfying

$$|x| + |y| = 1. \tag{1}$$

Thus

$$Q(x, y) = \lim_{n=\infty} \frac{1}{1 + (|x| + |y|)^n} = \begin{cases} \frac{1}{2}, \text{ on the edge} \\ 1, \text{ inside} \\ 0, \text{ outside} \end{cases} \tag{2}$$

of the square Q. Hence

$$L(x, y) = \frac{1}{2}\left[1 - \lim_{n=\infty} \frac{n\{1 - |x| - |y|\}}{1 + n\{1 - |x| - |y|\}}\right] = \begin{cases} \frac{1}{2}, \text{ on the edge,} \\ 0, \text{ off the edge.} \end{cases} \tag{3}$$

Thus

$$G(x, y) = Q(x, y) + L(x, y) = \begin{cases} 1, \text{ in } Q, \\ 0, \text{ without } Q. \end{cases}$$

2. We next show how to construct a function g which shall $= 0$ on one or more of the edges of Q. Let us call these sides e_1, e_2, e_3, e_4, as we go around the edge of Q beginning with the first quadrant. If $G = 0$ on e_ι, let us denote it by G_ι; if $G = 0$ on e_ι, e_κ let us denote it by $G_{\iota\kappa}$, etc. We begin by constructing G_1. We observe that

$$1 - \lim_{n=\infty} \frac{n|t|}{1 + n|t|} = \begin{cases} 1, \text{ for } t = 0, \\ 0, \text{ for } t \neq 0. \end{cases}$$

Now the equation of a right line l may be given the form

$$x \cos \alpha + y \sin \alpha = p$$

where $0 \leq \alpha < 2\pi$, $p \geq 0$. Hence

$$Z(x, y) = 1 - \lim_{n=\infty} \frac{n|x \cos \alpha + y \sin \alpha - p|}{1 + n|x \cos \alpha + y \sin \alpha - p|} = \begin{cases} 1, \text{ on } l, \\ 0, \text{ off } l. \end{cases}$$

If now we make l coincide with e_1, we see that

$$E_1(x, y) = 2\, Z(x, y)\, L(x, y) = \begin{cases} 1, \text{ on } e_1, \\ 0, \text{ off } e_1. \end{cases}$$

Hence

$$G_1(x, y) = G(x, y) - E_1(x, y) = \begin{cases} 1, \text{ in } Q \text{ except on } e_1, \\ 0, \text{on } e_1 \text{ and without } Q. \end{cases}$$

In the same way,

$$G_{\iota j} = G - (E_\iota + E_j),$$

$$G_{\iota j\kappa} = G - (E_\iota + E_j + E_\kappa),$$

$$G_{1234} = G - (E_1 + E_2 + E_3 + E_4).$$

By introducing a constant factor we can replace Q by a square Q_c whose sides are in the ratio $c: 1$ to those of Q.

Thus

$$Q(x, y) = \lim_{n=\infty} \frac{1}{1 + \left(\frac{|x|}{c} + \frac{|y|}{c}\right)^n} = \begin{cases} \frac{1}{2}, \text{ on the edge of } Q_c, \\ 1, \text{ inside}, \\ 0, \text{ outside}. \end{cases}$$

We can replace the square Q by a similar square whose center is a, b on replacing $|x|$, $|y|$ by $|x - a|$, $|y - b|$.

We have thus this result: by a limiting process, we can construct a function $g(x, y)$ having the value 1 inside Q, and on any of its edges, and $= 0$ outside Q, and on the remaining edges. Q has any point a, b as center, its edges have any length, and its sides are tipped at an angle of 45° to the axes.

We may take them parallel to the axes, if we wish, by replacing $|x|$, $|y|$ in our fundamental relation 1) by

$$|x - y| \quad , \quad |x + y|.$$

Finally let us remark that we may pass to m-way space, by replacing 1) by

$$|x_1| + |x_2| + \cdots + |x_m| = 1.$$

3. Let now $\mathfrak{Q} = \{\mathfrak{q}_n\}$ be a border set [328], of non-overlapping squares belonging to the complete apantactic set $\mathfrak{C}$, such that $\mathfrak{Q} + \mathfrak{C} = \mathfrak{R}$ the whole plane. We mark these squares in the plane and note which sides $\mathfrak{q}_n$ has in common with the preceding $\mathfrak{q}$'s. We take the $g_n(xy)$ function so that it is $= 1$ in $\mathfrak{q}_n$, except on these sides, and there 0. Then

$$G(x, y) = \Sigma g_n(xy)$$

has for each point only one term $\neq 0$, if x, y lies in $\mathfrak{Q}$, and no term $\neq 0$ if it lies in $\mathfrak{C}$.

Hence

$$G(xy) = \begin{cases} 1, \text{ for each point of } \mathfrak{Q}, \\ 0, \text{ for each point of } \mathfrak{C}. \end{cases}$$

Since $\mathfrak{C}$ is apantactic, each point of $\mathfrak{C}$ is a point of discontinuity of the 2° kind; each point of $\mathfrak{Q}$ is a point of continuity.

4. Let $f(xy)$ be a function defined over $\mathfrak{A}$ which contains the complete apantactic set $\mathfrak{C}$.

Then
$$F(xy) = \Sigma f(xy) g_n(xy) = \begin{cases} f(xy), \text{ in } \mathfrak{A} - \mathfrak{C}, \\ 0, \text{ in } \mathfrak{C}. \end{cases}$$

472. 1. Let $\mathfrak{A} = (0, 1)$, $\mathfrak{B}_n =$ the points $\dfrac{2m+1}{2^n}$ in $\mathfrak{A}$.

Then $\mathfrak{B}_n$, $\mathfrak{B}_\nu$ have no points in common.
Let $f_n(x) = 1$ in $\mathfrak{B}_n$, and $= 0$ in $B_n = \mathfrak{A} - \mathfrak{B}_n$.
Let $\mathfrak{B} = \{\mathfrak{B}_n\}$. Then
$$F(x) = \Sigma f_n(x) = \begin{cases} 1, \text{ in } \mathfrak{B}, \\ 0, \text{ in } B = \mathfrak{A} - \mathfrak{B}. \end{cases}$$

The function F is totally discontinuous in $\mathfrak{B}$, oscillating between 0 and 1. The series F does not converge uniformly in any subinterval of $\mathfrak{A}$.

2. Keeping the notation in 1, let
$$G(x) = \sum \frac{1}{n} f_n(x).$$

At each point of $\mathfrak{B}_n$, $G = \dfrac{1}{n}$, while $G = 0$ in B.

The function G is discontinuous at the points of $\mathfrak{B}$, but continuous at the points B. The series G converges uniformly in $\mathfrak{A}$, yet an infinity of terms are discontinuous in any interval in $\mathfrak{A}$.

473. Let the limited set $\mathfrak{A}$ be the union of an enumerable set of complete sets $\{\mathfrak{A}_n\}$. We show how to construct a function f, which is discontinuous at the points of $\mathfrak{A}$, but continuous elsewhere in an m-way space.

Let us suppose first that $\mathfrak{A}$ consists of but one set and is complete. A point all of whose coördinates are rational, let us call rational, the other points of space we will call non-rational. If $\mathfrak{A}$ has an inner rational point, let $f = 1$ at this point, on the frontier of $\mathfrak{A}$ let $f = 1$ also; at all other points of space let $f = 0$. Then each point a of $\mathfrak{A}$ is a point of discontinuity. For if x is a fron-

tier or an inner rational point of $\mathfrak{A}, f(x) = 1$, while in any $V(x)$ there are points where $f = 0$. If x is not in $\mathfrak{A}$, all the points of some $D(x)$ are also not in $\mathfrak{A}$. At these points $f = 0$. Hence f is continuous at such points.

We turn now to the general case. We have

$$\mathfrak{A} = A_1 + A_2 + A_3 + \cdots$$

where $A_1 = \mathfrak{A}_1$, $A_2 =$ points of $\mathfrak{A}_2$ not in $\mathfrak{A}_1$, etc. Let $f_1 = 1$ at the rational inner points of A_1, and at the frontier points of $\mathfrak{A}_1$; at all other points let $f_1 = 0$. Let $f_2 = \frac{1}{2}$ at the rational inner points of A_2, and at the frontier points of A_2 not in A_1; at all other points let $f_2 = 0$. At similar points of A_3 let $f_3 = \frac{1}{3}$, and elsewhere $= 0$, etc.

Consider now
$$F = \Sigma f_n(x_1 \cdots x_m).$$

Let $x = a$ be a point of $\mathfrak{A}$. If it is an inner point of some A_s, it is obviously a point of discontinuity of F. If not, it is a proper frontier point of one of the A's. Then in any $D(a)$ there are points of space not in $\mathfrak{A}$, or there are points of an infinite number of the A's. In either case a is a point of discontinuity. Similarly we see F is continuous at a point not in $\mathfrak{A}$.

2. We can obviously generalize the preceding problem by supposing $\mathfrak{A}$ to lie in a complete set $\mathfrak{B}$, such that each frontier point of $\mathfrak{A}$ is a limit point of $A = \mathfrak{B} - \mathfrak{A}$.

For we have only to replace our m-way space by $\mathfrak{B}$.

Functions of Class 1

474. 1. Baire has introduced an important classification of functions as follows:

Let $f(x_1 \cdots x_m)$ be defined over $\mathfrak{A}$; f and $\mathfrak{A}$ limited or unlimited. If f is continuous in $\mathfrak{A}$, we say its class is 0 in $\mathfrak{A}$, and write

$$\text{Class } f = 0 \quad , \quad \text{or } \text{Cl } f = 0 \quad , \quad \text{Mod } \mathfrak{A}.$$

If
$$f = \lim_{n=\infty} f_n(x_1 \cdots x_m),$$

each f_n being of class 0 in $\mathfrak{A}$, we say its class is 1, if f does not lie in class 0, mod $\mathfrak{A}$.

2. Let the series
$$F(x) = \Sigma f_n(x)$$
converge in $\mathfrak{A}$, each term f_n being continuous in $\mathfrak{A}$. Since
$$F(x) = \lim_{n=\infty} F_n(x),$$
we see F is of class 0, or class 1, according as F is continuous, or not continuous in $\mathfrak{A}$. A similar remark holds for infinite products
$$G(x) = \Pi g_n(x).$$

3. The derivatives of a function $f(x)$ give rise to functions of class 0 or 1. For let $f(x)$ have a unilateral differential coefficient $g(x)$ at each point of $\mathfrak{A}$. Both f and $\mathfrak{A}$ may be unlimited. To fix the ideas, suppose the right-hand differential coefficient exists. Let $h_1 > h_2 > \cdots \doteq 0$. Then
$$q_n(x) = \frac{f(x+h_n) - f(x)}{h_n}, \qquad x + h_n \text{ in } \mathfrak{A},$$
is a continuous function of x in $\mathfrak{A}$. But
$$q(x) = \lim_{n=\infty} q_n(x)$$
exists at each x in $\mathfrak{A}$ by hypothesis.

A similar remark applies to the partial derivatives
$$\frac{\partial f}{\partial x_1}, \cdots \frac{\partial f}{\partial x_m}$$
of a function $f(x_1 \cdots x_n)$.

4. Let
$$f(x) = \lim_{n=\infty} f_n(x_1 \cdots x_m),$$
each f_n being of class 1 in $\mathfrak{A}$. Then we say, $\mathrm{Cl}\, f = 2$ if f does not lie in a lower class. In this way we may continue. It is of course necessary to show that such functions actually exist.

475. *Example 1.*

Let
$$f(x) = \lim_{n=\infty} \frac{nx}{1+nx} = \begin{cases} 1, & \text{for } x > 0, \\ 0, & \text{for } x = 0. \end{cases}$$

This function was considered in I, 331. In any interval $\mathfrak{A} = (0 < b)$ containing the origin $x = 0$, $\mathrm{Cl}\, f = 1$; in any interval $(a < b)$, $a > 0$, not containing the origin, $\mathrm{Cl}\, f = 0$.

Example 2.

Let
$$f(x) = \lim_{n=\infty} \frac{nx}{e^{nx^2}} = 0, \quad \text{in } \mathfrak{A} = (-\infty, \infty).$$

The class of $f(x)$ is 0 in $\mathfrak{A}$. Although each f_n is limited in $\mathfrak{A}$, the graphs of f_n have peaks near $x = 0$ which $\doteq \infty$, as $n \doteq \infty$.

Example 3. If we combine the two functions in Ex. 1, 2, we get
$$f(x) = \lim_{n=\infty} \left\{ \frac{1}{1+nx} + \frac{1}{e^{nx^2}} \right\} nx = \begin{cases} 1, \text{ for } x \neq 0, \\ 0, \text{ for } x = 0. \end{cases}$$

Hence $\mathrm{Cl}\, f(x) = 1$ for any set $\mathfrak{B}$ embracing the origin; $= 0$ for any other set.

Example 4.

Let
$$f(x) = \lim_{n=\infty} xe^{\frac{1}{x+\frac{1}{n}}}, \quad \text{in } \mathfrak{A} = (0, 1).$$

Then
$$f(x) = 0 \quad , \quad \text{for } x = 0$$
$$= xe^{\frac{1}{x}} \quad , \quad \text{for } x > 0.$$

We see thus that f is continuous in $(0^*, 1)$, and has a point of infinite discontinuity at $x = 0$.

Hence
$$\text{Class } f(x) = 1, \quad \text{in } \mathfrak{A}$$
$$= 0, \quad \text{in } (0^*, 1).$$

Example 5.

Let
$$f(x) = \lim_{n=\infty} \frac{1}{x + \frac{1}{n}} \qquad \text{in } \mathfrak{A} = (0, \infty).$$

Then
$$f(x) = \frac{1}{x} \quad , \quad \text{for } x > 0$$
$$= +\infty \quad , \quad \text{for } x = 0.$$

Here
$$\lim_{n=\infty} f_n(x)$$
does not exist at $x = 0$. We cannot therefore speak of the class of $f(x)$ in $\mathfrak{A}$ since it is not defined at the point $x = 0$. It is defined in $\mathfrak{B} = (0^*, \infty)$, and its class is obviously 0, mod $\mathfrak{B}$.

Example 6.

Let
$$f(x) = \sin\frac{1}{x} \quad , \quad \text{for } x \neq 0$$
$$= \text{a constant } c \quad , \quad \text{for } x = 0.$$

We show that $\mathrm{Cl}\, f = 1$ in $\mathfrak{A} = (-0, \infty)$. For let

$$f_n(x) = c\left(1 - \frac{nx}{1+nx}\right) + \frac{nx}{1+nx} \sin\left[\frac{1}{x + \frac{1}{n}}\right]$$
$$= g_n(x) + h_n(x).$$

Now by Ex. 1,
$$\lim g_n(x) = \begin{cases} 0, \text{ for } x > 0, \\ c, \text{ for } x = 0\,; \end{cases}$$
while
$$\lim h_n(x) = \begin{cases} \sin\frac{1}{x}, \text{ for } x > 0, \\ 0, \text{ for } x = 0. \end{cases}$$

As each f_n is continuous in $\mathfrak{A}$, and
$$\lim f_n(x) = f(x) \text{ in } \mathfrak{A},$$
we see its class is ≤ 1. As f is discontinuous at $x = 0$, its class is not 0 in $\mathfrak{A}$.

Example 7. Let
$$f(x) = \lim_{n=\infty} \frac{1}{n} \cdot \sin\frac{1}{x}.$$

Here the functions $f_n(x)$ under the limit sign are not defined for $x = 0$. Thus f is not defined at this point. We cannot therefore speak of the class of f with respect to any set embracing the point $x = 0$. For any set $\mathfrak{B}$ not containing this point, $\mathrm{Cl}\, f = 0$, since $f(x) = 0$ in $\mathfrak{B}$.

Let us set
$$\phi(x) = \sin\frac{1}{x} \quad , \quad \text{for } x \neq 0$$
$$= \text{a constant } c \quad , \quad \text{for } x = 0.$$

Let
$$g(x) = \lim_{n=\infty} \frac{1}{n}\phi(x) = \lim_{n=\infty} \phi_n(x).$$

Here g is a continuous function in $\mathfrak{A} = (-\infty, \infty)$. Its class is thus 0 in $\mathfrak{A}$. On the other hand, the functions ϕ_n are each of class 1 in $\mathfrak{A}$.

Example 8.

$$\Gamma(x) = \frac{1}{x} \prod_1^\infty \frac{\left(1 + \frac{1}{n}\right)^x}{1 + \frac{x}{n}}$$

is defined at all the points of $(-\infty, \infty)$ except $0, -1, -2, \cdots$ These latter are points of infinite discontinuity. In its domain of definition, Γ is a continuous function. Hence $\mathrm{Cl}\, \Gamma(x) = 0$ with respect to this domain.

476. 1. If $\mathfrak{A}$, limited or unlimited, is the union of an enumerable set of complete sets, we say $\mathfrak{A}$ is *hypercomplete*.

Example 1. The points S^* within a unit sphere S, form a hypercomplete set. For let Σ_r have the same center as S, and radius $r < 1$. Obviously each Σ_r is complete, while $\{\Sigma_r\} = S^*$, r ranging over $r_1 < r_2 < \cdots \doteq 1$.

Example 2. An enumerable set of points $a_1, a_2 \cdots$ form a hypercomplete set. For each a_n may be regarded as a complete set, embracing but a single point.

2. *If $\mathfrak{A}_1, \mathfrak{A}_2 \cdots$ are limited hypercomplete sets, so is their union $\{\mathfrak{A}_n\} = \mathfrak{A}$.*

For each $\mathfrak{A}_m$ is the union of an enumerable set of complete sets $\mathfrak{A}_{m,n}$. Thus $\mathfrak{A} = \{\mathfrak{A}_{m,n}\}$ $m, n = 1, 2 \cdots$ is hypercomplete.

Let $\mathfrak{A}$ be complete. If $\mathfrak{B}$ is a complete part of $\mathfrak{A}$, $A = \mathfrak{A} - \mathfrak{B}$ is hypercomplete.

For let $\mathfrak{Q} = \{\mathfrak{q}_n\}$ be a border set of $\mathfrak{B}$, as in 328. The points A_n of A in each $\mathfrak{q}_n$ are complete, since $\mathfrak{A}$ is complete. Thus $A = \{A_n\}$, and A is hypercomplete.

Let $\mathfrak{A} = \{\mathfrak{A}_n\}$ be hypercomplete, each $\mathfrak{A}_n$ being complete. If $\mathfrak{B}$ is a complete part of $\mathfrak{A}$, $A = \mathfrak{A} - \mathfrak{B}$ is hypercomplete.

For let A_n denote the points of $\mathfrak{A}_n$ not in $\mathfrak{B}$. Then as above, A_n is hypercomplete. As $A = \{A_n\}$, A is also hypercomplete.

477. 1. $\mathfrak{E}_\epsilon$ *Sets.* If the limited or unlimited set $\mathfrak{A}$ is the union of an enumerable set of limited complete sets, in each of which $\text{Osc} f < \epsilon$, we shall say $\mathfrak{A}$ *is an* $\mathfrak{E}_\epsilon$ *set.* If, however small $\epsilon > 0$ is taken, $\mathfrak{A}$ is an $\mathfrak{E}_\epsilon$ set, we shall say $\mathfrak{A}$ is an $\mathfrak{E}_\epsilon$ set, $\epsilon \doteq 0$, which we may also express by $\mathfrak{E}_{\epsilon \doteq 0}$.

2. *Let $f(x_1 \cdots x_m)$ be continuous in the limited complete set $\mathfrak{A}$. Then $\mathfrak{A}$ is an $\mathfrak{E}_\epsilon$ set, $\epsilon \doteq 0$.*

For let $\epsilon > 0$ be taken small at pleasure and fixed. By I, 353, there exists a cubical division of space D, such that if $\mathfrak{A}_n$ denote the points of $\mathfrak{A}$ in one of the cells of D, $\text{Osc} f < \epsilon$ in $\mathfrak{A}_n$. As $\mathfrak{A}_n$ is complete, since $\mathfrak{A}$ is, $\mathfrak{A}$ is an $\mathfrak{E}_\epsilon$ set.

3. *An enumerable set of points $\mathfrak{A} = \{a_n\}$ is an $\mathfrak{E}_{\epsilon \doteq 0}$ set.*

For each a_n may be regarded as a complete set, embracing but a *single* point. But in a set embracing but one point, $\text{Osc} f = 0$.

4. *The union of an enumerable set of $\mathfrak{E}_\epsilon$ sets $\mathfrak{A} = \{\mathfrak{A}_m\}$ is an $\mathfrak{E}_\epsilon$ set.*

For each $\mathfrak{A}_m$ is the union of an enumerable set of limited sets $\mathfrak{A}_m = \{\mathfrak{A}_{m,n}\}$, $n = 1, 2, \cdots$ and $\text{Osc} f < \epsilon$ in each $\mathfrak{A}_{mn}$.

Thus
$$\mathfrak{A} = \{\mathfrak{A}_{mn}\} \quad , \quad m, n = 1, 2, \cdots$$

But an enumerable set of enumerable sets is an enumerable set. Hence $\mathfrak{A}$ is an $\mathfrak{E}_\epsilon$ set.

5. *Let $f(x_1 \cdots x_m)$ be continuous in the complete set $\mathfrak{A}$, except at the points $\mathfrak{D} = d_1, d_2 \cdots d_s$. Then $\mathfrak{A}$ is an $\mathfrak{E}_{\epsilon \doteq 0}$ set.*

For let $\epsilon > 0$ be taken small at pleasure and fixed. About each point of $\mathfrak{D}$ we describe a sphere of radius ρ. Let $\mathfrak{A}_\rho$ denote the points of $\mathfrak{A}$ not *within* one of these spheres. Obviously $\mathfrak{A}_\rho$ is complete. Let ρ range over $r_1 > r_2 > \cdots \doteq 0$. If we set $\mathfrak{A} = A + \mathfrak{D}$, obviously $A = \{\mathfrak{A}_{r_n}\}$. As f is continuous in $\mathfrak{A}_{r_n}$, it is an $\mathfrak{E}_\epsilon$ set. Hence $\mathfrak{A}$, being the union of A and $\mathfrak{D}$, is an $\mathfrak{E}_\epsilon$ set.

478. 1. *Let $\mathfrak{A}$ be an $\mathfrak{E}_\epsilon$ set. The points $\mathfrak{D}$ of $\mathfrak{A}$ common to the limited complete set $\mathfrak{B}$ form an $\mathfrak{E}_\epsilon$ set.*

For $\mathfrak{A}$ is the union of the complete sets $\mathfrak{A}_n$, in each of which $\text{Osc} f < \epsilon$. But the points of $\mathfrak{A}_n$ in $\mathfrak{B}$ form a complete set A_n, and of course $\text{Osc} f < \epsilon$ in A_n. As $\mathfrak{D} = \{A_n\}$, it is an $\mathfrak{E}_\epsilon$ set.

2. *Let $\mathfrak{A}$ be a limited $\mathfrak{E}_\epsilon$ set. Let $\mathfrak{B}$ be a complete part of $\mathfrak{A}$. Then $A = \mathfrak{A} - \mathfrak{B}$ is an $\mathfrak{E}_\epsilon$ set.*

For $\mathfrak{A}$ is the union of the complete sets $\mathfrak{A}_n$, in each of which $\text{Osc } f < \epsilon$. The points of $\mathfrak{A}_n$ not in $\mathfrak{B}$ form a set A_n, such that $\text{Osc } f < \epsilon$ in A_n also. But $A = \{A_n\}$, and each A_n being hypercomplete, is an $\mathfrak{E}_\epsilon$ set.

3. Let $f(x_1 \cdots x_m)$ be defined over $\mathfrak{A}$, either f or $\mathfrak{A}$ being limited or unlimited. The points of $\mathfrak{A}$ at which

$$\alpha \leq f \leq \beta \tag{1}$$

may be denoted by

$$(\alpha \leq f \leq \beta). \tag{2}$$

If in 1) one of the equality signs is missing, it will of course be dropped in 2).

479. 1. *Let $f_1, f_2, \cdots$ be continuous in the limited complete set $\mathfrak{A}$. If at each point of $\mathfrak{A}$, $\lim_{n=\infty} f_n$ exists, $\mathfrak{A}$ is an $\mathfrak{E}_{\epsilon \doteq 0}$ set and so is any complete $\mathfrak{B} < \mathfrak{A}$.*

For let $\lim_{n=\infty} f_n(x_1 \cdots x_m) = f(x_1 \cdots x_m)$ in $\mathfrak{A}$. Let us effect a division of norm $\epsilon/2$ of the interval $(-\infty, \infty)$ by interpolating the points $\cdots m_{-2}, m_{-1}, m_0 = 0, m_1, m_2 \cdots$

Let $\mathfrak{A}_\iota = (m_\iota < f < m_{\iota+2})$, then $\mathfrak{A} = \{\mathfrak{A}_\iota\}$.

Next let
$$\mathfrak{D}_{n,p} = \underset{q \geq p}{Dv} \left\{ m_\iota + \frac{1}{n} \leq f_q \leq m_{\iota+2} - \frac{1}{n} \right\}.$$

Then
$$\mathfrak{A}_\iota = \{\mathfrak{D}_{n,p}\} \quad , \quad n, p = 1, 2 \cdots \tag{1}$$

For let a be a point of $\mathfrak{A}_\iota$, and say $f(a) = \alpha$. Then

$$m_\iota < \alpha < m_{\iota+2}.$$

But
$$\alpha - \epsilon < f_q(a) < \alpha + \epsilon \quad , \quad q > \text{some } p,$$

and we may take ϵ and n so that

$$m_\iota + \frac{1}{n} \leq f_q(a) \leq m_{\iota+2} - \frac{1}{n}.$$

Hence a is in $\mathfrak{D}_{n,p}$.

Conversely, let a be a point of $\{\mathfrak{D}_{n,p}\}$. Then a lies in some $\mathfrak{D}_{n,p}$. Hence,

$$m_\iota + \frac{1}{n} \leq f_q(a) \leq m_{\iota+2} - \frac{1}{n} \quad , \quad q \geq p.$$

But as $f_n(a) \doteq f(a)$, we have

$$|f(a) - f_q(a)| < \epsilon \quad , \quad q > \text{some } p'.$$

Hence if ϵ is sufficiently small,

$$m_\iota < f(a) < m_{\iota+2},$$

and thus a is in $\mathfrak{A}_\iota$.

Thus 1) is established. But $\mathfrak{D}_{np}$ is a divisor of complete sets, and is therefore complete. Thus $\mathfrak{A}$ is the union of an enumerable set of complete sets $\{\mathfrak{B}_\iota\}$, in each of which $\operatorname{Osc} f < \epsilon$, ϵ small at pleasure.

Let now $\mathfrak{B}$ be any complete part of $\mathfrak{A}$. Let $\mathfrak{a}_\iota = Dv\,\{\mathfrak{B}, \mathfrak{B}_\iota\}$. Then $\mathfrak{a}_\iota$ is complete, and $\operatorname{Osc} f < \epsilon$, in $\mathfrak{a}_\iota$. Moreover, $\mathfrak{B} = \{\mathfrak{a}_\iota\}$.

Hence $\mathfrak{B}$ is an $\mathfrak{E}_{\epsilon \doteq 0}$ set.

2. *If* Class $f \leq 1$ *in limited complete* $\mathfrak{A}$, f *limited or unlimited,* $\mathfrak{A}$ *is an* $\mathfrak{E}_\epsilon$ *set.*

This is an obvious result from 1.

3. *Let* $f(x_1 \cdots x_m)$ *be a totally discontinuous function in the non-enumerable set* $\mathfrak{A}$. *Then* Class f *is not* 0 *or* 1 *in* $\mathfrak{A}$, *if* $\mathfrak{d} = \operatorname{Disc} f$ *at each point is* $\leq k > 0$.

For in any subset $\mathfrak{B}$ of $\mathfrak{A}$ containing the point x, $\operatorname{Osc} f \geq k$. Hence $\operatorname{Osc} f$ is not $\leq \epsilon$, in any part of $\mathfrak{A}$, if $\epsilon < k$. Thus $\mathfrak{A}$ cannot be an $\mathfrak{E}_\epsilon$ set.

4. *If* Class $f(x_1 \cdots x_m) \leq 1$ *in the limited complete set* $\mathfrak{A}$, *the set* $\mathfrak{B} = (a < f < b)$ *is a hypercomplete set,* a, b *being arbitrary numbers.*

For we have only to take $a = m_\iota$, $b = m_{\iota+2}$. Then $\mathfrak{B} = \mathfrak{A}_\iota$, which, as in 1, is hypercomplete.

480. (*Lebesgue.*) *Let the limited or unlimited function* $f(x_1 \cdots x_m)$ *be defined over the limited set* $\mathfrak{A}$. *If* $\mathfrak{A}$ *may be regarded as an* $\mathfrak{E}_{\epsilon \doteq 0}$ *set with respect to* f, *the class of* f *is* ≤ 1.

For let $\omega_1 > \omega_2 > \cdots \doteq 0$. By hypothesis $\mathfrak{A}$ is the union of a sequence of complete sets

$$\mathfrak{A}_{11} \quad , \quad \mathfrak{A}_{12} \quad , \quad \mathfrak{A}_{13} \cdots \qquad (S_1$$

in each of which $\operatorname{Osc} f \leq \omega_1$. $\mathfrak{A}$ is also the union of a sequence of complete sets

$$\mathfrak{B}_{11} \quad , \quad \mathfrak{B}_{12} \quad , \quad \mathfrak{B}_{13} \cdots \qquad (1$$

in each of which $\text{Osc}\, f \leq \omega_2$. If we superpose the division 1) of $\mathfrak{A}$ on the division S_1, each $\mathfrak{A}_{\iota\kappa}$ will fall into an enumerable set of complete sets, and together they will form an enumerable sequence

$$\mathfrak{A}_{21} \quad , \quad \mathfrak{A}_{22} \quad , \quad \mathfrak{A}_{23} \cdots \qquad (S_2$$

in each of which $\text{Osc}\, f \leq \omega_2$. Continuing in this way we see that $\mathfrak{A}$ is the union of the complete sets

$$\mathfrak{A}_{n1} \quad , \quad \mathfrak{A}_{n2} \quad , \quad \mathfrak{A}_{n3} \cdots \qquad (S_n$$

such that in each set of S_n, $\text{Osc}\, f < \omega_n$, and such that each set lies in some set of the preceding sequence S_{n-1}.

With each $\mathfrak{A}_{n,s}$ we associate a constant C_{ns}, such that

$$|f(x) - C_{ns}| \leq \omega_n \quad , \quad \text{in } \mathfrak{A}_{ns}, \qquad (2$$

and call C_{ns} the corresponding *field constant.*

We show now how to define a sequence of continuous functions $f_1, f_2 \cdots$ which $\doteq f$. To this end we effect a sequence of superimposed divisions of space $D_1, D_2 \cdots$ of norms $\doteq 0$. The vertices of the cubes of D_n we call the *lattice points* L_n. The cells of D_n containing a given lattice point l of L_n form a cube $\mathfrak{Q}$. Let $\mathfrak{A}_{1\iota_1}$ be the first set of S_1 containing a point of $\mathfrak{Q}$. Let $\mathfrak{A}_{2\iota_2}$ be the first set of S_2 containing a point of $\mathfrak{Q}$ lying in $\mathfrak{A}_{1\iota_1}$. Continuing in this way we get

$$\mathfrak{A}_{1\iota_1} \geq \mathfrak{A}_{2\iota_2} \geq \cdots \geq \mathfrak{A}_{n\iota_n}.$$

To $\mathfrak{A}_{n\iota_n}$ belongs the field constant $C_{n\iota_n}$; this we associate with the lattice point l and call it the corresponding *lattice constant.*

Let now $\mathfrak{C}$ be a cell of D_n containing a point of $\mathfrak{A}$. It has 2^n vertices or lattice points. Let P_s denote any product of s different factors $x_{r_1}, x_{r_2}, \cdots x_{r_s}$. We consider the polynomial

$$\phi = AP_n + \Sigma BP_{n-1} + \Sigma CP_{n-2} + \cdots + \Sigma KP_1 + L,$$

the summation in each case extending over all the distinct products of that type. The number of terms in ϕ is, by I, 96,

$$\binom{n}{n} + \binom{n}{1} + \binom{n}{2} + \cdots + 1 = 2^n.$$

We can thus determine the 2^n coefficients of ϕ so that the values of ϕ at the lattice points of $\mathfrak{C}$ are the corresponding lattice constants. Thus ϕ is a continuous function in $\mathfrak{C}$, whose greatest and least values are the greatest and least lattice constants belonging to $\mathfrak{C}$. Each cube $\mathfrak{C}$ containing a point of $\mathfrak{A}$ has associated with it a ϕ function.

We now define $f_n(x_1 \cdots x_m)$ by stating that its value in any cube $\mathfrak{C}$ of D_n, containing a point of $\mathfrak{A}$, is that of the corresponding ϕ function. Since ϕ is linear in each variable, two ϕ's belonging to adjacent cubes have the same values along their common points.

We show now that $f_n(x) \doteq f(x)$ at any point x of $\mathfrak{A}$, or that

$$\epsilon > 0, \quad \nu, \qquad |f(x) - f_n(x)| < \epsilon \quad , \quad n > \nu. \tag{3}$$

Let $\omega_e < \epsilon/8$. Let $\mathfrak{A}_{1\iota_1}$ be the first set in S_1 containing the point x, $\mathfrak{A}_{2\iota_2}$ the first set of S_2 lying in $\mathfrak{A}_{1\iota_1}$ and containing x. Continuing we get

$$\mathfrak{A}_{1\iota_1} \geq \mathfrak{A}_{2\iota_2} \geq \mathfrak{A}_{3\iota_3} \geq \cdots \geq \mathfrak{A}_{e\iota_e}.$$

Let $\mathfrak{P}_e$ be the union of the sets in S_1 preceding $\mathfrak{A}_{1\iota_1}$; of the sets in S_2 preceding $\mathfrak{A}_{2\iota_2}$ and lying in $\mathfrak{A}_{1\iota_1}$, and so on, finally the sets of S_e preceding $\mathfrak{A}_{e\iota_e}$, and lying in $\mathfrak{A}_{e-1,\iota_{e-1}}$. Their number being finite, $\delta = \text{Dist}\,(\mathfrak{A}_{e\iota_e}, \mathfrak{P}_e)$ is obviously > 0. We may therefore take $\nu > e$ so large that cubes of D_ν about the point x lie wholly in $D_\eta(x)$, $\eta < \delta$.

Consider now $f_n(x)$, $n > \nu$, and *let us suppose first* that x is not a lattice point of D_n. Let it lie within the cell $\mathfrak{C}$ of D_n. Then $f_n(x)$ is a mean of the values of

$$f_n(l) = C_{n, j_n},$$

where l is any one of the 2^n vertices of $\mathfrak{C}$, and C_{nj_n} is the corresponding lattice constant, which we know is associated with the set $\mathfrak{A}_{nj_n}$.

We observe now that each of the

$$\mathfrak{A}_{nj_n} \leq \mathfrak{A}_{e,\iota_e}. \tag{4}$$

For each set in S_n is a part of some set in any of the preceding sequences. Now $\mathfrak{A}_{nj_n}$ cannot be a part of $\mathfrak{A}_{1k}$, $k < \iota_1$, for none of

these points lie in $D_\eta(x)$. Hence $\mathfrak{A}_{nj_n}$ is a part of $\mathfrak{A}_{1\iota_1}$. For the same reason it is a part of $\mathfrak{A}_{2\iota_2}$, etc., which establishes 4).

Let now x' be a point of $\mathfrak{A}_{nj_n}$. Then

$$|C_{nj_n} - C_{e\iota_e}| \leq |C_{nj_n} - f(x')| + |f(x') - C_{e\iota_e}|$$

$$\leq \omega_n + \omega_e < \frac{\epsilon}{4}, \quad \text{by 2).} \tag{5}$$

From this follows, since $f_n(x)$ is a mean of these C_{nj_n}, that

$$|f_n(x) - C_{nj_n}| < \frac{\epsilon}{2}. \tag{6}$$

But now

$$|f(x) - f_n(x)| \leq |f(x) - C_{nj_n}| + |C_{nj_n} - f_n(x)|. \tag{7}$$

As x lies in $\mathfrak{A}_{e\iota_e}$,

$$|f(x) - C_{nj_n}| < |f(x) - C_{e\iota_e}| + |C_{e\iota_e} - C_{nj_n}|$$

$$\leq \omega_e + \frac{\epsilon}{4} < \frac{\epsilon}{2}, \tag{8}$$

by 2), 5). From 6), 8) we have 3) for the present case.

The case that x is a lattice point for some division and hence for all following, has really been established by the foregoing reasoning.

481. 1. *Let f be defined over the limited set $\mathfrak{A}$. If for arbitrary a, b, the sets $\mathfrak{B} = (a < f < b)$ are hypercomplete, then* Class $f \leq 1$.

For let us effect a division of norm $\epsilon/2$ of $(-\infty, \infty)$ as in 479, 1. Then $\mathfrak{A} = \{\mathfrak{A}_\iota\}$, where as before $\mathfrak{A}_\iota = (m_\iota < f < m_{\iota+2})$. But as Osc $f < \epsilon$ in $\mathfrak{A}_\iota$, and as each $\mathfrak{A}_\iota$ is hypercomplete by hypothesis, our theorem is a corollary of 480.

2. *For $f(x_1 \cdots x_m)$ to be of class ≤ 1 in the limited complete set $\mathfrak{A}$, it is necessary and sufficient that the sets $(a < f < b)$ are hypercomplete, a, b being arbitrary.*

This follows from 1 and 479, 2.

3. *Let limited $\mathfrak{A}$ be the union of an enumerable set of complete sets $\{\mathfrak{A}_n\}$, such that* Cl $f \leq 1$ *in each $\mathfrak{A}_n$, then* Cl $f \leq 1$ *in $\mathfrak{A}$.*

For by 479, 1, $\mathfrak{A}_n$ is the union of an enumerable set of complete sets in each of which $\mathrm{Osc} f \leq \epsilon$. Thus $\mathfrak{A}$ is also such a set, *i.e.* an $\mathfrak{E}_\epsilon$ set. We now apply 480, 1.

4. *If* $\mathrm{Class} f \leq 1$ *in the limited complete set* $\mathfrak{A}$, *its class is* ≤ 1, *in any complete part* $\mathfrak{B}$ *of* $\mathfrak{A}$.

This follows from 479, 1 and 480, 1.

482. 1. *Let* $f(x_1 \cdots x_m)$ *be defined over the complete set* $\mathfrak{A}$, *and have only an enumerable set* $\mathfrak{E}$ *of points of discontinuity in* $\mathfrak{A}$. *Then* $\mathrm{Class} f = 1$ *in* $\mathfrak{A}$.

For the points E of $\mathfrak{A}$ at which $\mathrm{Osc} f \geq \epsilon/2$ form a complete part of $\mathfrak{A}$, by 462, 3. But E, being a part of $\mathfrak{E}$, is enumerable and is hence an $\mathfrak{E}_\epsilon$ set by 477, 3. Let us turn to $\mathfrak{B} = \mathfrak{A} - E$. For each of its points b, there exists a $\delta > 0$, such that $\mathrm{Osc} f < \epsilon$ in the set $\mathfrak{b}$ of points of $\mathfrak{B}$ lying in $D_\delta(b)$. As $\mathfrak{A}$ is complete, so is $\mathfrak{b}$. As E is complete, there is an enumerable set of these $\mathfrak{b}$, call them $\mathfrak{b}_1, \mathfrak{b}_2 \cdots$, such that $\mathfrak{B} = \{\mathfrak{b}_s\}$. As $\mathfrak{A} = \mathfrak{B} + E$, it is the union of an enumerable set of complete sets, in each of which $\mathrm{Osc} f < \epsilon$. This is true however small $\epsilon > 0$ is taken. We apply now 480, 1.

2. We can now construct functions of class 2.

Example. Let $f_n(x_1 \cdots x_m) = 1$ at the rational points in the unit cube $\mathfrak{Q}$, whose coördinates have denominators $\leq n$. Elsewhere let $f_n = 0$. Since f_n has only a finite number of discontinuities in $\mathfrak{Q}$, $\mathrm{Cl} f_n = 1$ in $\mathfrak{Q}$. Let now

$$f(x_1 \cdots x_m) = \lim_{n=\infty} f_n.$$

At a non-rational point, each $f_n = 0$, $\therefore f = 0$. At a rational point, $f_n = 1$ for all $n >$ some s. Hence at such a point $f = 1$. Thus each point of $\mathfrak{Q}$ is a point of discontinuity and $\mathrm{Disc} f = 1$. Hence $\mathrm{Cl} f$ is not 1. As f is the limit of functions of class 1, its class is 2.

483. *Let* $f(x_1 \cdots x_m)$ *be continuous with respect to each* x_ι, *at each point of a limited set* $\mathfrak{A}$, *each of whose points is an inner point. Then* $\mathrm{Class} f \leq 1$.

For let $\mathfrak{A}$ lie within a cube $\mathfrak{Q}$. Then $A = \mathfrak{Q} - \mathfrak{A}$ is complete. We may therefore regard $\mathfrak{A}$ as a border set of A; that is, a set of non-overlapping cubes $\{\mathfrak{q}_n\}$. We show now that $\mathrm{Cl} f \leq 1$ in any one of these cubes as $\mathfrak{q}$. To this end we show that the points $\mathfrak{B}_m$ of $\mathfrak{q}$ at which

$$a + \frac{1}{m} \leq f \leq b - \frac{1}{m}$$

form a complete set. For let $b_1, b_2 \cdots$ be points of $\mathfrak{B}_m$, which $\doteq \beta$. We wish to show that β lies in $\mathfrak{B}_m$. Suppose first that $b_s, b_{s+1} \cdots$ have all their coördinates except one, say x, the same as the coördinates of β. Since

$$a + \frac{1}{m} \leq f(b_{s+p}) \leq b - \frac{1}{m},$$

therefore

$$a + \frac{1}{m} < \lim_{p=\infty} f(b_{s+p}) \leq b - \frac{1}{m}.$$

As f is continuous in x_1, and as only the coördinate x_1 varies in b_{s+p}, we have

$$a + \frac{1}{m} \leq f(\beta) \leq b - \frac{1}{m}.$$

Hence β lies in $\mathfrak{B}_m$.

We suppose next that $b_s, b_{s+1} \cdots$ have all their coördinates the same as β except two, say x_1, x_2.

We may place each b_n at the center of an interval $\mathfrak{i}$ of length δ, parallel to the x_1 axis, such that

$$a + \frac{1}{m} - \epsilon \leq f(x) \leq b - \frac{1}{m} + \epsilon,$$

since f is uniformly continuous in x_1, by I, 352. These intervals cut an ordinate in the x_1, x_2 plane through β, in a set of points c_{s+p} which $\doteq \beta$. Then as before,

$$a + \frac{1}{m} - \epsilon \leq f(\beta) \leq b - \frac{1}{m} + \epsilon.$$

As ϵ is small at pleasure, β lies in $\mathfrak{B}_m$. In this way we may continue.

As $\mathrm{Cl} f \leq 1$ in each $\mathfrak{q}_n$, it is in $\mathfrak{A}$, by 481, 3.

484. (*Volterra.*) *Let $f_1, f_2 \cdots$ be at most pointwise discontinuous in the limited complete set $\mathfrak{A}$. Then there exists a point of $\mathfrak{A}$ at which all the f_n are continuous.*

For if $\mathfrak{A}$ contains an isolated point, the theorem is obviously true, since every function is continuous at an isolated point. Let us therefore suppose that $\mathfrak{A}$ is perfect.

Let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. Let a_1 be a point of continuity of f_1. Then

$$\text{Osc} f_1 < \epsilon \quad , \quad \text{in some } \mathfrak{A}_1 = V_{\delta_1}(a_1).$$

In $\mathfrak{A}_1$ there is a point b of continuity of f_1. Hence $\text{Osc} f_1 < \epsilon_2$ in some $V_\eta(b)$, and we may take b so that $V_\eta(b) < \mathfrak{A}_1$. But in $V_\eta(b)$ there is a point a_2 at which f_2 is continuous. Hence

$$\text{Osc} f_1 < \epsilon_2 \quad , \quad \text{Osc} f_2 < \epsilon_1 \quad , \quad \text{in some } \mathfrak{A}_2 = V_{\delta_2}(a_2),$$

and we may take a_2 such that $\mathfrak{A}_2 < \mathfrak{A}_1$. Similarly there exists a point a_3 in $\mathfrak{A}_2$, such that

$$\text{Osc} f_1 < \epsilon_3 \quad , \quad \text{Osc} f_2 < \epsilon_2 \quad , \quad \text{Osc} f_3 < \epsilon_1 \quad , \quad \text{in some } \mathfrak{A}_3 = V_{\delta_3}(a_3),$$

and we may take a_3 so that $\mathfrak{A}_3 < \mathfrak{A}_2$.

In this way we may continue. As the sets $\mathfrak{A}_n$ are obviously complete, $Dv\{\mathfrak{A}_n\}$ contains at least one point α of $\mathfrak{A}$. But at this point each f_m is continuous.

485. 1. *Let $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ be complete, let $\mathfrak{B}$, $\mathfrak{C}$ be pantactic with reference to $\mathfrak{A}$. Then there exists no pair of functions f, g defined over $\mathfrak{A}$, such that if $\mathfrak{B}$ are the points of discontinuity of f in $\mathfrak{A}$, then $\mathfrak{B}$ shall be the points of continuity of g in $\mathfrak{A}$.*

This is a corollary of Volterra's theorem. For in any $V_\delta(a)$ of a point of $\mathfrak{A}$, there are points of $\mathfrak{B}$ and of $\mathfrak{C}$. Hence there are points of continuity of f and g. Hence f, g are at most pointwise discontinuous in $\mathfrak{A}$. Then by 484, there is a point in $\mathfrak{A}$ where f and g are both continuous, which contradicts the hypothesis.

2. *Let $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ be complete, and let $\mathfrak{B}$, $\mathfrak{C}$ each be pantactic with reference to $\mathfrak{A}$. If $\mathfrak{B}$ is hypercomplete, $\mathfrak{C}$ is not.*

For if $\mathfrak{B}$, $\mathfrak{C}$ were the union of an enumerable set of complete sets, 473 shows that there exists a function f defined over $\mathfrak{A}$ which has $\mathfrak{B}$ as its points of discontinuity ; and also a function g

which has $\mathfrak{E}$ as its points of discontinuity. But no such pair of functions can exist by 1.

3. *The non-rational points* $\mathfrak{J}$ *in any cube* $\mathfrak{Q}$ *cannot be hypercomplete.*

For the rational points in $\mathfrak{Q}$ are hypercomplete.

4. As an application of 2 we can state:

The limited function $f(x_1 \cdots x_m)$ *which is* ≤ 0 *at the irrational points of a cube* $\mathfrak{Q}$, *and* > 0 *at the other points* $\mathfrak{J}$ *of* $\mathfrak{Q}$, *cannot be of class* 0 *or* 1 *in* $\mathfrak{Q}$.

For if $\mathrm{Cl}\, f \leq 1$, the points of $\mathfrak{Q}$ where $f > 0$ must form a hypercomplete set, by 479, 4. But these are the points $\mathfrak{J}$.

486. 1. (*Baire.*) *If the class of* $f(x_1 \cdots x_m)$ *is* 1 *in the complete set* $\mathfrak{A}$, *it is at most pointwise discontinuous in any complete* $\mathfrak{B} \leq \mathfrak{A}$.

If $\mathrm{Cl}\, f = 1$ in $\mathfrak{A}$, it is ≤ 1 in any complete $\mathfrak{B} < \mathfrak{A}$ by 481, 4; we may therefore take $\mathfrak{B} = \mathfrak{A}$. Let a be any point of $\mathfrak{A}$. We shall show that in any $V = V_\delta(a)$ there is a point c of continuity of f. Let $\epsilon_1 > \epsilon_2 > \cdots = 0$. Using the notation of 479, 1, we saw that the sets $\mathfrak{A}_\iota = (m_\iota < f < m_{\iota+2})$ are hypercomplete. By 473, we can construct a function $\phi_\iota(x_1 \cdots x_m)$, defined over the m-way space $\mathfrak{R}_m$ which is discontinuous at the points $\mathfrak{A}_\iota$, and continuous elsewhere in $\mathfrak{R}_m$. These functions $\phi_1, \phi_2 \cdots$ are not all at most pointwise discontinuous in V. For then, by 484, there exists in V a point of continuity b, common to all the ϕ's. This point b must lie in some $\mathfrak{A}_\iota$, whose points are points of discontinuity of ϕ_ι.

Let us therefore suppose that ϕ_j is not at most pointwise discontinuous in V. Then there exists a point c_1 in V, and an η_1 such that $V_1 = V_{\eta_1}(c_1)$ contains no point of continuity of ϕ_j. Thus $V_1 \leq \mathfrak{A}_j$. But in $\mathfrak{A}_j$ and hence in V_1, $\mathrm{Osc}\, f < \epsilon_1$. The same reasoning shows that in V_1 there exists a $V_2 = V_{\eta_2}(c_2)$, such that $\mathrm{Osc}\, f < \epsilon_2$ in V_2. As $\mathfrak{A}$ is complete, $V_1 \geq V_2 \geq \cdots$ defines a point c in V at which f is continuous.

2. *If the class of* $f(x_1 \cdots x_m)$ *is 1 in the complete set* $\mathfrak{A}$, *its points of discontinuity* $\mathfrak{D}$ *form a set of the first category.*

For by 462, 3, the points $\mathfrak{D}_n$ of $\mathfrak{D}$ at which $\text{Osc} f \geq \frac{1}{n}$ form a complete set. Each $\mathfrak{D}_n$ is apantactic, since f is at most pointwise discontinuous, and $\mathfrak{D}_n$ is complete. Hence $\mathfrak{D} = \{\mathfrak{D}_n\}$ is the union of an enumerable set of apantactic sets, and is therefore of the 1° category.

487. 1. *Let f be defined over the limited complete set $\mathfrak{A}$. If* Class *f is not ≤ 1, there exists a perfect set $\mathfrak{D}$ in $\mathfrak{A}$, such that f is totally discontinuous in $\mathfrak{D}$.*

For if $\text{Cl} f$ is not ≤ 1 there exists, by 480, an ϵ such that for this ϵ, $\mathfrak{A}$ is not an $\mathfrak{E}_\epsilon$ set. Let now c be a point of $\mathfrak{A}$ such that the points $\mathfrak{a}$ of $\mathfrak{A}$ which lie within some cube $\mathfrak{q}$, whose center is c, form an $\mathfrak{E}_\epsilon$ set. Let $\mathfrak{B} = \{\mathfrak{a}\}$, $\mathfrak{C} = \{c\}$.

Then $\mathfrak{B} = \mathfrak{C}$. For obviously $\mathfrak{C} \leq \mathfrak{B}$, since each c is in some $\mathfrak{a}$. On the other hand, $\mathfrak{B} \leq \mathfrak{C}$. For any point b of $\mathfrak{B}$ lies within some $\mathfrak{q}$. Thus b is the center of a cube $\mathfrak{q}'$ within $\mathfrak{q}$. Obviously the points of $\mathfrak{A}$ within $\mathfrak{q}'$ form an $\mathfrak{E}_\epsilon$ set.

By Borel's theorem, each point c lies within an enumerable set of cubes $\{\mathfrak{c}_n\}$, such that each $\mathfrak{c}$ lies within some $\mathfrak{q}$. Thus the points $\mathfrak{a}_n$ of $\mathfrak{A}$ in $\mathfrak{c}_n$, form an $\mathfrak{E}_\epsilon$ set. As $\mathfrak{C} = \{\mathfrak{a}_n\}$, $\mathfrak{C}$ is an $\mathfrak{E}_\epsilon$ set.

Let $\mathfrak{D} = \mathfrak{A} - \mathfrak{C}$. If $\mathfrak{D}$ were 0, $\mathfrak{A} = \mathfrak{C}$ and $\mathfrak{A}$ would be an $\mathfrak{E}_\epsilon$ set contrary to hypothesis. Thus $\mathfrak{D} > 0$.

$\mathfrak{D}$ *is complete.* For if l were a limiting point of $\mathfrak{D}$ in $\mathfrak{C}$, l must lie in some $\mathfrak{c}$. But every point of $\mathfrak{A}$ in $\mathfrak{c}$ is a point of $\mathfrak{C}$ as we saw. Thus l cannot lie in $\mathfrak{C}$.

We show finally that at any point d of $\mathfrak{D}$,

$$\text{Osc} f \geq \epsilon, \text{ with respect to } \mathfrak{D}.$$

If not, $\text{Osc} f < \epsilon$ with respect to the points $\mathfrak{d}$ of $\mathfrak{D}$ within some cube $\mathfrak{q}$ whose center is d. Then $\mathfrak{d}$ is an $\mathfrak{E}_\epsilon$ set. Also the points $\mathfrak{e}$ of $\mathfrak{C}$ in $\mathfrak{q}$ form an $\mathfrak{E}_\epsilon$ set. Thus the points $\mathfrak{d} + \mathfrak{e}$, that is, the points of $\mathfrak{A}$ in $\mathfrak{q}$ form an $\mathfrak{E}_\epsilon$ set. Hence d belongs to $\mathfrak{C}$, and not to $\mathfrak{D}$. As $\text{Osc} f \geq \epsilon$ at each point of $\mathfrak{D}$, each point of $\mathfrak{D}$ is a point of discontinuity with respect to $\mathfrak{D}$. Thus f is totally discontinuous in $\mathfrak{D}$.

This shows that $\mathfrak{D}$ can contain no isolated points. Hence $\mathfrak{D}$ is perfect.

2. *Let f be defined over the limited complete set $\mathfrak{A}$. If f is at most pointwise discontinuous in any perfect $\mathfrak{B} \leq \mathfrak{A}$, its class is ≤ 1 in $\mathfrak{A}$.*

This is a corollary of 1. For if Class f were not 0, or 1, there exists a perfect set $\mathfrak{D}$ such that f is totally discontinuous in $\mathfrak{D}$.

488. *If the class of $f, g \leq 1$ in the limited complete set $\mathfrak{A}$, the class of their sum, difference, or product is ≤ 1. If $f > 0$ in $\mathfrak{A}$, the class of $\phi = 1/f$ is ≤ 1.*

For example, let us consider the product $h = fg$. If Cl h is not ≤ 1, there exists a perfect set $\mathfrak{D}$ in $\mathfrak{A}$, as we saw in 487, 1, such that h is totally discontinuous in $\mathfrak{D}$. But f, g being of class ≤ 1, are at most pointwise discontinuous in $\mathfrak{D}$ by 486. Then by 484, there exists a point of $\mathfrak{D}$ at which f, g are both continuous. Then h is continuous at this point, and is therefore not totally discontinous in $\mathfrak{D}$.

Let us consider now the quotient ϕ. If Cl ϕ is not ≤ 1, ϕ is totally discontinuous in some perfect set $\mathfrak{D}$ in $\mathfrak{A}$. But since $f > 0$ in $\mathfrak{D}$, f must also be totally discontinuous in $\mathfrak{D}$. This contradicts 486.

489. 1. *Let $F = \Sigma f_{\iota_1 \ldots \iota_s}(x_1 \cdots x_m)$ converge uniformly in the complete set $\mathfrak{A}$. Let the class of each term f_ι be ≤ 1, then* Class $F \leq 1$ *in $\mathfrak{A}$.*

For setting as usual [117],

$$F = F_\lambda + \bar{F}_\lambda \tag{1}$$

there exists for each $\epsilon > 0$, a fixed rectangular cell R_λ, such that

$$|\bar{F}_\lambda| < \epsilon, \qquad \text{as } x \text{ ranges over } \mathfrak{A}. \tag{2}$$

As the class of each term in F_λ is ≤ 1, Cl $F_\lambda \leq 1$ in $\mathfrak{A}$. Hence $\mathfrak{A}$ is an $\mathfrak{E}_\epsilon$ set with respect to F_λ.

From 1), 2) it follows that $\mathfrak{A}$ is an $\mathfrak{E}_\epsilon$ set with respect to F.

2. *Let $F = \Pi f_{\iota_1 \ldots \iota_s}(x_1 \cdots x_m)$ converge uniformly in the complete set $\mathfrak{A}$. If the class of each f_ι is ≤ 1, then* Cl $F \leq 1$ *in $\mathfrak{A}$.*

Semicontinuous Functions

490. Let $f(x_1 \cdots x_m)$ be defined over $\mathfrak{A}$. If a is a point of $\mathfrak{A}$, $\operatorname{Max} f$ in $V_\delta(a)$ exists, finite or infinite, and may be regarded as a function of δ. When finite, it is a monotone decreasing function of δ. Thus its limit as $\delta \doteq 0$ exists, finite or infinite. We call this limit the *maximum of f at $x = a$*, and we denote it by

$$\operatorname*{Max}_{x=a} f.$$

Similar remarks apply to the minimum of f in $V_\delta(a)$. Its limit, finite or infinite, as $\delta \doteq 0$, we call the *minimum of f at $x = a$*, and we denote it by

$$\operatorname*{Min}_{x=a} f.$$

The maximum and minimum of f in $V_\delta(a)$ may be denoted by

$$\operatorname*{Max}_{a,\,\delta} f \quad , \quad \operatorname*{Min}_{a,\,\delta} f.$$

Obviously,

$$\operatorname*{Max}_{x=a} (-f) = -\operatorname*{Min}_{x=a} f,$$

$$\operatorname*{Min}_{x=a} (-f) = -\operatorname*{Max}_{x=a} f.$$

491. *Example 1.*

$$f(x) = \frac{1}{x} \text{ in } (-1, 1) \quad , \quad \text{for } x \neq 0$$

$$= 0 \quad , \quad \text{for } x = 0.$$

Then

$$\operatorname*{Max}_{x=0} f = +\infty \quad , \quad \operatorname*{Min}_{x=0} f = -\infty.$$

Example 2.

$$f(x) = \sin\frac{1}{x} \text{ in } (-1, 1) \quad , \quad \text{for } x \neq 0$$

$$= 0 \quad , \quad \text{for } x = 0.$$

Then

$$\operatorname*{Max}_{x=0} f = 1 \quad , \quad \operatorname*{Min}_{x=0} f = -1.$$

Example 3.

$$f(x) = 1 \text{ in } (-1, 1) \quad , \quad \text{for } x \neq 0$$
$$= 2 \quad , \quad \text{for } x = 0.$$

Then

$$\operatorname*{Max}_{x=0} f = 2 \quad , \quad \operatorname*{Min}_{x=0} f = 1.$$

We observe that in Exs. 1 and 2,

$$\overline{\lim_{x=0}} f = \operatorname{Max}_{x=0} f \quad , \quad \underline{\lim_{x=0}} f = \operatorname{Min}_{x=0} f;$$

while in Ex. 3,

$$\overline{\lim_{x=0}} f = 1 \quad , \quad \text{and hence } \operatorname{Max}_{x=0} f > \overline{\lim_{x=0}} f.$$

Also

$$\underline{\lim_{x=0}} f = \operatorname{Min}_{x=0} f.$$

Example 4.

$$f(x) = (x^2+1)\sin\frac{1}{x} \text{ in } (-1, 1) \quad , \quad \text{for } x \neq 0$$

$$= -2 \quad , \quad \text{for } x = 0.$$

Here

$$\operatorname{Max}_{x=0} f = 1 \quad , \quad \operatorname{Min}_{x=0} f = -2,$$

$$\overline{\lim_{x=0}} f = 1 \quad , \quad \underline{\lim_{x=0}} f = -1.$$

Example 5. Let

$$f(x) = x \quad , \quad \text{for rational } x \text{ in } (0, 1)$$

$$= 1 \quad , \quad \text{for irrational } x.$$

Here

$$\operatorname{Max}_{x=0} f = 1 \quad , \quad \operatorname{Min}_{x=0} f = 0,$$

$$\overline{\lim_{x=0}} f = 1.$$

492. 1. *For M to be the maximum of f at $x = a$, it is necessary and sufficient that*

1° $\epsilon > 0, \quad \delta > 0, \quad f(x) < M + \epsilon,$ *for any x in $V_\delta(a)$;*

2° *there exists for each $\epsilon > 0$, and in any $V_\delta(a)$, a point α such that*

$$M - \epsilon < f(\alpha).$$

These conditions are *necessary*. For M is the limit of $\operatorname{Max} f$ in $V_\delta(a)$, as $\delta \doteq 0$. Hence

$$\epsilon > 0, \quad \delta > 0, \quad \operatorname{Max}_{a,\delta} f < M + \epsilon.$$

But for any x in $V_\delta(a)$,

$$f(x) \leq \operatorname{Max}_{a,\delta} f.$$

Hence
$$f(x) < M + \epsilon \quad , \quad x \text{ in } V_\delta(a),$$
which is condition 1°.

As to 2°, we remark that for each $\epsilon > 0$, and in any $V_\delta(a)$, there is a point α, such that
$$-\epsilon + \operatorname*{Max}_{a,\,\delta} f < f(\alpha).$$
But
$$M \leq \operatorname*{Max}_{a,\,\delta} f.$$
Hence
$$-\epsilon + M < f(\alpha),$$
which is 2°.

These conditions are *sufficient*. For from 1° we have
$$\operatorname*{Max}_{a,\,\delta} f \leq M + \epsilon,$$
and hence letting $\delta \doteq 0$,
$$\operatorname*{Max}_{x=a} f \leq M, \tag{1}$$
since $\epsilon > 0$ is small at pleasure.

From 2° we have
$$\operatorname*{Max}_{a,\,\delta} f \geq M - \epsilon,$$
and hence letting $\delta \doteq 0$,
$$\operatorname*{Max}_{x=a} f \geq M. \tag{2}$$

From 1), 2) we have $M = \operatorname*{Max}_{x=a} f$.

2. *For m to be the minimum of f at $x = a$, it is necessary and sufficient that*

1° $\epsilon > 0$, $\quad \delta > 0$, $\quad m - \epsilon < f(x)$, *for any x in $V_\delta(a)$;*

2° *that there exists for each* $\epsilon > 0$, *and in any $V_\delta(a)$, a point α such that*
$$f(\alpha) < m + \epsilon.$$

493. When $\operatorname*{Max}_{x=a} f = f(a)$, we say f is *supracontinuous* at $x = a$. When $\operatorname*{Min}_{x=a} f = f(a)$, we say f is *infracontinuous* at a. When f is supra (infra) continuous at each point of $\mathfrak{A}$, we say f is supra (infra) continuous in $\mathfrak{A}$. When f is either supra or infracontinuous at a and we do not care to specify which, we say it is *semicontinuous* at a.

The function which is equal to $\text{Max} f$ at each point x of $\mathfrak{A}$ we call the *maximal* function of f, and denote it by a dash above, *viz.* $\overline{f}(x)$. Similarly the *minimal* function $\underline{f}(x)$ is defined as the value of $\text{Min} f$ at each point of $\mathfrak{A}$.

Obviously
$$\underset{x=a}{\text{Osc}} f = \underset{x=a}{\text{Max}} f - \underset{x=a}{\text{Min}} f = \underset{x=a}{\text{Disc}} f.$$

We call
$$\omega(x) = \overline{f}(x) - \underline{f}(x)$$
the *oscillatory function*.

We have at once the theorem:

For f to be continuous at $x = a$, it is necessary and sufficient that
$$f(a) = \overline{f}(a) = \underline{f}(a).$$

For
$$\underset{a,\delta}{\text{Min}} f \le f(a) \le \underset{a,\delta}{\text{Max}} f.$$

Passing to the limit $x = a$, we have
$$\underset{x=a}{\text{Min}} f \le f(a) \le \underset{x=a}{\text{Max}} f,$$
or
$$\underline{f}(a) \le f(a) \le \overline{f}(a).$$

But for f to be continuous at $x = a$, it is necessary and sufficient that
$$\omega(a) = \underset{x=a}{\text{Osc}} f = 0.$$

494. 1. *For f to be supracontinuous at $x = a$, it is necessary and sufficient that for each $\epsilon > 0$, there exists a $\delta > 0$, such that*
$$f(x) < f(a) + \epsilon \quad , \quad \textit{for any } x \textit{ in } V_\delta(a). \tag{1}$$

Similarly the condition for infracontinuity is
$$f(a) - \epsilon < f(x) \quad , \quad \textit{for any } x \textit{ in } V_\delta(a). \tag{2}$$

Let us prove 1). It is *necessary*. For when f is supracontinuous at a,
$$f(a) = \underset{x=a}{\text{Max}} f(x).$$

Then by 492, 1,
$$\epsilon > 0 \quad , \quad \delta > 0 \quad , \quad f(x) < f(a) + \epsilon \quad , \quad \text{for any } x \text{ in } V_\delta(a),$$
which is 1).

It is *sufficient*. For 1) is condition 1° of 492, 1. The condition 2° is satisfied, since for α we may take the point a.

2. *The maximal function* $\overline{f}(x)$ *is supracontinuous; the minimal function* $\underline{f}(x)$ *is infracontinuous, in* $\mathfrak{A}$.

To prove that $\overline{f}$ is supracontinuous we use 1, showing that

$$\overline{f}(x) < \overline{f}(a) + \epsilon \quad , \quad \text{for any } x \text{ in some } V_\delta(a).$$

Now by 492, 1,

$$\epsilon' > 0, \ \delta > 0 \quad , \quad f(x) < \overline{f}(a) + \epsilon' \quad , \quad \text{for any } x \text{ in } V_\delta(a).$$

Thus if $\epsilon' < \epsilon$

$$\overline{f}(x) < \overline{f}(a) + \epsilon \quad , \quad \text{for any } x \text{ in } V_\eta(a) \quad , \quad \eta = \frac{\delta}{2}.$$

3. *The sum of two supra (infra) continuous functions in* $\mathfrak{A}$ *is a supra (infra) continuous function in* $\mathfrak{A}$.

For let f, g be supracontinuous in $\mathfrak{A}$; let $f + g = h$. Then by 1,

$$f(x) < f(a) + \frac{\epsilon}{2},$$

$$g(x) < g(a) + \frac{\epsilon}{2},$$

for any x in some $V_\delta(a)$; hence

$$h(x) < h(a) + \epsilon.$$

This, by 1, shows that h is supracontinuous at $x = a$.

4. *If* $f(x)$ *is supra (infra) continuous at* $x = a$, $g(x) = -f(x)$ *is infra (supra) continuous.*

Let us suppose that f is supracontinuous. Then by 1,

$$f(x) < f(a) + \epsilon \quad , \quad \text{for any } x \text{ in some } V_\delta(a).$$

Hence

$$-f(a) - \epsilon < -f(x),$$

or

$$g(a) - \epsilon < g(x) \quad , \quad \text{for any } x \text{ in } V_\delta(a).$$

Thus by 1, g is infracontinuous at a.

495. *If $f(x_1 \cdots x_m)$ is supracontinuous in the limited complete set $\mathfrak{A}$, the points $\mathfrak{B}$ of $\mathfrak{A}$ at which $f \geq c$ an arbitrary constant form a complete set.*

For let $f \geq c$ at $b_1, b_2 \cdots$ which $\doteq b$; we wish to show that b lies in $\mathfrak{B}$.

Since f is supracontinuous, by 494, 1,

$$f(x) < f(b) + \epsilon \quad , \quad \text{for any } x \text{ in some } V_\delta(b) = V.$$

But $c \leq f(b_n)$, by hypothesis; and b_n lies in V, for $n >$ some m. Hence

$$c \leq f(b_n) < f(b) + \epsilon,$$

or

$$c - \epsilon < f(b).$$

As $\epsilon > 0$ is small at pleasure,

$$f(b) \geq c,$$

and b lies in $\mathfrak{B}$.

496. 1. *The oscillatory function $\omega(x)$ is supracontinuous.*

For by 493,

$$\omega(x) = \text{Max} f - \text{Min} f$$
$$= \text{Max} f + \text{Max}\,(-f).$$

But these two maximal functions are supracontinuous by 494, 2. Hence by 494, 3, their sum ω is supracontinuous.

2. The oscillatory function ω is not necessarily infracontinuous, as is shown by the following

Example. $f = 1$ in $(-1, 1)$, except for $x = 0$, where $f = 2$. Then $\omega(x) = 0$, except at $x = 0$, where $\omega = 1$. Thus

$$\underset{x=0}{\text{Min}}\, \omega(x) = 0 \quad , \quad \text{while } \omega(0) = 1.$$

Hence $\omega(x)$ is not infracontinuous at $x = 0$.

3. *Let $\omega(x)$ be the oscillatory function of $f(x_1 \cdots x_m)$ in $\mathfrak{A}$. For f to be at most pointwise discontinuous in $\mathfrak{A}$, it is necessary that* Min $\omega = 0$ *at each point of $\mathfrak{A}$. If $\mathfrak{A}$ is complete, this condition is sufficient.*

It is *necessary*. For let a be a point of $\mathfrak{A}$. As f is at most pointwise discontinuous, there exists a point of continuity in any $V_\delta(a)$. Hence $\operatorname{Min} \omega(x) = 0$, in $V_\delta(a)$. Hence $\underset{x=a}{\operatorname{Min}}\, \omega(x) = 0$.

It is *sufficient*. For let $\epsilon_1 > \epsilon_2 > \cdots \doteq 0$. Since $\underset{x=a}{\operatorname{Min}}\, \omega(x) = 0$, there exists in any $V_\delta(a)$ a point α_1 such that $\omega(\alpha_1) < \frac{1}{2}\epsilon_1$. Hence $\omega(x) < \epsilon_1$ in some $V_{\delta_1}(\alpha_1) < V_\delta$. In V_{δ_1} there exists a point α_2 such that $\omega(x) < \epsilon_2$ in some $V_{\delta_2}(\alpha) < V_{\delta_1}$, etc. Since $\mathfrak{A}$ is complete and since we may let $\delta_n \doteq 0$,

$$V_{\delta_1} > V_{\delta_2} > \cdots \doteq \text{a point } \alpha \text{ of } \mathfrak{A},$$

at which f is obviously continuous. Thus in each $V_\delta(a)$ is a point of continuity of f. Hence f is at most pointwise discontinuous.

497. 1. *At each point x of $\mathfrak{A}$,*

$$\phi = \operatorname{Min}\{\bar{f}(x) - f(x)\}, \textit{ and } \psi = \operatorname{Min}\{f(x) - \underline{f}(x)\}$$

are both $= 0$.

Let us show that $\phi = 0$ at an arbitrary point a of $\mathfrak{A}$. By 494, 2, $\bar{f}(x)$ is supracontinuous; hence by 494, 1,

$$\bar{f}(x) < \bar{f}(a) + \epsilon \quad , \quad \text{for any } x \text{ in some } V_\delta(a) = V. \tag{1}$$

Also there exists a point α in V such that

$$-\epsilon + \bar{f}(a) < f(\alpha). \tag{2}$$

Also by definition

$$f(\alpha) \leq \bar{f}(\alpha). \tag{3}$$

If in 1) we replace x by α we get

$$\bar{f}(\alpha) < \bar{f}(a) + \epsilon. \tag{4}$$

From 2), 3), 4) we have

$$-\epsilon + \bar{f}(a) < f(\alpha) \leq \bar{f}(\alpha) < \bar{f}(a) + \epsilon,$$

or

$$0 \leq \bar{f}(\alpha) - f(\alpha) < 2\epsilon.$$

As $\epsilon > 0$ is small at pleasure, this gives

$$\phi(a) = 0.$$

2. *If f is semicontinuous in the complete set $\mathfrak{A}$, it is at most pointwise discontinuous in $\mathfrak{A}$.*

For

$$\begin{aligned} \omega(x) &= \bar{f}(x) - \underline{f}(x) \\ &= [\bar{f}(x) - f(x)] + [f(x) - \underline{f}(x)] \qquad (1 \\ &= \phi(x) + \psi(x). \end{aligned}$$

To fix the ideas let f be supracontinuous. Then $\phi = 0$ in $\mathfrak{A}$. Hence 1) gives

$$\text{Min } \omega(x) = \text{Min } \psi(x) = 0, \qquad \text{by 1.}$$

Thus by 496, 3, f is at most pointwise discontinuous in $\mathfrak{A}$.

CHAPTER XV

DERIVATES, EXTREMES, VARIATION

Derivates

498. Suppose we have given a one-valued continuous function $f(x)$ spread over an interval $\mathfrak{A} = (a < b)$. We can state various properties which it enjoys. For example, it is limited, it takes on its extreme values, it is integrable. On the other hand, we do not know 1° how it oscillates in $\mathfrak{A}$, or 2° if it has a differential coefficient at each point of $\mathfrak{A}$. In this chapter we wish to study the behavior of continuous functions with reference to these last two properties. In Chapters VIII and XI of volume I this subject was touched upon; we wish here to develop it farther.

499. In I, 363, 364, we have defined the terms difference quotient, differential coefficient, derivative, right- and left-hand differential coefficients and derivatives, unilateral differential coefficients and derivatives. The corresponding symbols are

$$\frac{\Delta f}{\Delta x} \quad , \quad f'(a) \quad , \quad f'(x) \quad , \quad Rf'(a) \quad ,$$

$$Lf'(a) \quad , \quad Rf'(x) \quad , \quad Lf'(x).$$

The unilateral differential coefficient and derivative may be denoted by

$$Uf'(a) \quad , \quad Uf'(x). \tag{1}$$

When

$$\lim_{h=0} \frac{\Delta f}{\Delta x}$$

does not exist, finite or infinite, we may introduce its upper and lower limits. Thus

$$\bar{f}'(a) = \overline{\lim_{h=0}} \frac{\Delta f}{\Delta x} \quad , \quad \underline{f}'(a) = \underline{\lim_{h=0}} \frac{\Delta f}{\Delta x} \tag{2}$$

always exist, finite or infinite. We call them the upper and lower differential coefficients at the point $x = a$. The aggregate of values

that 2) take on define the upper and lower derivatives of $f(x)$, as in I, 363.

In a similar manner we introduce the upper and lower right- and left-hand differential coefficients and derivatives,

$$\overline{R}f' \quad , \quad \underline{R}f' \quad , \quad \overline{L}f' \quad , \quad \underline{L}f'. \tag{3}$$

Thus, for example,

$$\overline{R}f'(a) = R\overline{\lim_{h=0}}\frac{f(a+h)-f(a)}{h},$$

finite or infinite. Cf. I, 336 *seq.*

If $f(x)$ is defined *only* in $\mathfrak{A} = (\alpha < \beta)$, the points a, $a+h$ must lie in $\mathfrak{A}$. Thus there is no upper or lower right-hand differential coefficient at $x = \beta$; also no upper or lower left-hand differential coefficient at $x = \alpha$. This fact must be borne in mind. We call the functions 3) *derivates* to distinguish them from the derivatives Rf', Lf'. When $\overline{R}f'(a) = \underline{R}f'(a)$, finite or infinite, $Rf'(a)$ exists also finite or infinite, and has the same value. A similar remark applies to the left-hand differential coefficient.

To avoid such repetition as just made, it is convenient to introduce the terms upper and lower unilateral differential coefficients and derivatives, which may be denoted by

$$\overline{U}f' \quad , \quad \underline{U}f'. \tag{4}$$

The symbol U should of course refer to the same side, if it is used more than once in an investigation.

When no ambiguity can arise, we may abbreviate the symbols 3), 4) thus:

$$\overline{R} \;,\; \underline{R} \;,\; \overline{L} \;,\; \underline{L} \;,\; \overline{U} \;,\; \underline{U}.$$

The value of one of these derivates as $\overline{R}$ at a point $x = \alpha$ may similarly be denoted by

$$\overline{R}(\alpha).$$

The difference quotient

$$\frac{f(a)-f(b)}{a-b}$$

may be denoted by

$$\Delta(a, b).$$

Example 1.
$$f(x)=x\sin\frac{1}{x}\quad,\quad x\neq 0 \text{ in } (-1,1)$$
$$=0\quad,\quad x=0.$$

Here for $x=0$, $$\frac{\Delta f}{\Delta x}=\frac{h\sin\frac{1}{h}}{h}=\sin\frac{1}{h}.$$

Hence
$$\overline{R}f'(0)=+1\quad,\quad \underline{R}f'(0)=-1,$$
$$\overline{L}f'(0)=+1\quad,\quad \underline{L}f'(0)=-1,$$
$$\overline{f}'(0)=+1\quad,\quad \underline{f}'(0)=-1.$$

Example 2.
$$f(x)=x^{\frac{1}{3}}\sin\frac{1}{x}\quad,\quad x\neq 0 \text{ in } (-1,1)$$
$$=0\quad,\quad x=0.$$

Here for $$x=0\quad,\quad \frac{\Delta f}{\Delta x}=\frac{\sin\frac{1}{h}}{h^{\frac{2}{3}}}.$$

Hence
$$\overline{R}f'(0)=+\infty\quad,\quad \underline{R}f'(0)=-\infty,$$
$$\overline{L}f'(0)=+\infty\quad,\quad \underline{L}f'(0)=-\infty,$$
$$\overline{f}'(0)=+\infty\quad,\quad \underline{f}'(0)=-\infty.$$

Example 3.
$$f(x)=x\sin\frac{1}{x}\quad,\quad \text{for } 0<x\leq 1$$
$$=x^{\frac{1}{3}}\sin\frac{1}{x}\quad,\quad \text{for } -1\leq x<0$$
$$=0\quad,\quad \text{for } x=0.$$

Here
$$\overline{R}f'(0)=+1\quad,\quad \underline{R}f'(0)=-1,$$
$$\overline{L}f'(0)=+\infty\quad,\quad \underline{L}f'(0)=-\infty,$$
$$\overline{f}'(0)=+\infty\quad,\quad \underline{f}'(0)=-\infty.$$

500. 1. Before taking up the general theory it will be well for the reader to have a few examples in mind to show him how complicated matters may get. In I, 367 *seq.*, we have exhibited functions which oscillate infinitely often about the points of a set

of the 1° species, and which may or may not have differential coefficients at these points.

The following theorem enables us to construct functions which do not possess a differential coefficient at the points of an enumerable set.

2. *Let* $\mathfrak{E} = \{e_n\}$ *be an enumerable set lying in the interval* $\mathfrak{A}$. *For each* x *in* $\mathfrak{A}$, *and* e_n *in* $\mathfrak{E}$, *let* $x - e_n$ *lie in an interval* $\mathfrak{B}$ *containing the origin. Let* $g(x)$ *be continuous in* $\mathfrak{B}$. *Let* $g'(x)$ *exist and be numerically* $\leq M$ *in* $\mathfrak{B}$, *except at* $x = 0$, *where the difference quotients are numerically* $\leq M$. *Let* $A = \Sigma a_n$ *converge absolutely. Then*

$$F(x) = \Sigma a_n g(x - e_n)$$

is a continuous function in $\mathfrak{A}$, *having a derivative in* $\mathfrak{C} = \mathfrak{A} - \mathfrak{E}$. *At the points of* $\mathfrak{E}$, *the difference quotient of* F *behaves essentially as that of* g *at the origin.*

For $g(x)$ being continuous in $\mathfrak{B}$, it is numerically < some constant in $\mathfrak{A}$. Thus F converges uniformly in $\mathfrak{A}$. As each term $g(x - e_n)$ is continuous in $\mathfrak{A}$, F is continuous in $\mathfrak{A}$.

Let us consider its differential coefficient at a point x of $\mathfrak{C}$. Since $g'(x - e_n)$ exists and is numerically $\leq M$,

$$F'(x) = \Sigma a_n g'(x - e_n) \quad , \quad \text{by 156, 2.}$$

Let now $x = e_m$, a point of $\mathfrak{E}$,

$$F(x) = a_m g(x - e_m) + \Sigma^* a_n g(x - e_n)$$ $\sum^* \equiv$ the deleted

$$= a_m g(x - e_m) + G(x).$$ i.e. for all pts ex e_m

The summation in Σ^* extends over all $n \neq m$. Hence by what has just been shown, G has a differential coefficient at $x = e_m$. Thus $\dfrac{\Delta F}{\Delta x}$ behaves at $x = e_m$, essentially as $\dfrac{\Delta g}{\Delta x}$ at $x = 0$. Hence

$$\underline{\overline{U}}F'(e_m) = a_m \underline{\overline{U}}g'(0) + G'(e_m). \tag{1}$$

501. *Example 1.* Let

$$g(x) = ax \quad , \quad x \geq 0$$
$$= bx \quad , \quad x < 0, \qquad b < 0 < a.$$

Then

$$F(x) = \sum \frac{1}{n^2} g(x - e_n)$$

is continuous in any interval $\mathfrak{A}$, and has a derivative

$$F'(x) = \sum \frac{1}{n^2} g'(x - e_n)$$

at the points of $\mathfrak{A}$ not in $\mathfrak{E}$. At the point e_m,

$$RF'(x) = a_m a + \sum{}^* \frac{1}{m^2} g'(e_m - e_n),$$

$$LF'(x) = a_m b + \sum{}^* \frac{1}{m^2} g'(e_m - e_n).$$

Let $\mathfrak{E}$ denote the rational points in $\mathfrak{A}$. The graph of $F(x)$ is a continuous curve having tangents at a pantactic set of points; and at another pantactic set, viz. the set $\mathfrak{E}$, angular points (I, 366).

A simple example of a g function is

$$g(x) = |x| = +\sqrt{x^2}.$$

Example 2. Let $g(x) = x^2 \sin \frac{\pi}{x}$, $x \neq 0$

$$= 0 \quad , \quad x = 0.$$

This function has a derivative

$$g'(x) = 2x \sin \frac{\pi}{x} - \pi \cos \frac{\pi}{x} \quad , \quad x \neq 0$$

$$= 0 \quad , \quad x = 0.$$

Thus if Σc_n is an absolutely convergent series, and $\mathfrak{E} = \{e_n\}$ an enumerable set in the interval $\mathfrak{A} = (0, 1)$,

$$F(x) = \Sigma c_n g(x - e_n)$$

is a continuous function whose derivative in $\mathfrak{A}$ is

$$F'(x) = \Sigma c_n g'(x - e_n).$$

Thus F has a derivative which is continuous in $\mathfrak{A} - \mathfrak{E}$, and at the point $x = e_m$

$$\text{Disc } F' = 2\, c_m \pi,$$

since

$$\underset{x=0}{\text{Disc}}\, g'(x) = 2\, \pi.$$ def. 454

If $\mathfrak{E}$ is the set of rational points in $\mathfrak{A}$, the graph of $F(x)$ is a continuous curve having at each point of $\mathfrak{A}$ a tangent which does not turn continuously as the point of contact ranges over the curve; indeed the points of abrupt change in the direction of the tangent are pantactic in $\mathfrak{A}$.

Example 3. Let $g(x) = x \sin \log x^2 \quad , \quad x \neq 0$

$$= 0 \quad , \quad x = 0.$$

Then
$$g'(x) = \sin \log x^2 + 2 \cos \log x^2 \quad , \quad x \neq 0.$$

At $x = 0$,
$$\frac{\Delta g}{\Delta x} = \sin \log h^2$$

which oscillates infinitely often between ± 1, as $h = \Delta x \doteq 0$. Let $\mathfrak{E} = \{e_n\}$ denote the rational points in an interval $\mathfrak{A}$. The series

$$F = \sum \frac{1}{n^2} (x - e_n) \sin \log (x - e_n)^2$$

satisfies the condition of our theorem. Hence $F(x)$ is a continuous function in $\mathfrak{A}$ which has a derivative in $\mathfrak{A} - \mathfrak{E}$. At $x = e_m$,

$$\overline{U}F'(x) = \frac{1}{m^2} + G'(e_m) \quad , \quad \underline{U}F'(x) = -\frac{1}{m^2} + G'(e_m).$$

Thus the graph of F is a continuous curve which has tangents at a pantactic set of points in $\mathfrak{A}$, and at another pantactic set it has neither right- nor left-hand tangents.

502. *Weierstrass' Function.* For a long time mathematicians thought that a continuous function of x must have a derivative, at least after removing certain points. The examples just given show that these exceptional points may be pantactic. Weierstrass called attention to a continuous function which has at no point a differential coefficient. This celebrated function is defined by the series

$$F(x) = \sum_0 a^n \cos b^n \pi x = \cos \pi x + a \cos b\pi x + a^2 \cos b^2 \pi x + \cdots \qquad (1$$

where $0 < a < 1$; b is an odd integer so chosen that

$$ab > 1 + \tfrac{3}{2}\pi. \qquad (2$$

The series F converges absolutely and uniformly in any interval $\mathfrak{A}$, since

$$|a^n \cos b^n \pi x| \leq a^n.$$

Hence F is a continuous function in $\mathfrak{A}$. Let us now consider the series obtained by differentiating 1) termwise,

$$G(x) = -\pi \Sigma (ab)^n \sin b^n \pi x.$$

If $ab < 1$, this series also converges absolutely and uniformly, and

$$F'(x) = G(x),$$

by 155, 1. In this case the function has a finite derivative in $\mathfrak{A}$. Let us suppose, however, that the condition 2) holds. We have

$$\frac{\Delta F}{\Delta x} = Q = \sum_0^\infty \frac{a^n}{h}\{\cos b^n \pi(x+h) - \cos b^n \pi x\} = Q_m + \overline{Q}_m. \tag{3}$$

Now

$$Q_m = \sum_0^{m-1} \frac{a^n}{h}\{\cos b^n \pi(x+h) - \cos b^n \pi x\}$$

$$= -\pi \sum_0^{m-1} \frac{(ab)^n}{h} \int_x^{x+h} \sin b^n \pi u du.$$

Since

$$\left| \int_x^{x+h} \sin b^n \pi u du \right| < \left| \int_x^{x+h} du \right| = |h|,$$

$$|Q_m| < \pi \sum_0^{m-1} (ab)^n = \pi \frac{1-(ab)^m}{1-ab} < \pi \frac{(ab)^m}{ab-1}, \text{ if } ab > 1.$$

Consider now

$$\overline{Q}_m = \sum_m^\infty \frac{a^n}{h}\{\cos b^n \pi(x+h) - \cos b^n \pi x\}.$$

Up to the present we have taken h arbitrary. Let us now take it as follows; the reason for this choice will be evident in a moment.

Let

$$b^m x = \iota_m + \xi_m,$$

where ι_m is the nearest integer to $b^m x$. Thus

$$-\tfrac{1}{2} \leq \xi_m \leq \tfrac{1}{2}.$$

Then

$$b^m(x+h) = \iota_m + \xi_m + hb^m = \iota_m + \eta_m.$$

We choose h so that

$$\eta_m = \xi_m + hb^m \text{ is } \pm 1, \text{ at pleasure.}$$

Then

$$h = \frac{\eta_m - \xi_m}{b^m} \doteq 0, \text{ as } m \doteq \infty;$$

moreover

$$\operatorname{sgn} h = \operatorname{sgn} \eta_m \quad , \quad \text{and } | \eta_m - \xi_m | \leq \tfrac{3}{2}.$$

This established, we note that

$$\cos b^n \pi (x+h) = \cos b^{n-m} \pi \cdot b^m (x+h) = \cos b^{n-m} (\iota_m + \eta_m)\pi$$

$$= \cos (\iota_m + \eta_m)\pi \quad , \quad \text{since } b \text{ is odd}$$

$$= (-1)^{\iota_m + 1} \quad , \quad \text{since } \eta_m \text{ is odd.}$$

Also

$$\cos b^n \pi x = \cos b^{n-m} (\iota_m + \xi_m)\pi$$

$$= (-1)^{\iota_m} \cos b^{n-m} \xi_m \pi.$$

Thus

$$\bar{Q}_m = e_m \sum_m^\infty \frac{a^n}{h} \{1 + \cos b^{n-m} \xi_m \pi\},$$

where

$$e_m = (-1)^{\iota_m + 1}.$$

Now each $\{\ \} \geq 0$ and in particular the first is > 0. Thus

$$\operatorname{sgn} \bar{Q}_m = \operatorname{sgn} \frac{e_m}{h} = \operatorname{sgn} e_m \eta_m,$$

and

$$| \bar{Q}_m | > \frac{a^m}{h} = \frac{(ab)^m}{\eta_m - \xi_m} \geq \tfrac{2}{3}(ab)^m.$$

Thus if 2) holds, $| \bar{Q}_m | > | Q_m |$. Hence from 3),

$$\operatorname{sgn} Q = \operatorname{sgn} \bar{Q}_m = \operatorname{sgn} e_m \eta_m,$$

and

$$| Q | > (ab)^m \left(\frac{2}{3} - \frac{\pi}{ab - 1} \right).$$

Let now $m \doteq \infty$. Since $\eta_m = \pm 1$ at pleasure, we can make $Q \doteq +\infty$, or to $-\infty$, or oscillate between $\pm\infty$, without becoming definitely infinite. *Thus $F(x)$ has at no point a finite or infinite differential coefficient. This does not say that the graph of F does not have tangents; but when they exist, they must be cuspidal tangents.*

503. 1. *Volterra's Function.*

In the interval $\mathfrak{A}=(0, 1)$, let $\mathfrak{H}=\{\eta\}$ be a Harnack set of measure $0<h<1$. Let $\Delta=\{\delta_n\}$ be the associate set of black intervals. In each of the intervals $\delta_n=(\alpha<\beta)$, we define an auxiliary function f_n as follows:

$$f_n(x)=(x-\alpha)^2 \sin\frac{1}{x-\alpha} \quad , \quad \text{in } (\alpha^*, \gamma), \tag{1}$$

where γ is the largest value of x corresponding to a maximum of the function on the right of 1), such that γ lies to the left of the middle point μ of δ_n. If the value of $f_n(x)$ at γ is g, we now make

$$f_n(x)=g \quad , \quad \text{in } (\gamma, \mu).$$

Finally $f_n(\alpha)=0$. This defines $f_n(x)$ for one half of the interval δ_n. We define $f_n(x)$ for the other half of δ_n by saying that if $x<x'$ are two points of δ_n at equal distances from the middle point μ, then

$$f_n(x)=f_n(x').$$

With Volterra we now define a function $f(x)$ in $\mathfrak{A}$ as follows:

$$\begin{aligned} f(x) &= f_n(x) \quad , \quad \text{in } \delta_n \quad , \quad n=1, 2, \cdots \\ &= 0 \quad , \quad \text{in } \mathfrak{H}. \end{aligned}$$

Obviously $f(x)$ is continuous in $\mathfrak{A}$.

At a point x of $\mathfrak{A}$ not in $\mathfrak{H}$, $f'(x)$ behaves as

$$2\,x \sin\frac{1}{x} - \cos\frac{1}{x},$$

as is seen from 1). Thus as x converges in δ_n toward one of its end points α, β, we see that $f'(x)$ oscillates infinitely often between limits which $\doteq \pm 1$. Thus

$$R\,\overline{\lim_{x=\alpha}} f'(x)=+1 \quad , \quad R\,\underline{\lim_{x=\alpha}} f'(x)=-1;$$

similar limits exist for the points β.

Let us now consider the differential coefficient at a point η of $\mathfrak{H}$. We have

$$\frac{\Delta f}{\Delta x}=\frac{f(\eta+k)-f(\eta)}{k}=\frac{f(\eta+k)}{k} \quad , \quad \text{since } f(\eta)=0.$$

If $\eta + k$ is a point of $\mathfrak{H}$, $f(\eta + k) = 0$. If not, $\eta + k$ lies in some interval δ_m. Let $x = e$ be the end point of δ_m nearest $\eta + k$. Then

$$\left|\frac{\Delta f}{\Delta x}\right| \leq \frac{|\eta + k - e|^2}{|k|} \leq |k| \doteq 0 \quad , \quad \text{as } k \doteq 0.$$

Thus $f'(\eta) = 0$. Hence Volterra's function $f(x)$ has a differential coefficient at each point of $\mathfrak{A}$; moreover $f'(x)$ is limited in $\mathfrak{A}$. Each point η of $\mathfrak{H}$ is a point of discontinuity of $f'(x)$, and

$$\operatorname*{Disc}_{x=\eta} f'(x) \geq 2.$$

Hence $f'(x)$ is not R-integrable, as $\overline{\overline{\mathfrak{H}}} = h > 0$.

We have seen, in I, 549, that not every limited R-integrable function has a primitive. Volterra's function illustrates conversely the remarkable fact that *Not every limited derivative is R-integrable.*

2. It is easy to show, however, that *The derivative of Volterra's function is L-integrable.*

For let $\mathfrak{A}_\lambda$ denote the points of $\mathfrak{A}$ at which $f'(x) \geq \lambda$. Then when $\lambda > 1/m$, $m = 1, 2, \cdots$ $\mathfrak{A}_\lambda$ consists of an enumerable set of intervals. Hence in this case $\mathfrak{A}_\lambda$ is measurable. Hence $\mathfrak{A}_\lambda$, $\lambda > 0$, is measurable. Now $\mathfrak{A}$, $\lambda \geq 0$, differs from the foregoing by adding the points $\mathfrak{Z}_n$ in each δ_n at which $f'(x) = 0$, and the points $\mathfrak{H}$. But each $\mathfrak{Z}_n$ is enumerable, and hence a null set, and $\mathfrak{H}$ is measurable, as it is perfect. Thus $\mathfrak{A}_\lambda$, $\lambda \geq 0$, is measurable. In the same way we see $\mathfrak{A}_\lambda$ is measurable when λ is negative. Thus $\mathfrak{A}_\lambda$ is measurable for any λ, and hence L-integrable.

504. 1. We turn now to general considerations and begin by considering the upper and lower limits of the sum, difference, product, and quotient of two functions at a point $x = a$.

Let us note first the following theorem :

Let $f(x_1 \cdots x_m)$ be limited or not in $\mathfrak{A}$ which has $x = a$ as a limiting point. Let $\Phi_\delta = \operatorname{Max} f$, $\phi_\delta = \operatorname{Min} f$ in $V_\delta^*(a)$. Then

$$\varliminf_{x=a} f = \lim_{\delta=0} \phi_\delta \quad , \quad \varlimsup_{x=a} f = \lim_{\delta=0} \Phi_\delta .$$

This follows at once from I, 338.

2. *Let $f(x_1 \cdots x_m)$, $g(x_1 \cdots x_m)$ be limited or not in $\mathfrak{A}$ which has $x = a$ as limiting point.*

Let
$$\underline{\lim}\, f = \alpha \quad , \quad \underline{\lim}\, g = \beta$$
$$\overline{\lim}\, f = A \quad , \quad \overline{\lim}\, g = B$$

as $x \doteq a$. Then, these limits being finite,

$$\alpha + \beta \leq \underline{\overline{\lim}}\,(f+g) \leq A + B, \tag{1}$$

$$\alpha - B \leq \underline{\overline{\lim}}\,(f-g) \leq A - \beta. \tag{2}$$

For in any $V_\delta^*(a)$,

$$\text{Min}\, f + \text{Min}\, g \leq \text{Min}\,(f+g) \leq \text{Max}\,(f+g) \leq \text{Max}\, f + \text{Max}\, g.$$

Letting $\delta \doteq 0$, we get 1).
Also in $V_\delta^*(a)$,

$$\text{Min}\, f - \text{Max}\, g \leq \text{Min}\,(f-g) \leq \text{Max}\,(f-g) \leq \text{Max}\, f - \text{Min}\, g.$$

Letting $\delta \doteq 0$, we get 2).

3. *If*
$$f(x) \geq 0 \quad , \quad g(x) \geq 0,$$
$$\alpha\beta \leq \underline{\overline{\lim}}\, fg \leq AB. \tag{3}$$

If
$$f(x) \geq 0 \quad , \quad \beta \leq 0 \leq B,$$
$$A\beta \leq \underline{\overline{\lim}}\, fg \leq AB. \tag{4}$$

4. *If*
$$f(x) \geq 0 \quad , \quad g(x) \geq k > 0,$$
$$\frac{\alpha}{B} \leq \underline{\overline{\lim}}\, \frac{f}{g} \leq \frac{A}{\beta}. \tag{5}$$

If
$$\alpha < 0 < A \quad , \quad g(x) \geq k > 0,$$
$$\frac{\alpha}{\beta} \leq \underline{\overline{\lim}}\, \frac{f}{g} \leq \frac{A}{\beta}. \tag{6}$$

The relations 3), 4), 5), 6) may be proved as in 2. For example, to prove 5), we observe that in $V_\delta^*(a)$,

$$\frac{\text{Min}\, f}{\text{Max}\, g} \leq \text{Min}\, \frac{f}{g} \leq \text{Max}\, \frac{f}{g} \leq \frac{\text{Max}\, f}{\text{Min}\, g}.$$

5.
$$\alpha + \beta \leq \underline{\lim}\,(f+g) \leq \alpha + B. \tag{7}$$

$$\alpha + B \leq \overline{\lim}\,(f+g) \leq A + B. \tag{8}$$

$$\alpha - B \leq \underline{\lim}\,(f-g) \leq \alpha - \beta. \tag{9}$$

$$A - B \leq \overline{\lim}\,(f-g) \leq A - \beta. \tag{10}$$

If
$$f(x) \geq 0 \quad , \quad g(x) \geq 0,$$

$$\alpha\beta \leq \underline{\lim}\, fg \leq \alpha B, \tag{11}$$

$$A\beta \leq \overline{\lim}\, fg \leq AB. \tag{12}$$

If
$$g(x) \geq k > 0,$$

$$\frac{\alpha}{B} \leq \underline{\lim}\, \frac{f}{g} \leq \frac{\alpha}{\beta}, \tag{13}$$

$$\frac{A}{B} \leq \overline{\lim}\, \frac{f}{g} \leq \frac{A}{\beta}. \tag{14}$$

6. *If* $\lim f$ *exists,*

$$\underline{\lim}\,(f+g) = \lim f + \underline{\lim}\, g, \tag{15}$$

$$\overline{\lim}\,(f+g) = \lim f + \overline{\lim}\, g. \tag{16}$$

If $\lim g$ *exists,*

$$\underline{\lim}\,(f-g) = \underline{\lim}\, f - \lim g, \tag{17}$$

$$\overline{\lim}\,(f-g) = \overline{\lim}\, f - \lim g. \tag{18}$$

Let $f(x) \geq 0$, $g(x) \geq 0$. *Let* $\lim g$ *exist. Then*

$$\underline{\lim}\, fg = \underline{\lim}\, f \cdot \lim g, \tag{19}$$

$$\overline{\lim}\, fg = \overline{\lim}\, f \cdot \lim g. \tag{20}$$

If also $g(x) \geq k > 0$,

$$\underline{\lim}\, f/g = \underline{\lim}\, f/\lim g, \tag{21}$$

$$\overline{\lim}\, f/g = \overline{\lim}\, f/\lim g. \tag{22}$$

505. The preceding results can be used to obtain relations between the derivates of the sum, difference, product, and quotient of two functions as in I, 373 seq.

1. Let $w(x) = u(x) + v(x)$. Then

$$\frac{\Delta w}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}. \tag{1}$$

Thus from 504, 1), we get the *theorem:*

$$\underline{U}u' + v'\underline{U} \leq \underline{U}w' \leq \overline{U}u' + \overline{U}v'. \tag{2}$$

If u has a unilateral derivative Uu',

$$\underline{U}w' = Uu' + \underline{U}v', \tag{3}$$

$$\overline{U}w' = Uu' + \overline{U}v'. \tag{4}$$

We get 3), 4) from 1), using 504, 15), 16).

2. *In the interval $\mathfrak{A}$, u, v are continuous, u is monotone increasing, v is ≥ 0, and v' exists. Then, if $w = uv$, we have*

$$\underline{U}w' = uv' + v\underline{U}u', \tag{1}$$

$$\overline{U}w' = uv' + v\overline{U}u'. \tag{2}$$

For from

$$\frac{\Delta w}{\Delta x} = (u + \Delta u)\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x},$$

we have

$$\underline{U}w' = uv' + U\,\underline{\lim}\, v\frac{\Delta u}{\Delta x}$$

$$= uv' + vU\,\underline{\lim}\,\frac{\Delta u}{\Delta x}$$

which gives 1). Similarly we establish 2).

506. 1. We show now how we may generalize the Law of the Mean, I, 393.

Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$. Let m, M be the minimum and maximum of one of the four derivates of f in $\mathfrak{A}$. Then for any $\alpha < \beta$ in $\mathfrak{A}$,

$$m \leq \frac{f(\beta) - f(\alpha)}{\beta - \alpha} \leq M. \tag{1}$$

To fix the ideas let us take $\overline{R}f'(x)$ as our derivate. Suppose now there exists a pair of points $\alpha < \beta$ in $\mathfrak{A}$, such that

$$\frac{f(\beta) - f(\alpha)}{\beta - \alpha} = M + e \quad , \quad e > 0.$$

We introduce the auxiliary function

$$\phi(x) = f(x) - (M + c)x, \tag{2}$$

where

$$0 < c < e = c + \delta.$$

Then $$\frac{\phi(\beta) - \phi(\alpha)}{\beta - \alpha} = \frac{f(\beta) - f(\alpha)}{\beta - \alpha} - (M + c) = \delta.$$

Hence $$\phi(\beta) - \phi(\alpha) = \delta(\beta - \alpha) = \eta.$$

Consider now the equation

$$\phi(\beta) - \phi(x) = \eta.$$

It is satisfied for $x = \alpha$. If it is satisfied for any other x in the interval $(\alpha\beta)$, there is a last point, say $x = \gamma$, where it is satisfied, by 458, 3.

Thus for $x > \gamma$, $$\phi(x) \text{ is } > \phi(\alpha).$$

Hence $$\overline{R}\phi'(\gamma) \geq 0. \tag{3}$$

Now from 2) we have

$$\overline{R}f'(\gamma) = \overline{R}\phi'(\gamma) + M + c$$
$$> M.$$

Hence M is not the maximum of $\overline{R}f'(x)$ in $\mathfrak{A}$. Similarly the other half of 1) is established. The case that m or M is infinite is obviously true.

2. *Let $f(x)$ be defined over $\mathfrak{A} = (a < b)$. Let $a_1 < a_2 < \cdots < a_n$ lie in $\mathfrak{A}$. Let m and M denote the minimum and maximum of the difference quotients*

$$\Delta(a_1, a_2) \quad , \quad \Delta(a_2, a_3) \quad , \quad \cdots \Delta(a_{n-1}, a_n).$$

Then $$m \leq \Delta(a_1, a_n) \leq M. \tag{1}$$

For let us first take three points $\alpha < \beta < \gamma$ in $\mathfrak{A}$. We have identically

$$\Delta(\alpha, \gamma) = \frac{\alpha - \beta}{\alpha - \gamma} \cdot \Delta(\alpha, \beta) + \frac{\beta - \gamma}{\alpha - \gamma} \cdot \Delta(\beta, \gamma).$$

Now the coefficients of Δ on the right lie between 0 and 1. Hence 1) is true in this case. The general case is now obvious.

507. 1. *Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$. The four derivates of f have the same extremes in $\mathfrak{A}$.*

To fix the ideas let

$$\text{Min } \underline{L} = m \quad , \quad \text{Min } \overline{R} = \mu, \qquad \text{in } \mathfrak{A}.$$

We wish to show that $m = \mu$. To this end *we first show that*

$$\mu \leq m. \tag{1}$$

For there exists an α in $\mathfrak{A}$, such that

$$\underline{L}(\alpha) < m + \epsilon.$$

There exists therefore a $\beta < \alpha$ in $\mathfrak{A}$, such that

$$q = \frac{f(\alpha) - f(\beta)}{\alpha - \beta} < m + \epsilon', \qquad 0 < \epsilon' < \epsilon.$$

Now by 506, 1,

$$\mu = \text{Min } \overline{R} \leq q.$$

Hence $$\mu \leq m,$$
as $\epsilon > 0$ is small at pleasure.

We show now that

$$m \leq \mu. \tag{2}$$

For there exists an α in $\mathfrak{A}$, such that

$$\overline{R}(\alpha) < \mu + \epsilon.$$

There exists therefore a $\beta > \alpha$ in $\mathfrak{A}$, such that

$$q = \frac{f(\alpha) - f(\beta)}{\alpha - \beta} < \mu + \epsilon', \qquad 0 < \epsilon >' \epsilon.$$

Thus by 506, 1,

$$m = \text{Min } \underline{L} \leq q.$$

Hence as before $m \leq \mu$. From 1), 2) we have $m = \mu$.

2. In 499, we emphasized the fact that the left-hand derivates are not defined at the left-hand end point of an interval, and the right-hand derivates at the right-hand end point of an interval for which we are considering the values of a function. The following example shows that our theorems may be at fault if this fact is overlooked.

Example. Let $f(x) = |x|$.

If we restrict x to lie in $\mathfrak{A} = (0, 1)$, the four derivates $= 1$ when they are defined. Thus the theorem 1 holds in this case. If, however, we regarded the left-hand derivates as defined at $x = 0$, and to have the value

$$Lf'(0) = -1,$$

as they would have if we considered values of f to the left of $\mathfrak{A}$, the theorem 1 would no longer be true.

For then $$\text{Min } \underline{\overline{L}} = -1 \quad , \quad \text{Min } \underline{\overline{R}} = +1,$$

and the four derivates do not have the same minimum in $\mathfrak{A}$.

3. *Let $f(x)$ be continuous about the point $x = c$. If one of its four derivates is continuous at $x = c$, all the derivates defined at this point are continuous, and all are equal.*

For their extremes in any $V_\delta(c)$ are the same. If now $\overline{R}$ is continuous at $x = c$,

$$\overline{R}(c) - \epsilon < \overline{R}(x) < \overline{R}(c) + \epsilon,$$

for any x in some $V_\delta(c)$.

4. *Let $f(x)$ be continuous about the point $x = c$. If one of its four derivates is continuous at $x = c$, the derivative exists at this point.*

This follows at once from 3.

Remark. We must guard against supposing that the derivative is continuous at $x = c$, or even exists in the vicinity of this point.

Example. Let $F(x)$ be as in 501, Ex. 1. Let

$$\mathfrak{A} = (0, 1) \text{ and } \mathfrak{E} = \left\{\frac{1}{n}\right\}.$$

Let $$H(x) = x^2 F(x).$$

Then $$RH'(x) = 2xF(x) + x^2 RF'(x),$$

$$LH'(x) = 2xF(x) + x^2 LF'(x).$$

Obviously both RH' and LH' are continuous at $x = 0$ and $H'(0) = 0$. But H' does not exist at the points of $\mathfrak{E}$, and hence

does not exist in any vicinity $(0, \delta)$ of the origin, however small $\delta > 0$ is taken.

5. *If one of the derivates of the continuous function $f(x)$ is continuous in an interval $\mathfrak{A}$, the derivative $f'(x)$ exists, and is continuous in $\mathfrak{A}$.*

This follows from 3.

6. *If one of the four derivates of the continuous function $f(x)$* is $= 0$ *in an interval $\mathfrak{A}$, $f(x) =$ const in $\mathfrak{A}$.*

This follows from 3.

508. 1. *If one of the derivates of the continuous function $f(x)$ is* ≥ 0 in $\mathfrak{A} = (a < b)$, *$f(x)$ is monotone increasing in $\mathfrak{A}$.*

For then $m = \text{Min } \overline{R}f' \geq 0$, in $(a < x)$. Thus by 506, 1,

$$f(x) - f(a) \geq 0.$$

2. *If one of the derivates of the continuous function $f(x)$ is ≥ 0 in $\mathfrak{A}$, $f(x)$ is monotone decreasing.*

3. *If one of the derivates of the continuous function $f(x)$ is ≥ 0 in $\mathfrak{A}$, without being constantly 0 in any little interval of $\mathfrak{A}$, $f(x)$ is an increasing function in $\mathfrak{A}$. Similarly f is a decreasing function in $\mathfrak{A}$, if one of the derivates is ≤ 0, without being constantly 0 in any little interval of $\mathfrak{A}$.*

The proof is analogous to I, 403.

509. 1. *Let $f(x)$ be continuous in the interval $\mathfrak{A}$, and have a derivative, finite or infinite, within $\mathfrak{A}$. Then the points where the derivative is finite form a pantactic set in $\mathfrak{A}$.*

For let $\alpha < \beta$ be two points of $\mathfrak{A}$. Then by the Law of the Mean,

$$f'(\gamma) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha} \quad , \quad \alpha < \gamma < \beta.$$

As the right side has a definite value, the left side must have. Thus in any interval (α, β) in $\mathfrak{A}$, there is a point γ where the differential coefficient is finite.

2. *Let $f(x)$ be continuous in the interval $\mathfrak{A}=(a<b)$. Then $Uf'(x)$ cannot be constantly $+\infty$, or constantly $-\infty$ in $\mathfrak{A}$.*

For consider

$$\phi(x)=f(x)-f(a)-\frac{x-a}{b-a}\{f(b)-f(a)\},$$

which is continuous, and vanishes for $x=a$, $x=b$. We observe that $\phi(x)$ differs from $f(x)$ only by a linear function. If now $Uf'(x)=+\infty$ constantly, obviously $U\phi'(x)=+\infty$ also. Thus ϕ is a univariant function in $\mathfrak{A}$. This is not possible, since ϕ has the same value at a and b.

3. *Let $f(x)$ be continuous in $\mathfrak{A}=(a<b)$, and have a derivative, finite or infinite, in $\mathfrak{A}=(a^*, b)$. Then*

$$\text{Min} f'(x) \leq \underline{\overline{R}} f'(a) \leq \text{Max} f'(x) \quad , \quad \textit{in } \mathfrak{A}.$$

For the Law of the Mean holds, hence

$$\frac{f(a+h)-f(a)}{h}=f'(\alpha) \quad , \quad a<\alpha<a+h.$$

Letting now $h \doteq 0$, we get the theorem.

Remark. This theorem answers the question: Can a continuous curve have a vertical tangent at a point $x=a$, if the derivatives remain $<M$ in $V^*(a)$? The answer is, No.

4. *Let $f(x)$ be continuous in $\mathfrak{A}=(a<b)$, and have a derivative, finite or infinite, in $\mathfrak{A}^*=(a^*, b)$. If $f'(a)$ exists, finite or infinite, there exists a sequence $\alpha_1>\alpha_2>\cdots \doteq a$ in $\mathfrak{A}$, such that*

$$f'(a)=\lim_{n=\infty} f'(\alpha_n). \tag{1}$$

For

$$\frac{f(a+h)-f(a)}{h}=f'(\alpha_h) \quad , \quad a<\alpha_h<a+h. \tag{2}$$

Let now h range over $h_1>h_2>\cdots \doteq 0$. If we set $\alpha_n=\alpha_{h_n}$, the relation 1) follows at once from 2), since $f'(a)$ exists by hypothesis.

510. 1. *A right-hand derivate of a continuous function $f(x)$ cannot have a discontinuity of the $1°$ kind on the right. A similar statement holds for the other derivates.*

For let $R(x)$ be one of the right-hand derivates. It it has a discontinuity of the 1° kind on the right at $x=a$, there exists a number l such that

$$l-\epsilon \leq R(x) \leq l+\epsilon \quad , \quad \text{in some } (a < a+\delta).$$

Then by 506, 1,

$$l-\epsilon \leq \frac{f(a+h)-f(a)}{h} \leq l+\epsilon \quad , \quad 0 < h < \delta.$$

Hence
$$R(a)=l,$$

and $R(x)$ is continuous on the right at $x=a$, which is contrary to hypothesis.

2. It can, however, have a discontinuity of the 1° kind on the left, as is shown by the following

Example. Let $f(x)=|x|=+\sqrt{x^2}$, in $\mathfrak{A}=(-1, 1)$.

Here
$$R(x)=+1 \quad , \quad \text{for } x \geq 0 \text{ in } \mathfrak{A}$$
$$=-1 \quad , \quad \text{for } x < 0.$$

Thus at $x=0$, R is continuous on the right, but has a discontinuity of the 1° kind on the left.

3. *Let $f(x)$ be continuous in $\mathfrak{A}=(a, b)$, and have a derivative, finite or infinite, in $\mathfrak{A}^*=(a^*, b^*)$. Then the discontinuities of $f'(x)$ in $\mathfrak{A}$, if any exist, must be of the second kind.*

This follows from 1.

Example.
$$f(x)=x^2 \sin\frac{1}{x} \quad , \quad \text{for } x \neq 0 \text{ in } \mathfrak{A}=(0, 1)$$
$$=0 \quad , \quad \text{for } x=0.$$

Then
$$f'(x)=2x \sin\frac{1}{x} - \cos\frac{1}{x} \quad , \quad x \neq 0$$
$$=0 \quad , \quad x=0.$$

The discontinuity of $f'(x)$ at $x=0$, is in fact of the 2° kind.

4. *Let $f(x)$ be continuous in $\mathfrak{A}=(a<b)$, except at $x=a$, which is a point of discontinuity of the 2° kind. Let $f'(x)$ exist, finite or infinite, in (a^*, b). Then $x=a$ is a point of infinite discontinuity of $f'(x)$.*

For if

$$p = R\,\overline{\lim_{x=a}} f(x) \quad , \quad q = R\,\underline{\lim_{x=a}} f(x),$$

there exists a sequence of points $\alpha_1 > \alpha_2 > \cdots \doteq a$, such that $f(\alpha_n) \doteq p$; and another sequence $\beta_1 > \beta_2 > \cdots \doteq a$, such that $f(\beta_n) \doteq q$. We may suppose

$$\alpha_n > \beta_n \quad , \quad \text{or } \alpha_n < \beta_n \quad , \quad n = 1, 2, \cdots$$

Then the Law of the Mean gives

$$Q_n = \frac{f(\alpha_n) - f(\beta_n)}{\alpha_n - \beta_n} = f'(\gamma_n),$$

where γ_n lies between α_n, β_n. Now the numerator $\doteq p - q$, while the denominator $\doteq 0$. Hence $Q_n \doteq +\infty$, or $-\infty$, as we choose.

5. *Let $f(x)$ have a finite unilateral differential coefficient U at each point of the interval $\mathfrak{A}$. Then U is at most pointwise discontinuous in $\mathfrak{A}$.*

For by 474, 3, U is a function of class 1. Hence, by 486, 1, it is at most pointwise discontinuous in $\mathfrak{A}$.

511. *Let $f(x)$ be continuous in the interval $(a < b)$. Let $R(x)$ denote one of the right-hand derivates of $f(x)$. If R is not continuous on the right at a, then*

$$l \leq R(a) \leq m, \tag{1}$$

where

$$l = R\,\underline{\lim}\, R(x) \quad , \quad m = R\,\overline{\lim}\, R(x) \quad , \quad x \doteq a.$$

To fix the ideas let R be the upper right-hand derivate. Let us suppose that $\alpha = \overline{R}f'(a)$ were $> m$. Let us choose η, and c such that

$$m + \eta < c < \alpha. \tag{2}$$

We introduce the auxiliary function

$$\phi(x) = cx - f(x).$$

Then

$$\overline{R}\phi'(x) = c - \underline{R}f'(x) \quad , \quad \underline{R}\phi'(x) = c - \overline{R}f'(x). \tag{3}$$

Now if $\delta > 0$ is sufficiently small,

$$\overline{R}f'(x) < m + \eta \quad , \quad \text{for any } x \text{ in } \mathfrak{A}^* = (a^*, a + \delta).$$

Thus 2), 3), show that

$$\underline{R}\phi'(x) \geq \sigma \quad , \quad \sigma > 0.$$

Hence $\phi(x)$ is an increasing function in $\mathfrak{A}^*$. But, on the other hand,

$$\bar{R}f'(a) = \underline{R}f'(a),$$

since $\alpha > m$. Hence

$$\bar{R}\phi'(a) = c - \bar{R}f'(a) = c - \alpha < 0.$$

Hence ϕ is a decreasing function at $x = a$. This is impossible since ϕ is continuous at a. Thus $\alpha \leq m$.

Similarly we may show that $l \leq \alpha$.

512. 1. *Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$, and have a derivative, finite or infinite. If $\alpha = f'(a)$, $\beta = f'(b)$, then $f'(x)$ takes on all values between α, β, as x ranges over $\mathfrak{A}$.*

For let $\alpha < \gamma < \beta$, and let

$$Q(x, h) = \frac{f(x+h) - f(x)}{h} \quad , \quad h > 0.$$

We can take h so small that

$$Q(a, h) < \gamma \quad , \quad \text{and } Q(b, -h) > \gamma.$$

Now

$$Q(b, -h) = Q(b - h, h).$$

Hence

$$Q(b - h, h) > \gamma.$$

If now we fix h, $Q(x, h)$ is a continuous function of x. As Q is $< \gamma$, for $x = \alpha$, and $> \gamma$, for $x = b - h$, it takes on the value γ for some x, say for $x = \xi$, between a, $b - h$. Thus

$$Q(\xi, h) = \gamma.$$

But by the Law of the Mean,

$$Q(\xi, h) = f'(\eta),$$

where

$$a < \xi < \eta < \xi + h < b.$$

Thus $f'(x) = \gamma$, at $x = \eta$ in $\mathfrak{A}$.

2. *Let $f(x)$ be continuous in the interval $\mathfrak{A}$, and admit a derivative, finite or infinite. If $f'(x) = 0$ in $\mathfrak{A}$, except possibly at an enumerable set $\mathfrak{E}$, then $f' = 0$ also in $\mathfrak{E}$.*

For if $f'(\alpha) = 0$, and $f'(\beta) = b \neq 0$, then $f'(x)$ ranges over all values in $(0, b)$, as x passes from α to β. But this set of values has the cardinal number $\mathfrak{c}$. Hence there is a set of values in (α, β) whose cardinal number is $\mathfrak{c}$, where $f'(x) \neq 0$. This is contrary to the hypothesis.

3. *Let $f(x)$, $g(x)$ be continuous and have derivatives, finite or infinite, in the interval $\mathfrak{A}$. If in $\mathfrak{A}$ there is an α for which*

$$f'(\alpha) > g'(\alpha),$$

and a β for which

$$f'(\beta) < g'(\beta),$$

then there is a γ for which

$$f'(\gamma) = g'(\gamma),$$

provided

$$\delta(x) = f(x) - g(x)$$

has a derivative, finite or infinite.

For by hypothesis

$$\delta'(\alpha) > 0 \quad , \quad \delta'(\beta) < 0.$$

Hence by 1 there is a point where $\delta' = 0$.

513. 1. *If one of the four derivates of the continuous function $f(x)$ is limited in the interval $\mathfrak{A}$, all four are, and they have the same upper and lower R-integrals.*

The first part of the theorem is obvious from 507, 1. Let us effect a division of $\mathfrak{A}$ of norm d. Then

$$\overline{\int_{\mathfrak{A}}} \overline{R} = \lim_{d=0} \Sigma M_\iota d_\iota \quad , \quad M_\iota = \text{Max } \overline{R}_\iota \text{ in } d_\iota.$$

But the maximum of the three other derivates in d_ι is also M_ι by 507, 1. Hence the last part of the theorem.

2. *Let $f(x)$ be continuous and have a limited unilateral derivate as $\overline{R}$ in $\mathfrak{A} = (a < b)$. Then*

$$\underline{\int_a^b} \overline{R} dx \overline{\overline{<}} f(b) - f(a) \leq \overline{\int_a^b} \overline{R} dx. \qquad (1$$

For let $a < a_1 < a_2 < \cdots < b$ determine a division of $\mathfrak{A}$, of norm d.

Then by 506, 1,

$$\text{Min } \overline{R} \leq \frac{f(a_{m+1}) - f(a_m)}{a_{m+1} - a_m} \leq \text{Max } \overline{R},$$

in the interval $(a_m, a_{m+1}) = d_m$.

Hence

$$\Sigma d_m \text{ Min } \overline{R} \leq f(b) - f(a) \leq \Sigma d_m \text{ Max } \overline{R}.$$

Letting $d \doteq 0$, we get 1).

3. *If $f(x)$ is continuous, and $\underline{\overline{U}}f'$ is limited and R-integrable in $\mathfrak{A} = (a < b)$, then*

$$\int_a^b \underline{\overline{U}}f' = f(b) - f(a).$$

514. 1. *Let $f(x)$ be limited in $\mathfrak{A} = (a < b)$, and*

$$F(x) = \underline{\overline{\int}}_a^x f dx \quad , \quad a \leq x \leq b.$$

Then

$$U \underline{\lim}_{x=u} f \leq \underline{\overline{U}}F'(u) \leq U \overline{\lim}_{x=u} f, \tag{1}$$

for any u within $\mathfrak{A}$.

To fix the ideas let us take a right-hand derivate at $x = u$. Then

$$h \text{ Min} f \leq \underline{\overline{\int}}_u^{u+h} f dx \leq h \text{ Max} f \quad , \quad \text{in } (u^*, u+h),\ h > 0.$$

Thus

$$\text{Min} f \leq \frac{\Delta F}{\Delta x} \leq \text{Max} f.$$

Letting $h \doteq 0$, we get

$$R \underline{\lim}_{x=u} f \leq \underline{\overline{R}}F'(u) \leq R \overline{\lim}_{x=u} f,$$

which is 1) for this case.

2. *Let $f(x)$ be limited in the interval $\mathfrak{A} = (a < b)$. If $f(x+0)$ exists,*

$$R \text{ derivative} \int_a^x f dx = f(x+0);$$

and if $f(x-0)$ exists, $\quad a < x < b$

$$L \text{ derivative} \int_a^x f dx = f(x-0).$$

3. *Let $f(x)$ be limited and R-integrable in $\mathfrak{A} = (a < b)$. The points where*

$$F(x) = \int_a^x f dx \quad , \quad a \leq x \leq b$$

does not have a differential coefficient in $\mathfrak{A}$ form a null set.

For

$$F'(x) = f(x) \qquad \text{by I, 537, 1,}$$

when f is continuous at x. But by 462, 6, the points where f is not continuous form a null set.

515. In I, 400, we proved the theorem :

Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$, and let its derivative $= 0$ within $\mathfrak{A}$. Then f is a constant in $\mathfrak{A}$. This theorem we have extended in 507, 6, to a derivate of $f(x)$. It can be extended still farther as follows :

1. (*L. Scheefer*). *If $f(x)$ is continuous in $\mathfrak{A} = (a < b)$, and if one of its derivates $= 0$ in $\mathfrak{A}$ except possibly at the points of an enumerable set $\mathfrak{E}$, then $f =$ constant in $\mathfrak{A}$.*

If f is a constant, the theorem is of course true. We show that the contrary case leads to an absurdity, by showing that Card $\mathfrak{E}$ would $= \mathfrak{c}$, the cardinal number of an interval.

For if f is not a constant, there is a point c in $\mathfrak{A}$ where $p = f(c) - f(a)$ is $\neq 0$. To fix the ideas let $p > 0$; also let us suppose the given derivate is $\overline{R} = \overline{R}f'(x)$.

Let

$$g(x, t) = f(x) - f(a) - t(x - a) \quad , \quad t > 0.$$

Obviously $|g|$ is the distance f is above or below the secant line,

$$y = t(x - a) + f(a).$$

Thus in particular for any t,

$$g(a, t) = 0 \quad , \quad g(c, t) = p - t(c - a).$$

Let $q > 0$ be an arbitrary but fixed number $< p$. Then

$$\begin{aligned} g(c, t) - q &= p - q - t(c - a) \\ &= (p - q)\left\{1 - t\frac{c - a}{p - q}\right\} > 0, \end{aligned}$$

if $t < T$, where

$$T = \frac{p - q}{c - a}.$$

Hence

$$g(c, t) > q$$

for any t in the interval $\mathfrak{T} = (\tau, T)$, $0 < \tau < T$. We note that

$$\text{Card } \mathfrak{T} = \mathfrak{c}.$$

Since for any t in $\mathfrak{T}$, $g(a, t) = 0$, and $g(c, t) > q$, let $x = e_t$ be the maximum of the points $< c$ where $g(x, t) = q$. Then $e < c$, and for any h such that $e + h$ lies in (e, c),

$$0 < \frac{g(e+h) - g(e)}{h} = \frac{f(e+h) - f(e)}{h} - t.$$

Hence

$$\overline{R}f'(e) > 0.$$

Thus for any t in $\mathfrak{T}$, e_t lies in $\mathfrak{E}$. As t ranges over $\mathfrak{T}$, let e_t range over $\mathfrak{E}_1 \leq \mathfrak{E}$. To each point e of $\mathfrak{E}_1$ corresponds but one point t of $\mathfrak{T}$. For

$$0 = g(e, t) - g(e, t') = (t - t')(e - a).$$

Hence

$$t = t' \quad , \quad \text{as } e > a.$$

Thus

$$\text{Card } \mathfrak{T} = \text{Card } \mathfrak{E}_1 \leq \text{Card } \mathfrak{E},$$

which is absurd.

2. *Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$. Let $\mathfrak{C}$ denote the points of $\mathfrak{A}$ where one of the derivates has one sign. If $\mathfrak{C}$ exists,*

$$\textit{Card } \mathfrak{C} = \mathfrak{c}, \textit{ the cardinal number of the continuum.}$$

The proof is entirely similar to that in 1. For let c be a point of $\mathfrak{C}$. Then there exists a $d > c$ such that

$$f(d) - f(c) = p > 0.$$

We now introduce the function

$$g(x, t) = f(x) - f(c) - t(x - c) \quad , \quad t > 0,$$

and reason on this as we did on the corresponding g in 1, using here the interval (c, d) instead of (a, b). We get

$$\text{Card } \mathfrak{E}_1 = \text{Card } \mathfrak{T} = \mathfrak{c}.$$

3. *Let $f(x)$, $g(x)$ be continuous in the interval $\mathfrak{A}$. Let a pair of corresponding derivates as $\overline{R}f'$, $\overline{R}g'$ be finite and equal, except possibly at an enumerable set $\mathfrak{E}$. Then $f = g + C$, in $\mathfrak{A}$, where C is a constant.*

For let
$$\phi = f - g \quad , \quad \psi = g - f.$$
Then in
$$A = \mathfrak{A} - \mathfrak{E},$$
$$\overline{R}\phi' \geq \overline{R}f' - \overline{R}g' = 0 \quad , \quad \overline{R}\psi' \geq 0.$$

But if $\overline{R}\phi' < 0$ at one point in $\mathfrak{A}$, it is < 0 at a set of points $\mathfrak{B}$ whose cardinal number is $\mathfrak{c}$. But $\mathfrak{B}$ lies in $\mathfrak{E}$. Hence $\overline{R}\phi$ is never < 0, in $\mathfrak{A}$. The same holds for ψ. Hence, by 508, ϕ and ψ are both monotone increasing. This is impossible unless $\phi =$ a constant.

516. The preceding theorem states that the continuous function $f(x)$ in the interval $\mathfrak{A}$ is known in $\mathfrak{A}$, aside from a constant, when $f'(x)$ is finite and known in $\mathfrak{A}$, aside from an enumerable set.

Thus $f(x)$ is known in $\mathfrak{A}$ when f' is finite and known at each *irrational* point of $\mathfrak{A}$.

This is not the case when f' is finite and known at each *rational* point only in $\mathfrak{A}$.

For the rational points in $\mathfrak{A}$ being enumerable, let them be
$$r_1, r_2, r_3 \cdots \tag{1}$$
Let
$$l = l_1 + l_2 + l_3 + \cdots$$
be a positive term series whose sum l is $< \widehat{\mathfrak{A}}$. Let us place r_1 within an interval δ_1 of length $\leq l_1$. Let r_{ι_2} be the first number in 1) not in δ_1. Let us place it within a non-overlapping interval δ_2 of length $\leq l_2$, etc.

We now define a function $f(x)$ in $\mathfrak{A}$ such that the value of f at any x is the length of all the intervals and part of an interval lying to the left of x. Obviously $f(x)$ is a continuous function of x in $\mathfrak{A}$. At each rational point $f'(x) = 1$. But $f(x)$ is not determined aside from a constant. For $\Sigma\delta_n \leq l$. Therefore when l is small enough we may vary the position and lengths of the δ-intervals, so that the resulting f's do not differ from each other only by a constant.

517. 1. *Let $f(x)$ be continuous in $\mathfrak{A} = (a < b)$ and have a finite derivate, say $\overline{R}f'$, at each point of $\mathfrak{A}$. Let $\mathfrak{C}$ denote the points of $\mathfrak{A}$ where $\overline{R}$ has one sign, say > 0. If $\mathfrak{C}$ exists, it cannot be a null set.*

For let c be a point of $\mathfrak{C}$, then there exists a point $d > c$ such that

$$f(d) - f(c) = p > 0. \tag{1}$$

Let $\mathfrak{C}_n$ denote the points of $\mathfrak{C}$ where

$$n - 1 \leq \overline{R}f' < n. \tag{2}$$

Then $\mathfrak{C} = \mathfrak{C}_1 + \mathfrak{C}_2 + \cdots$ Let $0 < q < p$. We take the positive constants $q_1, q_2 \cdots$ such that

$$q_1 + 2\,q_2 + 3\,q_3 + \cdots \leq q.$$

If now $\mathfrak{C}$ is a null set, each $\mathfrak{C}_m$ is also. Hence the points of $\mathfrak{C}_m$ can be inclosed *within* a set of intervals δ_{mn} such that $\sum_n \delta_{mn} < q_m$. Let now $q_m(x)$ be the sum of the intervals and parts of intervals $\delta_{m,n}$, $n = 1, 2 \cdots$ which lie in the interval $(a \leq x)$. Let

$$Q(x) = \sum_m mq_m(x).$$

Obviously $Q(x)$ is a monotone increasing function, and

$$0 \leq Q(x) \leq q. \tag{3}$$

Consider now

$$P(x) = f(x) - f(a) - Q(x).$$

We have at a point of $\mathfrak{A} - \mathfrak{C}$,

$$\frac{\Delta P}{\Delta x} = \frac{\Delta f}{\Delta x} - \frac{\Delta Q}{\Delta x} \leq \frac{\Delta f}{\Delta x}\ , \quad \Delta x > 0.$$

Hence at such a point

$$\overline{R}P' \leq \overline{R}f' \leq 0.$$

But at a point x of $\mathfrak{C}$, $\overline{R}P' < 0$ also. For x must lie in some $\mathfrak{C}_m$, and hence within some δ_{mn}. Thus $q_m(x)$ increases by at least Δx when x is increased to $x + \Delta x$. Hence $mq_m(x)$, and thus $Q(x)$ is increased at least $m\Delta x$. Thus

$$\frac{\Delta Q}{\Delta x} \geq m.$$

Thus

$$\overline{R}P' \leq \overline{R}f' - m < 0, \text{ by 2)},$$

since x lies in $\mathfrak{E}_m$. Thus $\overline{R}P' \leq 0$ at any point of $\mathfrak{A}$. Thus P is a monotone decreasing function in $\mathfrak{A}$, by 508, 2. Hence

$$P(c) - P(d) \geq 0.$$

Hence
$$f(c) - f(d) - \{Q(c) - Q(d)\} \geq 0,$$

or using 1), 3)

$$p - q \leq 0,$$

which is not so, as p is $> q$.

2. (*Lebesgue.*) *Let $f(x)$, $g(x)$ be continuous in the interval $\mathfrak{A}$, and have a pair of corresponding derivates as $\overline{R}f'$, $\overline{R}g'$ which are finite at each point of $\mathfrak{A}$, and also equal, the equality holding except possibly at a null set. Then $f(x) - g(x) =$ constant in $\mathfrak{A}$.*

The proof is entirely similar to that of 515, 3, the enumerable set $\mathfrak{E}$ being here replaced by a null set. We then make use of 1.

518. *Let $f'(x)$ be continuous in some interval $\Delta = (u - \delta, u + \delta)$. Let $f''(x)$ exist, finite or infinite, in Δ, but be finite at the point $x = u$. Then*

$$f'(u) = \lim_{h=0} Qf, \tag{1}$$

where

$$Qf(u) = \frac{f(u+h) + f(u-h) - 2f(u)}{h^2}, \quad h > 0.$$

Let us *first suppose* that $f''(u) = 0$. We have for $0 < h \leq \eta < \delta$,

$$Qf = \frac{1}{h}\left\{\frac{f(u+h) - f(u)}{h} - \frac{f(u-h) - f(u)}{-h}\right\}$$

$$= \frac{1}{h}\{f'(x') - f'(x'')\}, \quad u < x' < u + h, \quad u - h < x'' < u$$

$$= \frac{1}{h}[(x' - u)\{f''(u) + \epsilon'\} - (x'' - u)\{f''(u) + \epsilon''\}],$$

where $|\epsilon'|$, $|\epsilon''|$ are $< \epsilon/2$ for η sufficiently small.

Now
$$\frac{x' - u}{h} \leq 1, \quad \frac{|x'' - u|}{h} \leq 1,$$

while
$$f''(u) = 0, \quad \text{by hypothesis.}$$

Hence
$$|Qf| < \epsilon, \quad \text{for } 0 < h \leq \eta,$$

and 1) holds in this case.

Suppose now that $f''(u)=a\neq 0$. Let

$$g(x)=f(x)-q(x) \quad , \quad \text{where } q(x)=\tfrac{1}{2}ax^2+bx+c.$$

Since $$q''(u)=a \quad , \quad g''(u)=0.$$

Thus we are in the preceding case, and $\lim Qg=0$.

But $$Qg=Qf-Qq.$$

Hence $$\lim Qf=a.$$

Maxima and Minima

519. 1. In I, 466 and 476, we have defined the terms $f(x)$ as a maximum or a minimum at a point. Let us extend these terms as follows. Let $f(x_1 \cdots x_m)$ be defined over $\mathfrak{A}$, and let $x=a$ be an inner point of $\mathfrak{A}$.

We say f has a maximum at $x=a$ if 1°, $f(a)-f(x)\geq 0$, for any x in some $V(a)$, and 2°, $f(a)-f(x)>0$ for some x in any $V(a)$. If the sign $\geq$ can be replaced by $>$ in 1°, we will say f has a *proper* maximum at a, when we wish to emphasize this fact; and when $\geq$ cannot be replaced by $>$, we will say f has an *improper* maximum. A similar extension of the old definition holds for the minimum. A common term for maximum and minimum is *extreme.*

2. If $f(x)$ is a constant in some segment $\mathfrak{B}$, lying in the interval $\mathfrak{A}$, $\mathfrak{B}$ is called a *segment of invariability*, or a *constant segment* of f in $\mathfrak{A}$.

Example. Let $f(x)$ be continuous in $\mathfrak{A}=(0, 1^*)$.

Let
$$x=\cdot a_1a_2a_3\cdots \tag{1}$$

be the expression of a point of $\mathfrak{A}$ in the normal form in the dyadic system. Let
$$\xi=\cdot \alpha_1\alpha_2\alpha_3\cdots \tag{2}$$

be expressed in the triadic system, where $\alpha_n=a_n$, when $a_n=0$, and $=2$ when $a_n=1$. The points $\mathfrak{C}=\{\xi\}$ form a Cantor set, I, 272. Let $\{\mathfrak{J}_n\}$ be the adjoint set of intervals. We associate

now the point 1) with the point 2), which we indicate as usual by $x \sim \xi$. We define now a function $g(x)$ as follows:

$$g(\xi) = f(x) \quad , \quad \text{when } x \sim \xi.$$

This defines g for all the points of $\mathfrak{C}$. In the interval $\mathfrak{J}_n$, let g have a constant value. Obviously g is continuous, and has a pantactic set of intervals in each of which g is constant.

3. We have given criteria for maxima and minima in I, 468 seq., to which we may add the following:

Let $f(x)$ be continuous in $(a - \delta, a + \delta)$. If $\underline{R}f'(a) > 0$ and $\overline{L}f'(a) < 0$, finite or infinite, $f(x)$ has a minimum at $x = a$.

If $\overline{R}f'(a) < 0$ and $\underline{L}f'(a) > 0$, finite or infinite, $f(x)$ has a maximum at $x = a$.

For on the 1° hypothesis, let us take α such that $\underline{R} - \alpha > 0$. Then there exists a $\delta' > 0$ such that

$$\frac{f(a+h) - f(a)}{h} > \underline{R} - \alpha > 0 \quad , \quad 0 < h \leq \delta'.$$

Hence

$$f(a+h) > f(a) \quad , \quad a + h \text{ in } (a^*, a + \delta').$$

Similarly if β is chosen so that $\overline{L} + \beta < 0$, there exists a $\delta'' > 0$, such that

$$\frac{f(a-h) - f(a)}{-h} < \overline{L} + \beta.$$

Hence

$$f(a-h) > f(a) \quad , \quad a + h \text{ in } (a - \delta'', a^*).$$

520. *Example 1.* Let $f(x)$ oscillate between the x-axis and the two lines $y = x$ and $y = -x$, similar to

$$y = \left| x \sin \frac{\pi}{x} \right|.$$

In any interval about the origin, y oscillates infinitely often, having an infinite number of proper maxima and minima. At the point $x = 0$, f has an improper minimum.

Example 2. Let us take two parabolas P_1, P_2 defined by $y = x^2$, $y = 2x^2$. Through the points $x = \pm \frac{1}{2}, \pm \frac{1}{3} \cdots$ let us erect ordinates, and join the points of intersection with P_1, P_2, alternately by straight lines, getting a broken line oscillating between the

parabolas P_1, P_2. The resulting graph defines a continuous function $f(x)$ which has proper extremes at the points $\mathfrak{E} = \left\{ \pm \frac{1}{n} \right\}$. However, unlike Ex. 1, the limit point $x = 0$ of these extremes is also a point at which $f(x)$ has a proper extreme.

Example 3. Let $\{\delta\}$ be a set of intervals which determine a Harnack set $\mathfrak{H}$ lying in $\mathfrak{A} = (0, 1)$. Over each interval $\delta = (\alpha, \beta)$ belonging to the n^{th} stage, let us erect a curve, like a segment of a sine curve, of height $h_n \doteq 0$, as $n \doteq \infty$, and having horizontal tangents at α, β, and at γ, the middle point of the interval δ. At the points $\{\xi\}$ of $\mathfrak{A}$ not in any interval δ, let $f(x) = 0$. The function f is now defined in $\mathfrak{A}$ and is obviously continuous. At the points $\{\gamma\}$, f has a proper maximum; at points of the type α, β, ξ, f has an improper minimum. These latter points form the set $\mathfrak{H}$ whose cardinal number is $\mathfrak{c}$. The function is increasing in each interval (α, γ), and decreasing in each (γ, β). It oscillates infinitely often in the vicinity of any point of $\mathfrak{H}$.

We note that while the points where f has a proper extreme form an enumerable set, the points of improper extreme may form a set whose cardinal number is $\mathfrak{c}$.

Example 4. We use the same set of intervals $\{\delta\}$ but change the curve over δ, so that it has a constant segment $\eta = (\lambda, \mu)$ in its middle portion. As before $f = 0$, at the points ξ not in the intervals δ.

The function $f(x)$ has now no proper extremes. At the points of $\mathfrak{H}$, f has an improper minimum; at the points of the type λ, μ, it has an improper maximum.

Example 5. Weierstrass' Function. Let $\mathfrak{E}$ denote the points in an interval $\mathfrak{A}$ of the type

$$x = \frac{r}{b^s} \quad , \quad r, s, \text{ positive integers.}$$

For such an x we have, using the notation of 502,

$$b^m x = \iota_m + \xi_m = b^{m-s} r.$$

Hence $$\xi_m = 0 \quad , \quad \text{for } m \geq s.$$

Thus $$e_m = (-1)^{\iota_m + 1} = (-1)^{r+1}.$$

Hence $$\operatorname{sgn} \frac{\Delta F}{\Delta x} = \operatorname{sgn} Q = \operatorname{sgn} e_m \eta_m = \operatorname{sgn} (-1)^r h.$$

Thus $$\operatorname{sgn} Rf'(x) = +1 \quad , \quad \operatorname{sgn} Lf'(x) = -1,$$

if r is even, and reversed if r is odd. Thus at the points $\mathfrak{C}$, the curve has a vertical cusp. By 519, 3, F has a maximum at the points $\mathfrak{C}$, when r is odd, and a minimum when r is even. The points $\mathfrak{C}$ are pantactic in $\mathfrak{A}$.

Weierstrass' function has no constant segment δ, for then $f'(x) = 0$ in δ. But F' does not exist at any point.

521. 1. *Let $f(x_1 \cdots x_m)$ be continuous in the limited or unlimited set $\mathfrak{A}$. Let $\mathfrak{E}$ denote the points of $\mathfrak{A}$ where f has a proper extreme. Then $\mathfrak{E}$ is enumerable.*

Let us first suppose that $\mathfrak{A}$ is limited. Let $\delta > 0$ be a fixed positive number. There can be but a finite number of points α in $\mathfrak{A}$ such that

$$f(\alpha) > f(x) \quad , \quad \text{in } V_\delta^*(\alpha). \tag{1}$$

For if there were an infinity of such points, let β be a limiting point and $\eta < \frac{1}{2}\delta$. Then in $V_\eta(\beta)$ there exist points α', α'' such that $V_\delta(\alpha')$, $V_\delta(\alpha'')$ overlap. Thus in one case

$$f(\alpha') > f(\alpha''),$$

and in the other

$$f(\alpha') < f(\alpha''),$$

which contradicts the first.

Let now $\delta_1 > \delta_2 > \cdots \doteq 0$. There are but a finite number of points α for which 1) holds for $\delta = \delta_1$, only a finite number for $\delta = \delta_2$, etc. Hence $\mathfrak{E}$ is enumerable. The case that $\mathfrak{A}$ is unlimited follows now easily.

2. We have seen that Weierstrass' function has a pantactic set of proper extremes. However, according to 1, they must be enumerable. In Ex. 3, the function has a minimum at each point of the non-enumerable set $\mathfrak{H}$; but these minima are improper. On the other hand, the function has a proper maximum at the points $\{\gamma\}$, but these form an enumerable set.

522. 1. *Let $f(x)$ be continuous in the interval $\mathfrak{A}$. Let f have a proper maximum at $x = \alpha$, and $x = \beta$ in $\mathfrak{A}$. Then there is a point γ between α, β where f has a minimum, which need not however be a proper minimum.*

For say $\alpha < \beta$. In the vicinity of α, $f(x)$ is $< f(\alpha)$; also in the vicinity of β, $f(x)$ is $< f(\beta)$. Thus there are points $\mathfrak{B}$ in (α, β) where f is $<$ either $f(\alpha)$ or $f(\beta)$. Let μ be the minimum of the values of $f(x)$, as x ranges over $\mathfrak{B}$. There is a least value of x in (α, β) for which $f(x) = \mu$. We may take this as the point in question. Obviously γ is neither α nor β.

2. That at the point γ, f does not need to have a proper minimum is illustrated by Exs. 1, or 3.

3. *In $\mathfrak{A} = (a, b)$ let $f'(x)$ exist, finite or infinite. The points within $\mathfrak{A}$ at which f has an extreme proper or improper, lie among the zeros of $f'(x)$.*

This follows from the proof used in I, 468, 2, if we replace there < 0, by ≤ 0, and > 0, by ≥ 0.

4. *Let $f'(x)$ be continuous in the interval $\mathfrak{A}$, and let $f(x)$ have no constant segments in $\mathfrak{A}$. The points $\mathfrak{E}$ of $\mathfrak{A}$ where f has an extreme, form an apantactic set in $\mathfrak{A}$. Let $\mathfrak{Z}$ denote the zeros of $f'(x)$ in $\mathfrak{A}$. If $\mathfrak{B} = \{\mathfrak{b}_n\}$ is the border set of intervals lying in $\mathfrak{A}$ corresponding to $\mathfrak{Z}$, $f(x)$ is univariant in each $\mathfrak{b}_n$.*

For by 3, the points $\mathfrak{E}$ lie in $\mathfrak{Z}$. As $f'(x)$ is continuous, $\mathfrak{Z}$ is complete and determines the border set $\mathfrak{B}$. Within each $\mathfrak{b}_n$, $f'(x)$ has one sign. Hence $f(x)$ is univariant in $\mathfrak{b}_n$.

5. *Let $f(x)$ be a continuous function having no constant segment in the interval $\mathfrak{A}$. If the points $\mathfrak{E}$ where f has an extreme form a pantactic set in $\mathfrak{A}$, then the points $\mathfrak{B}$ where $f'(x)$ does not exist or is discontinuous, form also a pantactic set in $\mathfrak{A}$.*

For if $\mathfrak{B}$ is not pantactic in $\mathfrak{A}$, there is an interval $\mathfrak{C}$ in $\mathfrak{A}$ containing no point of $\mathfrak{B}$. Thus $f'(x)$ is continuous in $\mathfrak{C}$. But the points of $\mathfrak{E}$ in $\mathfrak{C}$ form an apantactic set in $\mathfrak{C}$ by 4. This, however, contradicts our hypothesis.

Example. Weierstrass' function satisfies the condition of the theorem 5. Hence the points where $F'(x)$ does not exist or is

discontinuous form a pantactic set. This is indeed true, since F' exists at no point.

6. *Let $f(x)$ be continuous and have no constant segment in the interval $\mathfrak{A}$. Let $f'(x)$ exist, finite or infinite. The points where $f'(x)$ is finite and is $\neq 0$ form a pantactic set in $\mathfrak{A}$.*

For let $\alpha < \beta$ be any two points in $\mathfrak{A}$. If $f(\alpha) = f(\beta)$, there is a point $\alpha < \gamma < \beta$ such that $f(\alpha) \neq f(\gamma)$, since f has no constant segment in $\mathfrak{A}$. Then the Law of the Mean gives

$$f'(\xi) = \frac{f(\alpha) - f(\gamma)}{\alpha - \gamma} \neq 0.$$

Thus in the arbitrary interval (α, β) there is a point ξ, where $f'(x)$ exists and is $\neq 0$.

7. *Let $f'(x)$ be continuous in the interval $\mathfrak{A}$. Then any interval $\mathfrak{B}$ in $\mathfrak{A}$ which is not a constant segment contains a segment $\mathfrak{C}$ in which f is univariant.*

For since f is not constant in $\mathfrak{B}$, there are two points a, b in $\mathfrak{B}$ at which f has different values. Then by the Law of the Mean

$$f(a) - f(b) = (a - b)f'(c) \quad , \quad c \text{ in } \mathfrak{B}.$$

Hence $f'(c) \neq 0$. As $f'(x)$ is continuous, it keeps its sign in some interval $(c - \delta, c + \delta)$, and f is therefore univariant.

523. *Let $f(x)$ be continuous in the interval $\mathfrak{A}$, and have in any interval in $\mathfrak{A}$ a constant segment or a point at which f has an extreme. If $f'(x)$ exists, finite or infinite, it is discontinuous infinitely often in any interval in $\mathfrak{A}$, not a constant segment. At a point of continuity of the derivative, $f'(x) = 0$.*

For if $f'(x)$ were continuous in an interval $\mathfrak{B}$, not a constant segment, f would be univariant in some interval $\mathfrak{C} \leq \mathfrak{B}$, by 522, 7. But this contradicts the hypothesis, which requires that any interval as $\mathfrak{C}$ has a constant segment. Hence $f'(x)$ is discontinuous in any interval, however small.

Let now $x = c$ be a point of continuity. Then if c lies in a constant segment, $f'(c) = 0$ obviously. If not, there is a sequence of points $e_1, e_2 \cdots \doteq e$ such that $f(x)$ has an extreme at e_n. But then $f'(e_n) = 0$, by 522, 3. As $f'(x)$ is continuous at $x = c$, $f'(c) = 0$ also.

524. (*König.*) *Let $f(x)$ be continuous in $\mathfrak{A}$ and have a pantactic set of cuspidal points $\mathfrak{C}$. Then for any interval $\mathfrak{B}$ of $\mathfrak{A}$, there exists a β such that $f(x)=\beta$ at an infinite set of points in $\mathfrak{B}$. Moreover, there is a pantactic set of points $\{\xi\}$ in $\mathfrak{B}$, such that k being taken at pleasure,*

$$\underline{f}'(x) \leq k \leq \overline{f}'(x). \tag{1}$$

For among the points $\mathfrak{C}$ there is an infinite pantactic set $\mathfrak{c}$ of proper maxima, or of proper minima. To fix the ideas, suppose the former. Let $x=c$ be one of these points within $\mathfrak{B}$. Then there exists an interval $\mathfrak{b} \leq \mathfrak{B}$, containing c, such that

$$f(c) > f(x) \quad , \quad \text{for any } x \text{ in } \mathfrak{b}.$$

Let
$$\mu = \text{Min} f(x), \quad \text{in } \mathfrak{b}.$$

Then there is a point $\bar{x}$ where f takes on this minimum value. The point c divides the interval $\mathfrak{b}$ into two intervals. Let $\mathfrak{l}$ be that one of these intervals which contains $\bar{x}$, the other interval we denote by $\mathfrak{m}$. Within $\mathfrak{m}$ let us take a point c_1 of $\mathfrak{c}$. Then in $\mathfrak{l}$ there is a point c_1' such that

$$f(c_1) = f(c_1').$$

The point c_1 determines an interval $\mathfrak{b}_1$, just as c determined $\mathfrak{b}$. Obviously $\mathfrak{b}_1 \leq \mathfrak{m}$, and $\mathfrak{b}_1$ falls into two segments $\mathfrak{l}_1$, $\mathfrak{m}_1$ as before $\mathfrak{b}$ did. Within $\mathfrak{m}_1$ we take a point of $\mathfrak{c}$. Then in $\mathfrak{l}$ there is a point c_2', and in $\mathfrak{l}_1$ a point c_2'', such that

$$f(c_2) = f(c_2') = f(c_2'').$$

In this way we may continue indefinitely. Let

$$c_1' \quad , \quad c_2' \quad , \quad c_3' \cdots$$

be the points obtained in this way which fall in $\mathfrak{l}$. Let c' be a limit point of this set. Let

$$c_1'' \quad , \quad c_2'' \quad , \quad c_3'' \cdots$$

be the points obtained above which fall in $\mathfrak{l}_1$, and let c'' be a limit point of this set. Continuing in this way we get a sequence of limiting points

$$c' \quad , \quad c'' \quad , \quad c''' \cdots \tag{2}$$

lying respectively in $\mathfrak{l}$, $\mathfrak{l}_1$, $\mathfrak{l}_2 \cdots$

Since f is continuous,

$$f(c')=f(c'')=f(c''')=\cdots \tag{3}$$

Thus if we set $f(c')=\beta$ we see that $f(x)$ takes on the value β at the infinite set of points 2), which lie in $\mathfrak{B}$.

Let $\gamma_1, \gamma_2 \cdots$ be a set of points in 2) which $\doteq \gamma$.

Then

$$\frac{f(\gamma)-f(\gamma_1)}{\gamma-\gamma_1}=\frac{f(\gamma)-f(\gamma_2)}{\gamma-\gamma_2}=\cdots=0. \tag{4}$$

Thus if $f'(x)$ exists at $x=\gamma$, the equations 3) show that $f'(\gamma) = 0$. If f' does not exist at γ, they show that

$$\underline{f}' \leq 0 \leq \overline{f}' \quad , \quad \text{at } x=\gamma.$$

Let now k be taken at pleasure. Then

$$g(x)=f(x)-kx$$

is constituted as f, and

$$\overline{\underline{g}}'(x)=\overline{\underline{f}}'(x)-k.$$

This gives 1).

525. 1. *Lineo-Oscillating Functions.* The oscillations of a continuous function fall into two widely different classes, according as $f(x)$ becomes monotone on adding a linear function $l(x)=ax+b$, or does not.

The former are called lineo-oscillating functions. A continuous function which does not oscillate in $\mathfrak{A}$, or if it does is lineo-oscillating, we say is *at most* a lineo-oscillating function.

Example 1. Let

$$f(x)=\sin x \quad , \quad l(x)=x.$$

If we set

$$y=f(x)+l(x)$$

and plot the graph, we see at once that y is an increasing function. At the point $x=\pi$, the slope of the tangent to $f(x)=\sin x$ is greatest negatively, *i.e.* $\sin x$ is decreasing here fastest. But the angle that the tangent to $\sin x$ makes at this point is $-45°$, while the slope of the line $l(x)$ is constantly 45°. Thus at $x=\pi$, y has a point of inflection with horizontal tangent.

If we take $l(x)=ax$, $a>1$, y is an increasing function, increasing still faster than before.

All this can be verified by analysis. For setting

$$y = \sin x + ax \quad , \quad a > 1,$$

we get

$$y' = a + \cos x,$$

and

$$y' > 0.$$

Thus y is a lineo-oscillating function in any interval.

Example 2.

$$f(x) = x^2 \sin \frac{1}{x} \quad , \quad x \neq 0$$

$$= 0 \quad , \quad x = 0.$$

$$l(x) = ax + b \quad , \quad y = f(x) + l(x).$$

Then

$$y' = 2x \sin \frac{1}{x} - \cos \frac{1}{x} + a \quad , \quad x \neq 0$$

$$= a \quad , \quad x = 0.$$

Hence, if $a > 1 + 2\pi$, y is an increasing function in $\mathfrak{A} = (-\pi, \pi)$. The function f oscillates infinitely often in $\mathfrak{A}$, but is a lineo-oscillating function.

Example 3.

$$f(x) = x \sin \frac{1}{x} \quad , \quad x \neq 0$$

$$= 0 \quad , \quad x = 0.$$

$$l(x) = ax + b \quad , \quad y = f(x) + l(x).$$

Here

$$y' = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} + a \quad , \quad x \neq 0.$$

For $x = 0$, y' does not exist, finitely or infinitely.

Obviously, however great a is taken, y has an infinity of oscillations in any interval about $x = 0$. Hence f is not a lineo-oscillating function in such an interval.

2. *If one of the four derivates of the continuous function $f(x)$ is limited in the interval $\mathfrak{A}$, $f(x)$ is at most lineo-oscillating in $\mathfrak{A}$.*

For say $\underline{R}f' > -\alpha$ in $\mathfrak{A}$. Let $0 < \alpha < \beta$,

and

$$g(x) = f(x) + \beta x.$$

Then

$$\underline{g}'(x) = \beta + \underline{f}'(x) > 0.$$

Hence g is monotone increasing by 508, 1.

3. *Let $f(x)$ be at most lineo-oscillating in the interval $\mathfrak{A}$. If Uf' does not exist finitely at a point x in $\mathfrak{A}$, it is definitely infinite at the point. Moreover, the sign of the ∞ is the same throughout $\mathfrak{A}$.*

For if f is monotone in $\mathfrak{A}$, the theorem is obviously true. If not, let

$$g(x) = f(x) + ax$$

be monotone. Then

$$Uf' = Ug' - a,$$

and this case is reduced to the preceding.

Remark. This shows that no continuous function whose graph has a vertical cusp can be lineo-oscillating. All its vertical tangents correspond to points of inflection, as in

$$y = x^{\frac{1}{3}}.$$

Variation

526. 1. *Let $f(x)$ be continuous in the interval $\mathfrak{A}$, and have limited variation. Let D be a division of $\mathfrak{A}$ of norm d. Then using the notation of* 443,

$$\lim V_D f = Vf \quad , \quad \lim P_D f = Pf \quad , \quad \lim N_D f = Nf. \tag{1}$$

For there exists a division Δ such that

$$V - \frac{\epsilon}{2} < V_\Delta \leq V,$$

where for brevity we have dropped f after the symbol V. Let now Δ divide $\mathfrak{A}$ into ν segments whose minimum length call λ. Let D be a division of $\mathfrak{A}$ of norm $d \leq d_0 < \lambda$. Then not more than one point of Δ, say a_κ, can lie in any interval as $(a_\iota, a_{\iota+1})$ of D. Let $E = D + \Delta$, the division obtained by superposing Δ on D. Then μ denoting some integer $\leq \nu$,

$$V_E - V_D = \sum_{\kappa=1}^{\mu} \{ |f(a_\kappa) - f(a_\iota)| + |f(a_{\iota+1}) - f(a_\kappa)| - |f(a_{\iota+1}) - f(a_\iota)| \}.$$

If now d_0 is taken sufficiently small, Osc$f$ in any interval of D is as small as we choose, say $< \frac{\epsilon}{6\nu}$. Then

$$|V_E - V_D| \leq \frac{\mu\epsilon}{2\nu} \leq \frac{\epsilon}{2}.$$

But since E is got by superposing Δ on D,

$$V_\Delta < V_E \leq V.$$

Hence for any D of norm $< d_0$,

$$|V_D - V| < \epsilon,$$

which proves the first relation in 1. The other two follow at once now from 443.

527. *If $f(x)$ is continuous and has limited variation in the interval $\mathfrak{A} = (a < b)$, then*

$$P(x) \quad , \quad N(x) \quad , \quad V(x)$$

are also continuous functions of x in $\mathfrak{A}$.

Let us show that $V(x)$ is continuous; the rest of the theorem follows at once by 443.

By 526, there exists a d_0, such that for any division D of norm $d < d_0$,
$$V(b) = V_D(b) + \epsilon' \quad , \quad 0 \leq \epsilon' < \epsilon/3.$$

Then *a fortiori*, for any $x < b$ in $\mathfrak{A}$,

$$V(x) = V_D(x) + \epsilon_1 \quad , \quad 0 \leq \epsilon_1 < \epsilon/3. \tag{1}$$

In the division D, we may take x as one of the end points of an interval, and $x + h$ as the other end point. Then

$$V(x+h) = V_D(x) + |f(x+h) - f(x)| + \epsilon_2 \quad , \quad 0 \leq \epsilon_2 < \epsilon/3. \tag{2}$$

On the other hand, if d_0 is taken sufficiently small,

$$|f(x+h) - f(x)| < \frac{\epsilon}{3} \quad , \quad \text{for } 0 < h < \delta. \tag{3}$$

From 1), 2), 3) we have

$$0 \leq V(x+h) - V(x) < \epsilon \quad , \quad \text{for any } 0 \leq h < \delta. \tag{4}$$

But in the division D, x is the right-hand end point of some interval as $(x-k, x)$. The same reasoning shows that

$$|V(x-k)-V(x)|<\epsilon \quad , \quad \text{for any } 0 \leq k < \delta. \tag{5}$$

From 4), 5) we see $V(x)$ is continuous.

528. 1. *If one of the derivates of the continuous function $f(x)$ is numerically $\leq M$ in the interval $\mathfrak{A}$, the variation V of f is $\leq M\overline{\mathfrak{A}}$.*

For by definition
$$V = \text{Max } V_D,$$

with respect to all divisions $D = \{d_\iota\}$ of $\mathfrak{A}$. Here

$$V_D = \Sigma \, | f(a_\iota) - f(a_{\iota+1}) \, |.$$

Now by 506, 1,
$$-M \leq \frac{f(a_\iota) - f(a_{\iota+1})}{a_\iota - a_{\iota+1}} \leq M,$$

or
$$| f(a_\iota) - f(a_{\iota+1}) \, | \leq M d_\iota .$$

Hence
$$V_D \leq M \Sigma d_\iota \leq M \overline{\mathfrak{A}}.$$

2. *Let $f(x)$ be limited and R-integrable in $\mathfrak{A} = (a < b)$. Then*

$$F(x) = \int_a^x f dx \quad , \quad a \leq x \leq b$$

has limited variation in $\mathfrak{A}$.

For let D be a division of $\mathfrak{A}$ into the intervals $d_\iota = (a_\iota, a_{\iota+1})$.

Then
$$V_D \cdot F = \Sigma \, | F(a_{\iota+1}) - F(a_\iota) \, | = \Sigma \, | \int_{a_\iota}^{a_{\iota+1}} f dx \, |$$

$$\leq \Sigma \int_{a_\iota}^{a_{\iota+1}} | f | \, dx \leq M \Sigma \int_{a_\iota}^{a_{\iota+1}} dx = M \Sigma d_\iota = M \overline{\mathfrak{A}}.$$

Thus
$$\text{Max } V_D \cdot F \leq M \overline{\mathfrak{A}},$$

and F has limited variation.

529. 1. *If $f(x)$ has limited variation in the interval $\mathfrak{A}$, the points $\mathfrak{K}$ where* $\text{Osc} f \geq k$, *are finite in number.*

For suppose they were not. Then however large G is taken, we may take n so large that $nk > G$. There exists a division D

of $\mathfrak{A}$, such that there are at least n intervals, each containing a point of $\mathfrak{K}$ within it. Thus for the division D,

$$\Sigma \operatorname{Osc} f \geq nk > G.$$

Thus the variation of f is large at pleasure, and therefore is not limited.

2. *If f has limited variation in the interval $\mathfrak{A}$, its points of discontinuity form an enumerable set.*

This follows at once from 1.

530. 1. *Let $D_1, D_2 \cdots$ be a sequence of superposed divisions, of norms $d_n \doteq 0$, of the interval $\mathfrak{A}$. Let Ω_{D_n} be the sum of the oscillations of f in the intervals of D_n. If $\underset{n}{\operatorname{Max}}\, \Omega_{D_n}$ is finite, $f(x)$ has limited variation in $\mathfrak{A}$.*

For suppose f does not have limited variation in $\mathfrak{A}$. Then there exists a sequence of divisions $E_1, E_2 \cdots$ such that if Ω_{E_n} is the sum of the oscillations of f in the intervals of E_n, then

$$\Omega_{E_1} < \Omega_{E_2} < \cdots \doteq +\infty. \tag{1}$$

Let us take ν so large that no interval of D_ν contains more than one interval of E_n or at most parts of two E_n intervals. Let $F_n = E_n + D_\nu$. Then an interval δ of D_ν is split up into at most two intervals δ', δ'' in F_n. Let ω, ω', ω'' denote the oscillation of f in δ, δ', δ''. Then the term ω in D_ν goes over into

$$\omega' + \omega'' \leq 2\,\omega$$

in Ω_{F_n}. Hence if $\operatorname{Max} \Omega_{D_n} = M$,

$$\Omega_{E_n} \leq 2\,\Omega_{D_\nu} \leq 2\,M,$$

which contradicts 1).

2. Let $V_{D_n} = \Sigma\, |f(a_\iota) - f(a_{\iota+1})|$, the summation extended over the intervals $(a_\iota, a_{\iota+1})$ of the division D_n. If $\underset{n}{\operatorname{Max}}\, V_{D_n}$ is finite with respect to a sequence of superposed divisions $\{D_n\}$, we cannot say that f has limited variation.

Example. For let $f(x) = 0$, at the rational points in the interval $\mathfrak{A} = (0, 1)$, and $= 1$, at the irrational points. Let D_n be

obtained by interpolating the points $\frac{2m+1}{2^n}$ in $\mathfrak{A}$. Then $f=0$ at the end points a_ι, $a_{\iota+1}$ of the intervals of D_n. Hence $V_{D_n}=0$. On the other hand, $f(x)$ has not limited variation in $\mathfrak{A}$ as is obvious.

531. *Let* $F(x)=\lim_{t=\tau} f(x,t)$, τ *finite or infinite, for* x *in the interval* $\mathfrak{A}$. *Let* $\mathrm{Var}\, f(x,t) \leq M$ *for each* t *near* τ.

Then $F(x)$ *has limited variation in* $\mathfrak{A}$.

To fix the ideas let τ be finite. Let

$$F=f(x,t)+g(x,t).$$

Then for a division D of $\mathfrak{A}$,

$$V_D F \leq V_D f + V_D g.$$

But

$$V_D g = \Sigma \mid g(a_m) - g(a_{m+1}) \mid,$$

where (a_m, a_{m+1}) are the intervals of D.

But for some $t=t'$ near τ, each

$$g(a_\iota, t') < \frac{\eta}{2s},$$

where s is the number of intervals in the division D.

Thus

$$V_D g < \eta.$$

Hence

$$V_D F < M + \eta,$$

and F has limited variation.

532. *Let* $f(x)$, $g(x)$ *have limited variation in the interval* $\mathfrak{A}$, *then their sum, difference, and product have limited variation.*

If also

$$|g| \geq \gamma > 0 \quad , \quad \text{in } \mathfrak{A}$$

then f/g *has limited variation.*

Let us show, for example, that $h=fg$ has limited variation. For let

$$\begin{array}{ll} \mathrm{Min}\, f = m \quad , & \mathrm{Min}\, g = n \\ \mathrm{Osc}\, f = \omega \quad , & \mathrm{Osc}\, g = \tau \end{array} \quad \text{in the interval } d_\iota.$$

Then
$$f = m + \alpha\omega \quad , \quad g = n + \beta\tau \quad , \quad \text{in } d_\iota,$$
$$0 \leq \alpha \leq 1 \quad , \quad 0 \leq \beta \leq 1.$$

Thus
$$fg = mn + m\beta\tau + n\alpha\omega + \alpha\beta\omega\tau.$$

Now
$$mn - |m|\tau - |n|\omega - \omega\tau \leq fg \leq mn + |m|\tau + |n|\omega + \omega\tau.$$

Hence
$$\eta = \text{Osc}\, h \leq 2\{\tau|m| + \omega|n| + \omega\tau\}.$$

But
$$|m|, |n|, \tau \leq \text{some } K.$$

Thus
$$V_D h \leq 4\, K\Sigma\omega + 2\, K\Sigma\tau,$$
$$< \text{some } G,$$

and h has limited variation.

533. 1. Let us see what change will be introduced if we replace the finite divisions D employed up to the present by divisions E, which divide the interval $\mathfrak{A} = (a < b)$ into an infinite enumerable set of intervals $(a_\iota, a_{\iota+1})$.

Let
$$W_E = \sum_1^\infty |f(a_m) - f(a_{m+1})|, \tag{1}$$

and
$$W = \text{Max } V_E,$$

for the class of finite or infinite enumerable divisions $\{E\}$.

Obviously
$$W \geq V;$$

hence if W is finite, so is V.

We show that if V is finite, so is W. For suppose W were infinite. Then for any $G > 0$, there exists a division E, and an n, such that the sum of the first n terms in 1) is $\geq G$, or
$$W_{E,n} \geq G. \tag{2}$$

Let now D be the finite division determined by the points a_1, $a_2 \cdots a_{n+1}$ which figure in 2).

Then
$$V_D \geq G,$$

hence $V = \infty$, which is contrary to our hypothesis.

We show now that V and W are equal, when finite. For let E be so chosen that

$$W - \frac{\epsilon}{2} < W_E \leq W.$$

Now
$$W_E = W_{E,n} + \epsilon' \quad , \quad | \epsilon' | < \epsilon/2$$

if n is sufficiently large.

Let D correspond to the points $a_1 \, a_2 \cdots$ in $W_{E,n}$. Then

$$V_D \geq W_{E,n},$$

and hence
$$V_D + \epsilon' \geq W_{E,n} + \epsilon' = W_E.$$

Hence
$$W - V_D < \epsilon.$$

We may therefore state the theorem :

2. *If f has limited variation in the interval $\mathfrak{A}$ with respect to the class of finite divisions D, it has with respect to the class of enumerable divisions E, and conversely. Moreover*

$$\text{Max } V_D = \text{Max } V_E.$$

534. Let us show that *Weierstrass' function F*, considered in 502, *does not have limited variation in any interval* $\mathfrak{A} = (\alpha < \beta)$ *when* $ab > 1$. Since F is periodic, we may suppose $\alpha > 0$. Let

$$\frac{k}{b^m}, \ \frac{k+1}{b^m}, \cdots \frac{k+\mu}{b^m}$$

be the fractions of denominator b^m which lie in $\mathfrak{A}$.

These points effect a division D_m of $\mathfrak{A}$, and

$$V_{D_m} = \left| F\left(\frac{k}{b^m}\right) - F(\alpha) \right| + \sum_{j=0}^{\mu-1} \left| F\left(\frac{k+j+1}{b^m}\right) - F\left(\frac{k+j}{b^m}\right) \right|$$
$$+ \left| F(\beta) - F\left(\frac{k+\mu}{b^m}\right) \right|.$$

If l is the minimum of the terms F_j under the Σ sign,

$$V_{D_m} \geq \mu l. \tag{1}$$

Now
$$\frac{k-1}{b^m} < \alpha \quad , \quad \frac{k+\mu+1}{b^m} > \beta.$$

Hence
$$\overline{\mathfrak{A}} < \frac{\mu+2}{b^m} \quad , \quad \mu > b^m \overline{\mathfrak{A}} - 2. \tag{2}$$

On the other hand, using the notation and results of 502,

$$b^m x = \iota_m + \xi_m \quad , \quad h = \frac{\eta_m - \xi_m}{b^m};$$

and also

$$\left|\frac{F(x+h) - F(x)}{h}\right| \geq a^m b^m\left(\frac{2}{3} - \frac{\pi}{ab-1}\right). \tag{3}$$

Let us now take

$$\xi_m = 0, \qquad \eta_m = +1, \qquad \iota_m = k + j.$$

Then

$$x = \frac{k+j}{b^m} \quad , \quad h = \frac{1}{b^m}.$$

Hence from 3),

$$F_j \geq a^m\left(\frac{2}{3} - \frac{\pi}{ab-1}\right).$$

Thus

$$V_{D_m} \geq a^m\left(\frac{2}{3} - \frac{\pi}{ab-1}\right)(b^m \overline{\mathfrak{A}} - 2) \quad , \quad \text{by 1), 2).}$$

As $a < 1$, and $ab > 1$, we see that

$$V_{D_m} \doteq +\infty, \text{ as } m \doteq \infty.$$

Non-intuitional Curves

535. 1. Let $f(x)$ be continuous in the interval $\mathfrak{A}$. The graph of f is a continuous curve C. If f has only a finite number of oscillations in $\mathfrak{A}$, and has a tangent at each point, we would call C an ordinary or intuitional curve. It might even have a finite number of angle points, *i.e.* points where the right-hand tangent is different from the left-hand one [cf. I, 366]. But if there were an infinity of such points, or an infinity of points in the vicinity of each of which f oscillates infinitely often, the curve grows less and less clear to the intuition as these singularities increase in number and complexity. Just where the dividing point lies between curves whose peculiarities can be clearly seen by the intuition, and those which cannot, is hard to say. Probably different persons would set this point at different places.

For example, one might ask: Is it possible for a continuous curve to have tangents at a pantactic set of points, and no tangent at another pantactic set? If one were asked to picture such a curve to the imagination, it would probably prove an impossibility.

Yet such curves exist, as Ex. 3 in 501 shows. Such curves might properly be called *non-intuitional.*

Again we might ask of our intuition: Is it possible for a continuous curve to have a tangent at every point of an interval $\mathfrak{A}$, which moreover turns abruptly at a pantactic set of points? Again the answer would not be forthcoming. Such curves exist, however, as was shown in Ex. 2 in 501.

We wish now to give other examples of non-intuitional curves. Since their singularity depends on their derivatives or the nature of their oscillations, they may be considered in this chapter.

Let us first show how to define curves, which, like Weierstrass' curve, have a pantactic set of cusps. To effect this we will extend the theorem of 500, 2, so as to allow $g(x)$ to have a cusp at $x = 0$.

536. *Let* $\mathfrak{E} = \{e_n\}$ *denote the rational points in the interval* $\mathfrak{A} = (-a, a)$. *Let* $g(x)$ *be continuous in* $\mathfrak{B} = (-2a, 2a)$, *and* $= 0$, *at* $x = 0$. *Let* $\mathfrak{B}^*$ *denote the interval* $\mathfrak{B}$ *after removing the point* $x = 0$. *Let* g *have a derivative in* $\mathfrak{B}^*$, *such that*

$$|g'(x)| \leq \frac{M}{|x|^\alpha} \quad , \quad \alpha > 0. \tag{1}$$

Then

$$F(x) = \sum \frac{1}{n^{2+\alpha+\beta}} g(x - e_n) = \Sigma a_n g(x - e_n) \quad , \quad \beta > 0$$

is a continuous function in $\mathfrak{A}$, *and* $\frac{\Delta F}{\Delta x}$ *behaves at* $x = e_m$ *essentially as* $\frac{\Delta g}{\Delta x}$ *does at the origin.**

To simplify matters, let us suppose that $\mathfrak{E}$ does not contain the origin. Having established this case, it is easy to dispose of the general case. We begin by ordering the e_n as in 233. Then obviously if

$$e_n = \frac{p}{q} \quad , \quad q > 0 \quad , \quad p \text{ positive or negative,}$$

we have

$$n \geq q.$$

Let

$$e_{mn} = e_m - e_n. \qquad \text{If } e_m = \frac{r}{s},$$

$$|e_{mn}| = \left|\frac{p}{q} - \frac{r}{s}\right| \geq \frac{1}{qs} \geq \frac{1}{mn}. \tag{2}$$

* Cf. Dini, *Theorie der Functionen, etc.*, p. 192 *seq.* Leipzig, 1892.

Let $E(x)$ be the F series after deleting the m^{th} term. Then

$$F(x) = a_m g(x - e_m) + E(x).$$

We show that E has a differential coefficient at $x = e_m$, obtained by differentiating E termwise. To this end we show that as $h \doteq 0$,

$$D(h) = \sum a_n \frac{g(e_{mn} + h) - g(e_{mn})}{h} \quad , \quad m \neq n \tag{3}$$

converges to

$$G = \Sigma a_n g'(e_{mn}) \quad , \quad m \neq n. \tag{4}$$

That is, we show

$$\epsilon > 0 \quad , \quad \eta > 0 \quad , \quad | D(h) - G | < \epsilon \quad , \quad 0 < | h | < \eta. \tag{5}$$

Let us break up the sums 3), 4) which figure in 5), into three parts

$$\sum_1^\infty = \sum_1^r + \sum_{r+1}^s + \sum_{s+1}^\infty. \tag{6}$$

Thus

$$| D - G | \leq | D_r - G_r | + | D_{r,s} - G_{r,s} | + | \overline{D}_s - \overline{G}_s | \tag{7}$$
$$\leq A + B + C.$$

Since $g'(e_{mn})$ exists, the first term may be made as small as we choose for an arbitrary but fixed r; thus

$$A < \frac{\epsilon}{3}.$$

Let us now turn to B. We have

$$B \leq | D_{rs} | + | G_{rs} |,$$

$$\frac{g(e_{mn} + h) - g(e_{mn})}{h} = g'(e_{mn} + h') \quad , \quad | h' | < | h |$$

provided $g'(x)$ exists in the interval $(e_{mn}, e_{mn} + h)$.

But by 2),

$$| e_{mn} + h' | \geq | e_{mn} | - | h | \geq \frac{1}{2\,mn} \geq \frac{1}{2\,ms} \quad , \quad \text{for } r < n \leq s$$

if

$$\eta < \frac{1}{2\,ms}. \tag{8}$$

Thus by 1),

$$| g'(e_{mn} + h') | \leq 2^a M m^a n^a < M_1 n^a \quad , \quad M_1 \text{ a constant.}$$

Hence *a fortiori*, $$| g'(e_{mn}) | < M_1 n^{\alpha}. \tag{9}$$

Now the sum
$$H = \sum \frac{1}{n^{1+\mu}}$$

converges if $\mu > 0$. Hence $H_{p,q}$ and $\overline{H}_p$ may be made as small as we choose, by taking p sufficiently large. Let us note that by 91,
$$\overline{H}_p < \frac{1}{\mu} \frac{1}{p^{\mu}}. \tag{10}$$

Thus if $\mu = \text{Min}(\alpha, \beta)$,
$$B \leq | D_{rs} | + | G_{rs} | \leq 2 \sum_{r+1}^{s} \frac{M_1 n^{\alpha}}{n^{2+\alpha+\beta}} = 2 M_1 H_{r,s} < \frac{\epsilon}{3},$$
for a sufficiently large r.

We consider finally C. We have
$$C \leq | \overline{D}_s | + | \overline{G}_s |$$
$$\leq \frac{1}{| h |} \sum_{s+1}^{\infty} a_n | g(e_{mn} + h) | + \frac{1}{| h |} \sum_{s+1}^{\infty} a_n | g(e_{mn}) | + | \overline{G}_s |$$
$$\leq C_1 + C_2 + C_3.$$

From 9) we see that
$$C_3 < M_1 \overline{H}_s < \frac{\epsilon}{6},$$

for s sufficiently large. Since $g(x)$ is continuous in $\mathfrak{B}$,
$$| g(x) | < N.$$

Hence
$$C_1 \text{ and } C_2 \leq \frac{1}{| h |} \sum_{s+1}^{\infty} \frac{N}{n^{2+\alpha+\beta}} \leq \frac{N}{| h |} \frac{1}{1 + \alpha + \beta} \cdot \frac{1}{s^{1+\alpha+\beta}}$$
$$\leq \frac{N}{1 + \alpha + \beta} \cdot \frac{1}{s^{\alpha+\beta}} \quad ,$$

if $s \geq \frac{1}{| h |}$, on using 10).

Taking s still larger if necessary, we can make
$$C_1, \; C_2 < \frac{\epsilon}{6}.$$

Thus
$$C < \frac{\epsilon}{3}.$$

The reader now sees why we broke the sum 6) into three parts. As $h \doteq 0$, the middle term contains an increasing number of terms. But whatever given value h has, s has a finite value.

Thus as A, B, C are each $< \epsilon/3$, the relation 5) is established.

Hence E has a differential coefficient at $x = e_m$, and as

$$\frac{\Delta F}{h} = a_m \frac{\Delta(0)}{h} + \frac{\Delta E}{h},$$

our theorem is established.

537. *Example 1.* Let

$$g(x) = \sqrt[3]{x^2}.$$

Then for $x \neq 0$, $g'(x) = \dfrac{2}{3} \dfrac{1}{\sqrt[3]{x}}$. Here $\alpha = \frac{1}{3}$.

For $x = 0$, $$Rg'(x) = +\infty \quad , \quad Lg'(x) = -\infty.$$

Thus

$$F(x) = \sum \frac{\sqrt[3]{(x - e_n)^2}}{n^{\frac{2}{3}+\beta}} \quad , \quad \beta > 0$$

is a continuous function, and at the rational points e_m in the interval $\mathfrak{A}$,

$$RF'(x) = +\infty \quad , \quad LF'(x) = -\infty.$$

Hence the graph of F has a pantactic set of cuspidal tangents in $\mathfrak{A}$. The curve is not monotone in any interval of $\mathfrak{A}$, however small.

Example 2. Let

$$g(x) = x \sin \frac{1}{x} \quad , \quad x \neq 0$$

$$= 0 \quad , \quad x = 0.$$

Then

$$g'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} \quad , \quad x \neq 0.$$

Here $\alpha = 1$. For $x = 0$,

$$\overline{g}'(x) = +1 \quad , \quad \underline{g}'(x) = -1.$$

Then
$$F(x) = \sum \frac{1}{n^{3+\beta}} (x - e_n) \sin \frac{1}{x - e_n} \quad , \quad \beta > 0$$

is a continuous function in $\mathfrak{A}$, and at the rational point e_m,

$$\overline{F}'(x) = \frac{1}{n^{3+\beta}} + E'(e_m)$$

$$\underline{F}'(x) = -\frac{1}{n^{3+\beta}} + E'(e_m),$$

where E is the series obtained from F by deleting the m^{th} term.

538. *Pompeiu Curves.** Let us now show the existence of curves which have a tangent at each point, and a pantactic set of vertical inflectional tangents.

We first prove the theorem (*Borel*):

Let
$$B(x) = \sum \frac{a_n}{x - e_n} = \sum \frac{a_n}{r_n} \quad , \quad a_n > 0,$$

where $\mathfrak{E} = \{e_n\}$ *is an enumerable set in the interval* $\mathfrak{A}$, and

$$A = \Sigma \sqrt{a_n}$$

is convergent. Then B *converges absolutely and uniformly in a set* $\mathfrak{B} < \mathfrak{A}$, *and* $\widehat{\mathfrak{B}}$ *is as near* $\widehat{\mathfrak{A}}$ *as we choose.*

The points $\mathfrak{D}$ *where adjoint* B *is divergent form a null set.*

For let us enclose each point e_n in an interval δ_n of length $\frac{2\sqrt{a_n}}{k}$, with e_n as center.

The sum of these intervals is

$$\leq \sum \frac{2\sqrt{a_n}}{k} = \frac{2A}{k} < \epsilon,$$

for $k > 0$ sufficiently large. Let now k be fixed. A point x of $\mathfrak{A}$ will not lie in any δ_n if

$$r_n = |x - e_n| > \frac{\sqrt{a_n}}{k}.$$

Then at such a point,

$$\text{Adjoint } B < \Sigma a_n \frac{k}{\sqrt{a_n}} = k \Sigma \sqrt{a_n} = kA.$$

* *Math. Annalen*, v. 63 (1907), p. 326.

As $\hat{\mathfrak{B}} > \hat{\mathfrak{A}} - \epsilon$, the points $\mathfrak{D}$ where B does not converge absolutely form a null set.

539. 1. We now consider the function

$$F(x) = \sum_1^\infty a_n(x - e_n)^{\frac{1}{3}} = \Sigma f_n(x) \tag{1}$$

where $\mathfrak{E} = \{e_n\}$ is an enumerable pantactic set in an interval $\mathfrak{A}$, and

$$A = \Sigma a_n \tag{2}$$

is a convergent positive term series.

Then *F is a continuous function of x in $\mathfrak{A}$.* For $|x - e_n|^{\frac{1}{3}}$ is < some M in $\mathfrak{A}$.

Let us note that each $f_n(x)$ is an increasing function and the curve corresponding to it has a vertical inflectional tangent at the point $x = e_n$.

We next show that *$F(x)$ is an increasing function in $\mathfrak{A}$.* For let $x' < x''$. Then

$$f_n(x') < f_n(x'').$$

Hence

$$F_n(x') < F_n(x'').$$

Thus

$$\overline{F}_n(x') \leq \overline{F}_n(x'').$$

Hence

$$F(x') < F(x'').$$

2. Let us now consider the convergence of

$$D(x) = \frac{1}{3}\sum \frac{a_n}{(x - e_n)^{\frac{2}{3}}} \tag{3}$$

obtained by differentiating F termwise at the points of $\mathfrak{A} - \mathfrak{E}$. Let $\mathfrak{D}$ denote the points in $\mathfrak{A}$ where

$$B = \sum \frac{a_n}{|x - e_n|} \tag{4}$$

diverges. We have seen $\mathfrak{D}$ is a null set if

$$\Sigma\sqrt{a_n} \tag{5}$$

is convergent. Let $\mathfrak{A} = \mathfrak{D} + \mathfrak{C}$. Let x be a point of $\mathfrak{C}$, *i.e.* a point where 4) is convergent. We break 3) into two parts

$$D = D_1 + D_2,$$

such that in D_1, each $\xi_n < 1$. Then D_2 is obviously convergent, since each of its terms

$$\frac{a_n}{\xi_n^{\frac{2}{3}}} \le a_n, \qquad \text{where } \xi_n = |x - e_n|,$$

and the series 2) is convergent.

The series D_1 is also convergent. For as $\xi_n < 1$, the term

$$\frac{a_n}{\xi_n^{\frac{2}{3}}} < \frac{a_n}{\xi_n}$$

and the series 4) converges by hypothesis, at a point x in $\mathfrak{C}$. *Hence $D(x)$ is convergent at any point in $\mathfrak{C}$, and $\overline{\mathfrak{C}} = \overline{\mathfrak{A}}$ when 5) is convergent.*

3. Let C denote the points in $\mathfrak{A}$ where 3) converges. Let $\mathfrak{A} = C + \Delta$.

We next show that $F'(x) = D(x)$, for x in C. For taking x at pleasure in C but fixed,

$$Q(h) = \frac{\Delta F}{\Delta x} = \Sigma\, a_n \frac{(x + h - e_n)^{\frac{1}{3}} - (x - e_n)^{\frac{1}{3}}}{h}, \quad \Delta x = h. \tag{6}$$

We now apply 156, 2, showing that Q is uniformly convergent in $(0^*, \eta)$. By direct multiplication we find that

$$\frac{(a+b)^{\frac{1}{3}} - a^{\frac{1}{3}}}{b} = \frac{1}{(a+b)^{\frac{2}{3}} + a^{\frac{1}{3}}(a+b)^{\frac{1}{3}} + a^{\frac{2}{3}}}.$$

Thus 6) gives

$$Q(h) = \sum \frac{a_n}{(x + h - e_n)^{\frac{2}{3}} + (x + h - e_n)^{\frac{1}{3}}(x - e_n)^{\frac{1}{3}} + (x - e_n)^{\frac{2}{3}}}.$$

Let us set

$$h_n = \left(\frac{x + h - e_n}{x - e_n}\right)^{\frac{1}{3}}.$$

Then

$$Q(h) = \sum \frac{1}{1 + h_n + h_n^2} \cdot \frac{a_n}{(x - e_n)^{\frac{2}{3}}} \le 2 \sum \frac{a_n}{\xi^{\frac{2}{3}}}, \tag{7}$$

for $0 < |h| \leq \eta$, η sufficiently small. As the series on the right is independent of h, Q converges uniformly in $(0^*, \eta)$. Thus by 156, 2

$$F' = D \quad , \quad \text{for any } x \text{ in } C.$$

4. *Let now x be a point of Δ, not in $\mathfrak{E}$. At such a point we show that*

$$F'(x) = +\infty, \tag{8}$$

and thus the curve F has a vertical inflectional tangent. For as D is divergent at x, there exists for each M an m, such that

$$D_m > 2M.$$

But the middle term in 7) shows that for $|h| <$ some η' each term in Q_m is $> \frac{1}{2}$ the corresponding term in D_m. Thus

$$Q_m(h) > M \quad , \quad 0 < |h| < \eta'.$$

Since each term of Q is > 0, as 7) shows,

$$Q(h) > M.$$

Hence 8) is established.

5. Let us finally consider the points $x = e_m$. If Φ denotes the series obtained from F by deleting the m^{th} term, we have

$$\frac{\Delta F}{\Delta x} = \frac{a_m}{h^{\frac{2}{3}}} + \frac{\Delta \Phi}{\Delta x} \quad , \quad \text{for } x = e_m.$$

As F is increasing, the last term is ≥ 0.

Hence
$$F'(x) = +\infty \quad , \quad \text{in } \mathfrak{E}.$$

As a result we see the curve F has at each point a tangent. At an enumerable pantactic set V, it has points of inflection with vertical tangents.

7. Let us now consider the *inverse of the function F*, which we denote by

$$x = G(t). \tag{9}$$

As x in 1) ranges over the interval $\mathfrak{A}$, $t = F(x)$ will range over an interval $\mathfrak{B}$, and by I, 381, the inverse function 9) is a one-valued continuous function of t in $\mathfrak{B}$ which has a tangent at each

point of $\mathfrak{B}$. If W are the points in $\mathfrak{B}$ which correspond to the points V in $\mathfrak{A}$, then the tangent is parallel to the t-axis at the points W, or $G'(t)=0$, at these points. The points W are pantactic in $\mathfrak{B}$.

Let Z denote the points of $\mathfrak{B}$ at which $G'(t)=0$. We show that Z is of the 2° category, and therefore

$$\text{Card } Z = \mathfrak{c}.$$

For $G'(t)$ being of class ≤ 1 in $\mathfrak{B}$, its points of discontinuity δ form a set of the 1° category, by 486, 2. On the other hand, the points of continuity of G' form precisely the set Z, since the points W are pantactic in $\mathfrak{B}$ and $G'=0$ in W. In passing let us note that the points Z in $\mathfrak{B}$ correspond 1–1 to a set of points $\mathfrak{Z}$ at which the series 3) diverges. For at these points the tangent to F is vertical. But at any point of convergence of 3), we saw in 2 that the tangent is not vertical.

Finally we observe that 3) shows that

$$\text{Min } D(x) > \frac{1}{3}\cdot\frac{1}{\widehat{\mathfrak{A}}^{\frac{2}{3}}}\Sigma a_n \quad , \quad \text{in } \mathfrak{A}.$$

Hence

$$\text{Max } G'(t) \leq \frac{3\,\widehat{\mathfrak{A}}^{\frac{2}{3}}}{\Sigma a_n}.$$

Summing up, we have this result:

8. *Let the positive term series* $\Sigma\sqrt{a_n}$ *converge. Let* $\mathfrak{E}=\{e_n\}$ *be an enumerable pantactic set in the interval* $\mathfrak{A}$. *The Pompeiu curves defined by*

$$F(x) = \Sigma a_n(x-e_n)^{\frac{1}{3}}$$

have a tangent at each point in $\mathfrak{A}$, *whose slope is given by*

$$F'(x) = \frac{1}{3}\sum \frac{a_n}{(x-e_n)^{\frac{2}{3}}},$$

when this series is convergent, i.e. for all x *in* $\mathfrak{A}$ *except a null set. At a point set* $\mathfrak{Z}$ *of the* 2° *category which embraces* $\mathfrak{E}$, *the tangents are vertical. The ordinates of the curve* F *increase with* x.

540. 1. *Faber Curves.** Let $F(x)$ be continuous in the interval $\mathfrak{A}=(0, 1)$. Its graph we denote by F. For simplicity let

* *Math. Annalen*, v. 66 (1908), p. 81.

$F(0)=0$, $F(1)=l_0$. We proceed to construct a sequence of broken lines or polygons,

$$L_0, L_1, L_2 \cdots \tag{1}$$

which converge to the curve F as follows:

As first line L_0 we take the segment joining the end points of F. Let us now divide $\mathfrak{A}$ into n_1 equal intervals

$$\delta_{11}, \delta_{12} \cdots \delta_{1, n_1} \tag{2}$$

of length

$$\delta_1 = \frac{1}{n_1},$$

and having

$$a_{11}, a_{12}, a_{13} \cdots \tag{3}$$

as end points. As second line L_1 we take the broken line or polygon joining the points on F whose abscissæ are the points 3). We now divide each of the intervals 2) into n_2 equal intervals, getting the $n_1 n_2$ intervals

$$\delta_{21}, \delta_{22}, \delta_{23} \cdots \tag{4}$$

of length

$$\delta_2 = \frac{1}{n_1 n_2},$$

and having

$$a_{21}, a_{22}, a_{23} \cdots \tag{5}$$

as end points. In this way we proceed on indefinitely. Let us call the points

$$A = \{a_{mn}\}$$

terminal points. The number of intervals in the r^{th} division is

$$\nu_r = n_1 \cdot n_2 \cdots n_r.$$

If $L_m(x)$ denote the one-valued continuous function in $\mathfrak{A}$ whose value is the ordinate of a point on L_m, we have

$$F(a_{mn}) = L_m(a_{mn}), \tag{6}$$

since the vertices of L_m lie on the curve F.

2. *For each x in $\mathfrak{A}$,*

$$\lim_{m=\infty} L_m(x) = F(x). \tag{7}$$

For if x is a terminal point, 7) is true by 6).

If x is not a terminal point, it lies in a sequence of intervals

$$\delta_1 > \delta_2 > \cdots$$

belonging to the 1°, 2° ⋯ division of $\mathfrak{A}$.

Let

$$\delta_m = (a_{m,n}, a_{m,n+1}).$$

Since $F(x)$ is continuous, there exists an s, such that

$$|F(x) - F(a_{m,n})| \leq \frac{\epsilon}{2}, \quad m > s \tag{8}$$

for any x in δ_m. As $L_m(x)$ is monotone in δ_m,

$$\begin{aligned} |L_m(x) - L_m(a_{mn})| &\leq |L_m(a_{mn}) - L_m(a_{m,n+1})| \\ &\leq |F_m(a_{mn}) - F_m(a_{m,n+1})| \\ &\leq \frac{\epsilon}{2}, \quad \text{by 8)}. \end{aligned}$$

Thus

$$|L_m(x) - F_m(a_{mn})| \leq \frac{\epsilon}{2}. \tag{9}$$

Hence from 8), 9),

$$|F(x) - L_m(x)| < \epsilon \quad , \quad m \geq s$$

which is 7).

3. *We can write* 7) *as a telescopic series.* For

$$L_1 = L_0 + (L_1 - L_0)$$

$$L_2 = L_1 + (L_2 - L_1) = L_0 + (L_1 - L_0) + (L_2 - L_1)$$

etc. Hence

$$F(x) = \lim L_n(x) = L_0(x) + \sum_1^\infty \{L_n(x) - L_{n-1}(x)\}.$$

If we set

$$f_0(x) = L_0(x) \quad , \quad f_n(x) = L_n(x) - L_{n-1}(x), \tag{10}$$

we have

$$F(x) = \sum_0^\infty f_n(x), \tag{11}$$

and

$$F_n(x) = \sum_0^n f_s(x) = L_n(x). \tag{12}$$

The function $f_n(x)$, as 10) shows, is the difference between the ordinates of two successive polygons L_{n-1}, L_n at the point x. It may be positive or negative. In any case its graph is a polygon

f_n which has a vertex on the x-axis at the end point of each interval δ_{n-1}. Let l_{ns} be the value of $f_n(x)$ at the point $x = a_{ns}$, that is, at a point corresponding to one of the vertices of f_n. We call l_{ns} the *vertex differences* of the polygon L_n.

Let
$$p_n = \underset{s}{\mathrm{Min}}\, | l_{ns} | \quad , \quad q_n = \underset{s}{\mathrm{Max}}\, | l_{ns} | .$$

Then
$$| f_n(x) | \leq q_n \quad , \quad \text{in } \mathfrak{A}. \tag{13}$$

In the foregoing we have supposed $F(x)$ given. Obviously if the vertex differences were given, the polygons 1) could be constructed successively.

We now show:

If
$$\Sigma q_n \tag{14}$$
is convergent,
$$F(x) = \Sigma f_n(x)$$
is uniformly convergent in $\mathfrak{A}$, *and is a continuous function in* $\mathfrak{A}$.

For by 13), 14), F converges uniformly in $\mathfrak{A}$. As each $f_n(x)$ is continuous, F is continuous in $\mathfrak{A}$.

The functions so defined may be called *Faber functions.*

541. 1. We now investigate the *derivatives of Faber's functions*, and begin by proving the theorem:

If
$$\underset{s}{\Sigma} n_1 \cdots n_s q_s = \Sigma \nu_s q_s \tag{1}$$
converge, the unilateral derivatives of $F(x)$ *exist in* $\mathfrak{A} = (0, 1)$. *Moreover they are equal, except possibly at the terminal points* $A = \{a_{mn}\}$.

For let x be a point not in A. Let x', x'' lie in $V = V_\eta^*(x)$; let $x' - x = h'$, $x'' - x = h''$.

Let
$$Q = \frac{F(x') - F(x)}{h'} - \frac{F(x'') - F(x)}{h''}.$$

Then $F'(x)$ exists at x, if
$$\epsilon > 0 \quad , \quad \eta > 0 \quad , \quad | Q | < \epsilon \quad , \quad \text{for any } x', x'' \text{ in } V. \tag{2}$$

Now

$$|Q| \leq \left| \frac{F_m(x') - F_m(x)}{h'} - \frac{F_m(x'') - F_m(x)}{h''} \right| + \left| \frac{\overline{F}_m(x') - \overline{F}_m(x)}{h'} \right| + \left| \frac{\overline{F}_m(x'') - \overline{F}_m(x)}{h''} \right|$$

$$\leq Q_1 + Q_2 + Q_3.$$

But

$$\left| \frac{f_s(x') - f_s(x)}{x' - x} \right| \leq \frac{2\, q_s}{\delta_s}.$$

Hence

$$Q_2 \leq 2 \sum_{s=m+1}^{\infty} \nu_s q_s < \frac{\epsilon}{2}\quad,\quad m \text{ sufficiently large.}$$

Similarly

$$Q_3 < \frac{\epsilon}{2}.$$

Finally, if η is taken sufficiently small, x, x', x'' will correspond to the side of the polygon L_m. Hence using 540, 12), we see that $Q_1 = 0$. Thus 2) holds, and $F'(x)$ exists at x.

If x is a terminal point a_{mn}, and the two points x', x'' are taken on the same side of a_{mn}, the same reasoning shows that the unilateral derivatives exist at a_{mn}. They may, however, be different.

2. *Let* $n_1 = n_2 = \cdots = 2$. *For the differential coefficient* $F'(x)$ *to exist at the terminal point* x, *it is necessary that*

$$\overline{\lim}\ 2^n q_n < \infty. \tag{3}$$

If

$$\overline{\lim}\ 2^n p_n = \infty, \tag{4}$$

the points where the differential coefficient does not exist form a pantactic set in $\mathfrak{A}$.

Let us *first* prove 3). Let $b < a < c$ be terminal points. Then they belong to every division after a certain stage. We will therefore suppose that b, c are consecutive points in the n^{th} division, and a is a point of the $n + 1^{\text{st}}$ division falling in the interval $\delta_n = (b, c)$. If a differential coefficient is to exist at a,

$$\frac{F(a) - F(b)}{a - b} \text{ and } \frac{F(a) - F(c)}{a - c} \tag{5}$$

must be numerically less than some M, as $n \doteq \infty$, and hence their sum Q remains numerically $< 2\,M$.

Now

$$F(a) = L_{n+1}(a) \quad , \quad F(b) = L_n(b) \quad , \quad F(c) = L_n(c),$$

$$|a-b| = |a-c| = \delta_n = \frac{1}{2^{n+1}}.$$

Thus
$$Q = 2^{n+1}\{2\, L_{n+1}(a) - [L_n(b) + L_n(c)]\}$$

$$= 2 \cdot 2^{n+1}\left\{L_{n+1}(a) - \frac{L_n(b) + L_n(c)}{2}\right\},$$

or
$$|\, Q \,| = 4 \cdot 2^n l_{n,s} \quad , \quad \text{supposing } a = a_{ns}.$$

Hence
$$2^n q_n < M,$$

which establishes 3).

Let us now consider 4). By hypothesis there exists a sequence $n_1 < n_2 < \cdots \doteq \infty$, such that

$$2^{n_m} p_{n_m} > G \quad , \quad m = 1, 2 \cdots,$$

G being large at pleasure. Hence at least one of the difference quotients 5) belonging to this sequence of divisions is numerically large at pleasure.

3. *If*
$$\lambda = \Sigma l_{ms} \tag{1}$$

is absolutely convergent, the functions $F(x)$ have limited variation in $\mathfrak{A}$.

For $f_m(x)$ is monotone in each interval δ_{ms}. Hence in δ_{ms},

$$\text{Var}\, f_m = |\, l_{ms} - l_{m,s+1} \,| \leq |\, l_{ms} \,| + |\, l_{m,s+1} \,|.$$

Hence in $\mathfrak{A}$,
$$\text{Var}\, f_m(x) \leq 2 \sum_s l_{ms}.$$

Hence
$$\text{Var}\, F_n(x) \leq 2 \sum_{m=1}^{n} \sum_s l_{ms} = 2\,\lambda \quad , \quad \text{in } \mathfrak{A}.$$

We apply now 531.

542. *Faber Functions without Finite or Infinite Derivatives.*

To simplify matters let us consider the following example. The method employed admits easy generalization and gives a *class* of functions of this type. We use the notation of the preceding sections.

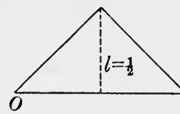


FIG. 1

Let $f_0(x)$ have as graph Fig. 1. We next divide $\mathfrak{A}=(0, 1)$ into $2^{1!}$ equal parts δ_{11}, δ_{12} and take $f_1(x)$ as in Fig. 2. We now divide $\mathfrak{A}$ into $2^{2!}$ equal parts δ_{21}, δ_{22}, δ_{23}, δ_{24} and take $f_2(x)$ as in Fig. 3. The height of the peaks is $l_2=\dfrac{l}{10^2}$.

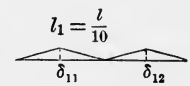


FIG. 2

In the m^{th} division $\mathfrak{A}$ falls into $2^{m!}$ equal parts

$$\delta_{m1}, \delta_{m2} \cdots$$

one of which may be denoted by

$$\delta_m = (a_{mn}, a_{m,n+1}).$$

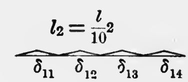


FIG. 3

Its length may be denoted by the same letter, thus

$$\delta_m = \frac{1}{2^{m!}}.$$

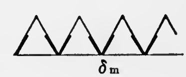


FIG. 4

In Fig. 4, δ_m is an interval of the $m-1^{\text{st}}$ division.

The maximum ordinate of $f_m(x)$ is $l_m=\dfrac{l}{10^m}=\dfrac{1}{2}\cdot\dfrac{1}{10^m}$. The part of the curve whose points have an ordinate $\leq \frac{1}{2} l_m$ have been marked more heavily. The x of such points, form class 1. The other x's make up class 2. With each x in class 1, we associate the points $\alpha_m < \beta_m$ corresponding to the peaks of f_m adjacent to x. Thus $\alpha_m < x < \beta_m$. If x is in class 2, the points α_m, β_m are the adjacent valley points, where $f_m = 0$.

Let now x be a point of class 1. The numerators in

$$\frac{f_m(\beta_m)-f(x)}{\beta_m - x} \qquad \frac{f_m(\alpha_m)-f(x)}{\alpha_m - x} \tag{1}$$

have like signs, while their denominators are of opposite sign. Thus the signs of the quotients 1) are different. Similarly if x belongs to class 2, the signs of 1) are opposite. Hence for any x,

the signs of 1) are opposite. It will be convenient to let e_m denote either α_m or β_m. We have

$$|x - e_m| < \delta_m,$$

$$|f_m(x) - f_m(e_m)| \geq \tfrac{1}{2} l_m. \tag{2}$$

Hence
$$\left|\frac{f_m(x) - f_m(e_m)}{x - e_m}\right| > \frac{1}{4}\frac{2^{m!}}{10^m}. \tag{3}$$

On the other hand, for any $x \neq x'$ in δ_m,

$$\left|\frac{f_m(x') - f_m(x)}{x' - x}\right| \leq \frac{2\, l_m}{\delta_m}.$$

Hence setting $x' = e_n$, and letting $n > m$,

$$|f_m(e_n) - f_m(x)| \leq \frac{l_m}{\delta_m} |e_n - x| < l_m \cdot \frac{\delta_n}{\delta_m}$$

$$\leq \frac{1}{10^m} \cdot \frac{2^{m!}}{2^{n!}} < \frac{1}{10^m} \cdot \frac{2^{n-1!}}{2^{n!}}$$

$$< \frac{1}{10^n} \cdot \frac{1}{10^m}. \tag{4}$$

For if $\log_2 a$ be the logarithm of a with the base 2,

$$n - 1! > \frac{n}{n-1} \log_2 10 \quad , \quad \text{for } n \text{ sufficiently large.}$$

Hence
$$(n-1)!(n-1) > \log_2 10^n.$$

Thus
$$\frac{2^{n!}}{2^{(n-1)!}} > 10^n \quad , \quad \text{or } \frac{2^{n-1!}}{2^{n!}} < \frac{1}{10^n},$$

and this establishes 4).

Let us now extend the definition of the functions $f_n(x)$ by giving them the period 1. The corresponding Faber function $F(x)$ defined by 540, 12) will admit 1 as period. We have now

$$F(e_n) - F(x) = \{f_n(e_n) - f_n(x)\} + \{F_{n-1}(e_n) - F_{n-1}(x)\}$$
$$+ \{\overline{F}_{n+1}(e_n) - \overline{F}_{n+1}(x)\} = T_1 + T_2 + T_3.$$

From 2) we have
$$T_1 \geq \tfrac{1}{2} l_n.$$

As to T_2, we have, using 4) and taking n sufficiently large,

$$|T_2| < \frac{1}{10^n}\sum_{m=1}^{n-1}\frac{1}{10^m} < \frac{1}{10^n}\sum_1^{\infty}\frac{1}{10^m} = \frac{1}{9}\cdot\frac{1}{10^n}$$

$$< \tfrac{2}{9}\, l_n.$$

Similarly

$$|T_3| \leq \sum_{m=n+1}^{\infty} |f_m(e_n) - f_m(x)| \leq \sum_{m=n+1}^{\infty} \{f_m(e_n) + f_m(x)\}$$

$$\leq \sum_{n+1}^{\infty} 2\, l_m = \frac{1}{9}\cdot\frac{1}{10^n}$$

$$\leq \tfrac{2}{9}\, l_n.$$

Thus finally

$$|F(e_n) - F(x)| > l_n(\tfrac{1}{2} - \tfrac{2}{9} - \tfrac{2}{9})$$

$$> \tfrac{1}{18}\, l_n.$$

As

$$|T_1| > |T_2| + |T_3|$$

$$\text{sgn}\, \frac{F(e_n) - F(x)}{e_n - x} = \text{sgn}\, \frac{f_n(e_n) - f_n(x)}{e_n - x}.$$

Thus

$$\left|\frac{F(e_n) - F(x)}{e_n - x}\right| > \frac{1}{18}\frac{l_n}{\delta_n}$$

$$> \frac{1}{36}\frac{2^{n!}}{10^n} \doteq \infty.$$

As e_n may be at pleasure α_n or β_n, and as the signs of 1) are opposite, we see that

$$\overline{F}'(x) = +\infty \quad , \quad \underline{F}'(x) = -\infty\, ;$$

and $F(x)$ has neither a finite nor an infinite differential coefficient at any point.

CHAPTER XVI

SUB- AND INFRA-UNIFORM CONVERGENCE

Continuity

543. In many places in the preceding pages we have seen how important the notion of uniform convergence is when dealing with iterated limits. We wish in this chapter to treat a kind of uniform convergence first introduced by *Arzelà*, and which we will call *subuniform.* By its aid we shall be able to give conditions for integrating and differentiating series termwise much more general than those in Chapter V.

We refer the reader to Arzelà's two papers, "*Sulle Serie di Funzioni*," *R. Accad. di Bologna*, ser. V, vol. 8 (1899). Also to a fundamental paper by *Osgood*, *Am. Journ. of Math.*, vol. 19 (1897), and to another by *Hobson*, *Proc. Lond. Math. Soc.*, ser. 2, vol. 1 (1904).

544. 1. Let $f(x_1 \cdots x_m, t_1 \cdots t_n) = f(x, t)$ be a function of two sets of variables. Let $x = (x_1 \cdots x_m)$ range over $\mathfrak{X}$ in an m-way space, and $t = (t_1 \cdots t_n)$ range over $\mathfrak{T}$ in an n-way space. As x ranges over $\mathfrak{X}$ and t over $\mathfrak{T}$, the point $(x_1 \cdots t_1 \cdots) = (x, t)$ will range over a set $\mathfrak{A}$ lying in a space $\mathfrak{R}_p$, $p = m + n$.

Let τ, finite or infinite, be a limiting point of $\mathfrak{T}$.

Let
$$\lim_{t=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m) \qquad \text{in } \mathfrak{X}.$$

Let the point x range over $\mathfrak{B} \leq \mathfrak{X}$, while t remains fixed, then the point (x, t) will range over a *layer of ordinate* t, which we will denote by $\mathfrak{L}_t$. We say x belongs to or is associated with this layer.

We say now that $f \doteq \phi$, *subuniformly* in $\mathfrak{X}$ when for each $\epsilon > 0$, $\eta > 0$:

1° There exists a finite number of layers $\mathfrak{L}_t$ whose ordinates t lie in $V_\eta^*(\tau)$.

2° Each point x of $\mathfrak{X}$ is associated with one or more of these layers. Moreover if $x = a$ belongs to the layer $\mathfrak{L}_t$, all the points x in some $V_\delta(a)$ also belong to $\mathfrak{L}_t$.

3° $$|f(x, t) - \phi(x)| < \epsilon$$

while (x, t) ranges over *any* one of the layers $\mathfrak{L}_t$. When $m = 1$, that is when there is but a single variable x which ranges over an interval, the layers reduce to segments. For this reason Arzelà calls the convergence *uniform in segments.*

2. In case that subuniform convergence is applied to the series

$$F(x_1 \cdots x_m) = \Sigma f_n(x_1 \cdots x_m)$$

convergent in $\mathfrak{A}$, we may state the definition as follows:

F converges subuniformly in $\mathfrak{A}$ when

1° For each $\epsilon > 0$, and for each ν there exists a finite set of layers of ordinates $\geq \nu$, call them

$$\mathfrak{L}_1, \mathfrak{L}_2 \cdots \tag{2}$$

such that each point x of $\mathfrak{A}$ belongs to one or more of them, and if $x = a$ belongs to $\mathfrak{L}_m$, then all the points of $\mathfrak{A}$ near a also belong to $\mathfrak{L}_m$.

2° $$|\overline{F}_n(x_1 \cdots x_m)| < \epsilon$$

as the point (x, n) ranges over any one of the layers 2).

545. *Example.* Let

$$F(x) = \sum_1^\infty \left\{ \frac{nx}{1 + n^2x^2} - \frac{(n-1)x}{1 + (n-1)^2x^2} \right\} \qquad \text{in } \mathfrak{A} = (-1, 1).$$

Here

$$F_n(x) = \frac{nx}{1 + n^2x^2}, \quad F(x) = 0.$$

The series converges uniformly in $\mathfrak{A}$, except at $x = 0$. The convergence is therefore not uniform in $\mathfrak{A}$; it is, however, subuniform. For

$$|\overline{F}_n(x)| = \frac{n\,|x|}{1 + n^2x^2}.$$

Hence taking m at pleasure and fixed,

$$|\bar{F}_m| < \epsilon \quad , \quad x \text{ in } s_1 = (-\delta, \delta),$$

δ sufficiently small. On the other hand,

$$|\bar{F}_n| \leq \frac{n}{1 + \frac{1}{4} n^2 \delta^2} \qquad \text{in } (-1, -\tfrac{1}{2}\delta) + (\tfrac{1}{2}\delta, 1) = s_2 + s_3.$$

Thus for n sufficiently large,

$$|\bar{F}_n| < \epsilon.$$

Hence we need only three segments s_1, s_2, s_3 to get subuniform convergence.

546. 1. *Let* $f(x_1 \cdots x_m, t_1 \cdots t_n) \doteq \phi(x_1 \cdots x_m)$ *in* $\mathfrak{X}$, *as* $t \doteq \tau$, *finite or infinite. Let* $f(x, t)$ *be continuous in* $\mathfrak{X}$ *for each* t *near* τ. *For* ϕ *to be continuous at the point* $x = a$ *in* $\mathfrak{X}$, *it is necessary that for each* $\epsilon > 0$, *there exists an* $\eta > 0$, *and a* d_t *for each* t *in* $V_\eta^*(\tau)$ *such that*

$$|f(x, t) - \phi(x)| < \epsilon \tag{1}$$

for each t *in* V_η *and for any* x *in* $V_{d_t}(a)$.

It is sufficient if there exists a single $t = \beta$ *in* $V_\eta^*(\tau)$ *for which the inequality* 1) *holds for any* x *in some* $V_\delta(a)$.

It is *necessary*. For since ϕ is continuous at $x = a$,

$$|\phi(x) - \phi(a)| < \frac{\epsilon}{3} \quad , \quad \text{for any } x \text{ in some } V_\delta(a).$$

Also since $f \doteq \phi$,

$$|f(a, t) - \phi(a)| < \frac{\epsilon}{3} \quad , \quad \text{for any } t \text{ in some } V_\eta^*(\tau).$$

Finally, since f is continuous in x for any t near τ,

$$|f(x, t) - f(a, t)| < \frac{\epsilon}{3} \quad , \quad \text{for any } x \text{ in some } V_{\delta_t}(a).$$

Adding these three inequalities we get 1), on taking

$$d_t < \delta, \delta_t.$$

It is sufficient. For by hypothesis

$$|f(x, \beta) - \phi(x)| > \frac{\epsilon}{3} \quad , \quad \text{for any } x \text{ in some } V_{\delta'}(a);$$

and hence in particular.

$$|f(a, \beta) - \phi(a)| < \frac{\epsilon}{3}.$$

Also since $f(x, \beta)$ is continuous in x,

$$|f(x, \beta) - f(a, \beta)| < \frac{\epsilon}{3} \quad , \quad \text{for any } x \text{ in some } V_{\delta''}(a).$$

Thus if $\delta < \delta', \delta''$, these unequalities hold simultaneously. Adding them we get

$$|\phi(x) - \phi(a)| < \epsilon \quad , \quad \text{for any } x \text{ in } V_\delta(a),$$

and thus ϕ is continuous at $x = a$.

2. As a corollary we get:

Let $$F(x) = \Sigma f_{\iota_1 \cdots \iota_n}(x_1 \cdots x_m)$$

converge in $\mathfrak{A}$, each term being continuous in $\mathfrak{A}$. For $F(x)$ to be continuous at the point $x = a$ in $\mathfrak{A}$, it is necessary that for each $\epsilon > 0$, and for any cell $R_\mu >$ some R_λ, there exists a δ_μ such that

$$|\overline{F}_\mu(x)| < \epsilon \quad , \quad \textit{for any } x \textit{ in } V_{\delta_\mu}(a).$$

It is sufficient if there exists an R_λ and a $\delta > 0$ such that

$$|\overline{F}_\lambda(x)| < \epsilon \quad , \quad \textit{for any } x \textit{ in } V_\delta(a).$$

547. 1. *Let* $\lim_{x=\tau} f(x_1 \cdots x_m, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$ *in $\mathfrak{X}$, τ finite or infinite. Let $f(x, t)$ be continuous in $\mathfrak{X}$ for each t near τ.*

1° *If $f \doteq \phi$ subuniformly in $\mathfrak{X}$, ϕ is continuous in $\mathfrak{X}$.*

2° *If $\mathfrak{X}$ is complete, and ϕ is continuous in $\mathfrak{X}$, $f \doteq \phi$ subuniformly in $\mathfrak{X}$.*

To prove 1°. Let $x = a$ be a point of $\mathfrak{X}$. Let $\epsilon > 0$ be taken at pleasure and fixed. Then there is a layer $\mathfrak{L}_\beta$ to which the point a belongs and such that

$$|f(x, t) - \phi(x)| < \epsilon, \tag{1}$$

when (x, t) ranges over the points of $\mathfrak{L}_\beta$. But then 1) holds for $t=\beta$ and x in some $V_\delta(a)$. Thus the condition of 546, 1 is satisfied.

To prove 2°. Since ϕ is continuous at $x=a$, the relation 1) holds by 546, 1, for each t in $V_\eta^*(\tau)$ and for any x in $V_{d_t}(a)$. With the point a let us associate a cube $C_{a,t}$ lying in $D_{dt}(a)$ and having a as center. Then each point of $\mathfrak{X}$ lies within a cube. Hence by Borel's theorem there exists a finite number of these cubes C, such that each point of $\mathfrak{X}$ lies within one of them, say

$$C_{a_1 t_1} \quad , \quad C_{a_2 t_2} \cdots \tag{2}$$

But the cubes 2) determine a set of layers

$$\mathfrak{L}_{t_1} \quad , \quad \mathfrak{L}_{t_2} \cdots \tag{3}$$

such that 1) holds as (x, t) ranges over the points of $\mathfrak{A}$ in each layer of 3). Thus the convergence of f to ϕ is subuniform in $\mathfrak{X}$.

2. As a corollary we have the theorem:

Let

$$F(x_1 \cdots x_m) = \Sigma f_{\iota_1 \cdots \iota_n}(x_1 \cdots x_m)$$

converge in $\mathfrak{X}$, each f_ι being continuous in $\mathfrak{X}$. If F converges subuniformly in $\mathfrak{X}$, F is continuous in $\mathfrak{X}$. If $\mathfrak{X}$ is complete and F is continuous in $\mathfrak{X}$, F converges subuniformly in $\mathfrak{X}$.

548. 1. *Let*

$$F(x) = \Sigma f_{\iota_1 \cdots \iota_n}(x_1 \cdots x_m)$$

converge in $\mathfrak{A}$.

Let the convergence be uniform in $\mathfrak{A}$ except possibly for the points of a complete discrete set $\mathfrak{B} = \{b\}$. For each b, let there exist a λ_0 such that for any $\lambda \geq \lambda_0$,

$$\lim_{x=b} \overline{F}_\lambda(x) = 0.$$

Then F converges subuniformly in $\mathfrak{A}$.

For let D be a cubical division of norm d of the space $\mathfrak{R}_m$ in which $\mathfrak{A}$ lies. We may take d so small that $\overline{\mathfrak{B}}_D$ is small at pleasure. Let B_D denote the cells of D containing points of $\mathfrak{A}$ but none of $\mathfrak{B}$. Then by hypothesis F converges uniformly in B_D. Thus there exists a μ_0 such that for any $\mu \geq \mu_0$,

$$|\overline{F}_\mu(x)| < \epsilon \quad , \quad \text{for any } x \text{ of } \mathfrak{A} \text{ in } B_D.$$

At a point b of $\mathfrak{B}$, there exists by hypothesis a $V_\delta(b)$ and a λ_0 such that for each $\lambda \geq \lambda_0$

$$|\overline{F}_\lambda(x)| < \epsilon \quad , \quad \text{for any } x \text{ in } V_\delta(b).$$

Let $C_{b,\lambda}$ be a cube lying in $D_\delta(b)$, having b as center. Since $\mathfrak{B}$ is complete there exists a finite number of these cubes

$$C_{b_1\lambda_1} \quad , \quad C_{b_2\lambda_2} \cdots \tag{1}$$

such that each point of $\mathfrak{B}$ lies within one of them.

Moreover

$$|\overline{F}_{\lambda_\kappa}(x)| < \epsilon,$$

for any x of $\mathfrak{A}$ lying in the κ^{th} cube of 1).

As B_D embraces but a finite number of cubes, and as the same is true of 1), there is a finite set of layers $\mathfrak{L}$ such that

$$|\overline{F}_\nu(x)| < \epsilon \quad , \quad \text{in each } \mathfrak{L}.$$

The convergence is thus subuniform, as λ, μ are arbitrarily large.

2. The reasoning of the preceding section gives us also the theorem:

Let

$$\lim_{t=\tau} f(x_1 \cdots x_m, \ t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$$

in $\mathfrak{X}$, τ finite or infinite. Let the convergence be uniform in $\mathfrak{X}$ except possibly for the points of a complete discrete set $\mathfrak{E} = \{e\}$. For each point e, let there exist an η such that setting $\epsilon(x, t) = f(x, t) - \phi(x)$,

$$\lim_{x=e} \epsilon(x, t) = 0 \quad , \quad \textit{for any } t \textit{ in } V_\eta^*(\tau).$$

Then $f \doteq \phi$ subuniformly in $\mathfrak{X}$.

3. As a special case of 1 we have the theorem:

Let

$$F(x) = f_1(x) + f_2(x) + \cdots$$

converge in $\mathfrak{A}$, and converge uniformly in $\mathfrak{A}$, except at $x = \alpha_1, \cdots x = \alpha_s$. At $x = \alpha_\iota$ let there exist a ν_ι such that

$$\lim_{x=\alpha_\iota} \overline{F}_{n_\iota}(x) = 0 \quad , \quad n_\iota \geq \nu_\iota \quad , \quad \iota = 1, 2 \cdots s.$$

Then F converges subuniformly in $\mathfrak{A}$.

4. When
$$\lim_{t=\tau} f(x, t) = \phi(x)$$
we will often set
$$f(x, t) = \phi(x) + \epsilon(x, t),$$
and call ϵ the *residual function.*

549. *Example 1.*

$$f(x, n) = \frac{n^\lambda x^\alpha}{e^{n^\mu x^\beta}} \doteq \phi(x) = 0 \quad , \quad \text{for } n \doteq \infty \text{ in } \mathfrak{A} = (0 < a),$$
$$\alpha, \beta, \lambda \geq 0 \quad , \quad \mu > 0.$$

The convergence is subuniform in $\mathfrak{A}$. For $x = 0$ is the only possible point of non-uniform convergence, and for any m,

$$|\epsilon(x, m)| = \frac{m^\lambda x^\alpha}{e^{m^\mu x^\beta}} \doteq 0 \quad , \quad \text{as } x \doteq 0.$$

Example 2. $$f(x, n) = \frac{n^\lambda x^\alpha}{c + n^\mu x^\beta} \doteq \phi(x) = 0 \quad , \quad \text{as } n \doteq \infty,$$

$$x \text{ in } \mathfrak{A} = (0 < a) \quad , \quad \alpha, \beta, \lambda, \mu > 0 \quad , \quad \mu > \lambda \quad , \quad c > 0.$$

The convergence is uniform in $\mathfrak{B} = (e < a)$, where $e > 0$. For

$$|\epsilon(x, n)| \leq \frac{n^\lambda a^\alpha}{c + n^\mu e^\beta} \quad , \quad \text{in } \mathfrak{B}$$
$$< \frac{a^\alpha}{e^\beta} \cdot \frac{n^\lambda}{n^\mu}$$
$$< \epsilon \quad , \quad \text{for } n > \text{some } m.$$

Thus the convergence is uniform in $\mathfrak{A}$, except possibly at $x = 0$. The convergence is subuniform in $\mathfrak{A}$. For obviously for a given n

$$\lim_{x=0} f(x, n) = 0.$$

550. 1. *Let* $\lim_{t=\tau} f(x_1 \cdots x_m t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$ *in* $\mathfrak{X}$, τ *finite or infinite.*

Let the convergence be uniform in $\mathfrak{X}$ *except at the points*

$$\mathfrak{B} = (b_1, b_2, \cdots b_p).$$

For the convergence to be sub-uniform in $\mathfrak{X}$, it is necessary that for each b in $\mathfrak{B}$, and for each $\epsilon > 0$, there exists a $t = \beta$ near τ, such that

$$\overline{\lim_{x=b}} \, | \epsilon(x, t) | \geq \epsilon. \tag{1}$$

For if the convergence is subuniform, there exists for each ϵ and $\eta > 0$ a finite set of layers $\mathfrak{L}_t$, t in $V_\eta^*(\tau)$ such that

$$| \epsilon(x, t) | < \epsilon \quad , \quad x \text{ in } \mathfrak{L}_t.$$

Now the point $x = b$ lies in one of these layers, say in $\mathfrak{L}_\beta$. Then

$$| \epsilon(x, \beta) | < \epsilon \quad , \quad \text{for all } x \text{ in some } V^*(b).$$

But then 1) holds.

2. *Example.* Let

$$F(x) = \sum_0^\infty x^n (1 - x).$$

This is the series considered in 140, Ex. 2.

F converges uniformly in $\mathfrak{A} = (-1, 1)$, except at $x = 1$.

As

$$\overline{F}_m(x) = -x^m,$$

we see that

$$\lim_{x=1} \overline{F}_m(x) = -1.$$

Hence F is not subuniformly convergent in $\mathfrak{A}$.

Integrability

551. 1. *Infra-uniform Convergence.* It often happens that

$$f(x_1 \cdots x_m t_1 \cdots t_n) \doteq \phi(x_1 \cdots x_m)$$

subuniformly in $\mathfrak{X}$ except possibly at certain points $\mathfrak{E} = \{e\}$ forming a discrete set. To be more specific, let Δ be a cubical division of $\mathfrak{R}_m$ in which $\mathfrak{X}$ lies, of norm δ. Let X_Δ denote those cells containing points of $\mathfrak{X}$, but none of $\mathfrak{E}$. Since $\mathfrak{E}$ is discrete, $\overline{X}_\Delta \doteq \overline{\mathfrak{X}}$. Suppose now $f \doteq \phi$ subuniformly in any X_Δ; we shall say the convergence is *infra-uniform* in $\mathfrak{X}$. When there are no exceptional points, infra-uniform convergence goes over into subuniform convergence.

This kind of convergence Arzelà calls uniform convergence by segments, in general.

2. We can make the above definition independent of the set $\mathfrak{E}$, and this is desirable at times.

Let $\mathfrak{X} = (X, \mathfrak{x})$ be an unmixed division of $\mathfrak{X}$ such that $\bar{\mathfrak{x}}$ may be taken small at pleasure. If $f \doteq \phi$ subuniformly in each X, we say the convergence is infra-uniform in $\mathfrak{X}$.

3. Then to each $\epsilon, \eta > 0$, and a given X, there exists a set of layers $\mathfrak{l}_1, \mathfrak{l}_2 \cdots, t$ in $V_\eta^*(\tau)$, such that the residual function $\epsilon(x, t)$ is numerically $< \epsilon$ for each of these layers. As the projections of these layers $\mathfrak{l}$ do not in general embrace all the points of $\mathfrak{X}$, we call them *deleted layers*.

4. The points $\mathfrak{x}$ we shall call the *residual points*.

5. *Example 1.*
$$F = \sum_0^\infty \frac{x^2}{(1 + nx^2)(1 + (n+1)x^2)}.$$

This series was studied in 150. We saw that it converges uniformly in $\mathfrak{A} = (0, 1)$, except at $x = 0$.

As
$$\bar{F}_n(x) = \frac{1}{1 + nx^2} \quad , \quad x \neq 0$$

and as this $\doteq 1$ as $x \doteq 0$ for an arbitrary but fixed n, F does not converge subuniformly in $\mathfrak{A}$, by 550. The series converges infra-uniformly in $\mathfrak{A}$, obviously.

6. *Example 2.*
$$F = \sum_0^\infty x^n(1 - x).$$

This series was considered in 550, 2. Although it does not converge subuniformly in an interval containing the point $x = 1$, the convergence is obviously infra-uniform.

552. 1. *Let* $\lim_{x=\tau} f(x_1 \cdots x_m t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$ *be limited in* $\mathfrak{X}$, τ *finite or infinite. For each t near* τ, *let f be limited and R-integrable in* $\mathfrak{X}$. *For* ϕ *to be R-integrable in* $\mathfrak{X}$, *it is sufficient that* $f \doteq \phi$ *infra-uniformly in* $\mathfrak{X}$. *If* $\mathfrak{X}$ *is complete, this condition is necessary.*

It is sufficient. We show that for each ϵ, $\omega > 0$ there exists a division D of $\mathfrak{R}_m$ such that the cells in which

$$\text{Osc}\ \phi \geq \omega \tag{1}$$

have a volume $< \sigma$. For setting as usual

$$f = \phi + \epsilon,$$

we have in any point set,

$$\text{Osc}\ \phi \leq \text{Osc} f + \text{Osc}\ \epsilon.$$

Using the notation of 551,

$$| \epsilon(x, t) | < \frac{\omega}{4}$$

in the finite set of deleted layers $\mathfrak{l}_1, \mathfrak{l}_2 \cdots$ corresponding to $t = t_1, t_2 \cdots$ For each of these ordinates $t_\iota, f(x, t_\iota)$ is integrable in $\mathfrak{X}$. There exists, therefore, a rectangular division D of $\mathfrak{R}_m$, such that those cells in which

$$\text{Osc} f(x, t_\iota) \geq \frac{\omega}{2}$$

have a content $< \frac{\sigma}{2}$, whichever ordinate t_ι is used. Let E be a division of $\mathfrak{R}_m$ such that the cells containing points of the residual set $\mathfrak{x}$ have a content $< \sigma/2$. Let $F = D + E$. Then those cells of F in which

$$\text{Osc} f(x, t_\iota) \geq \frac{\omega}{2}, \quad \text{or} \quad \text{Osc}\ | \epsilon(x, t_\iota) | \geq \frac{\omega}{2}$$

$\iota = 1, 2 \cdots$ have a content $< \sigma$. Hence those cells in which 1) holds have a content $< \sigma$.

It is necessary, if $\mathfrak{X}$ is complete. For let

$$t_1, t_2 \cdots \doteq \tau.$$

Since ϕ and $f(x, t_n)$ are integrable, the points of discontinuity of $\phi(x)$ and of $f(x, t_n)$ are null sets by 462, 6. Hence if $\mathfrak{C}$, $\mathfrak{C}_t$ denote the points of continuity of $\phi(x)$ and $f(x, t)$ in $\mathfrak{X}$,

$$\widehat{\mathfrak{C}} = \widehat{\mathfrak{C}}_t = \widehat{\mathfrak{X}},$$

since $\mathfrak{X}$ is measurable, as it is complete.

Let $\mathfrak{G} = \text{Qdv}\,\{\mathfrak{C}_t\}$,

then $\hat{\hat{\mathfrak{G}}} = \hat{\hat{\mathfrak{X}}}$

by 410, 6.

Let $\mathfrak{D} = Dv(\mathfrak{C}, \mathfrak{G})$,

then $$\hat{\hat{\mathfrak{D}}} = \hat{\hat{\mathfrak{X}}}, \tag{1}$$

as we proceed to show. For if $G = \mathfrak{X} - \mathfrak{G}$,

$$\mathfrak{C} = Dv(\mathfrak{C}, \mathfrak{G}) + Dv(\mathfrak{C}, G) = \mathfrak{D} + Dv(\mathfrak{C}, G).$$

But G is a null set. Hence Meas $Dv(\mathfrak{C}, G) = 0$, and thus $\hat{\hat{\mathfrak{C}}} = \hat{\hat{\mathfrak{X}}} = \hat{\hat{\mathfrak{D}}}$, which is 1).

Let now ξ be a point of $\mathfrak{D}$, let it lie in $\mathfrak{C}_{t_1}, \mathfrak{C}_{t_2} \cdots$ where $t_1, t_2 \cdots$ form a monotone sequence $\doteq \tau$. Then since

$$f(\xi, t_n) \doteq \phi(\xi),$$

there is an m such that

$$|\,\epsilon(\xi, t_n)\,| < \frac{\epsilon}{3} \quad , \quad \text{for any } n > m. \tag{2}$$

But ξ lying in $\mathfrak{D}$, it lies in $\mathfrak{C}$ and $\mathfrak{C}_{t_n}$.

Thus $$|\,\phi(x) - \phi(\xi)\,| < \frac{\epsilon}{3},$$

$$|f(x, t_n) - f(\xi, t_n)\,| < \frac{\epsilon}{3},$$

for any x in $V_\delta(\xi)$. Hence

$$|\,\epsilon(x, t_n) - \epsilon(\xi, t_n)\,| < \frac{2\,\epsilon}{3} \quad , \quad x \text{ in } V_\delta(\xi). \tag{3}$$

Now

$$\epsilon(x, t_n) = \epsilon(x, t_n) - \epsilon(\xi, t_n) + \epsilon(\xi, t_n).$$

Hence from 2), 3),

$$|\,\epsilon(x, t_n)\,| < \epsilon \quad , \quad \text{for any } x \text{ in } V_\delta(\xi).$$

Thus associated with the point ξ, there is a cube Γ lying in $D_\delta(\xi)$, having ξ as center. As $D = \mathfrak{X} - \mathfrak{D}$ is a null set, each of its points can be enclosed within cubes C, such that the resulting enclosure

$\mathfrak{E}$ has a measure $< \sigma$, small at pleasure. Thus each point of $\mathfrak{X}$ lies within a cube. By Borel's theorem there exists a finite set of these cubes

$$\Gamma_1, \Gamma_2 \cdots \Gamma_r \quad ; \quad C_1, C_2 \cdots C_s,$$

such that each point of $\mathfrak{X}$ lies within one of them. But corresponding to the Γ's, are layers

$$\mathfrak{L}_1, \mathfrak{L}_2, \cdots \mathfrak{L}_r$$

such that in each of them

$$|\epsilon(x, t)| < \epsilon.$$

Thus $f \doteq \phi$ subuniformly in $X = (\Gamma_1, \Gamma_2 \cdots \Gamma_r)$. Let $\mathfrak{x}$ be the residual set. Obviously $\bar{\mathfrak{x}} < \sigma$. Thus the convergence is infra-uniform.

2. As a corollary we have:

Let $$F(x) = \Sigma f_{\iota_1 \cdots \iota_n}(x_1 \cdots x_m)$$

converge in $\mathfrak{A}$. *Let* F *be limited, and each* f_ι *be limited and* R*-integrable in* $\mathfrak{A}$. *For* F *to be* R*-integrable in* $\mathfrak{A}$, *it is sufficient that* F *converges infra-uniformly in* $\mathfrak{A}$.

If $\mathfrak{A}$ *is complete, this condition is necessary.*

553. *Infinite Peaks.* 1. Let $\lim\limits_{t=\tau} f(x_1 \cdots x_m t_1 \cdots t_n) = \phi(x)$ in $\mathfrak{X}$, τ finite or infinite. Although $f(x, t)$ is limited in $\mathfrak{X}$ for each t near τ, and although $\phi(x)$ is also limited in $\mathfrak{X}$, we cannot say that

$$|f(x, t)| < \text{some } M \tag{1}$$

for any x in $\mathfrak{X}$ and any t near τ, as is shown by the following

Example. Let $f(x, t) = \dfrac{tx}{e^{tx^2}} \doteq \phi(x) = 0$, as $t \doteq \infty$ for x in $\mathfrak{X} = (-\infty, \infty)$.

It is easy to see that the peak of f becomes infinitely high as $n \doteq \infty$.

In fact, for $x = \dfrac{1}{\sqrt{t}}$, $f = \dfrac{\sqrt{t}}{e}$. Thus the peak is at least as high as $\dfrac{\sqrt{t}}{e}$, which $\doteq \infty$.

The origin is thus a point in whose vicinity the peaks of the family of curves $f(x, t)$ are infinitely high. In general, if the peaks of

$$f(x_1 \cdots x_m\ t_1 \cdots t_n)$$

in the vicinity V_δ of $x = \xi$ become infinitely high as $t \doteq \tau$, however small δ is taken, we say ξ *is a point with infinite peaks.*

On the other hand, if the relation 1) holds for all x and t involved, we shall say $f(x, t)$ is *uniformly limited.*

2. *If* $\lim\limits_{t=\tau} f(x_1 \cdots x_m\ t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$, *and if* $f(x, t)$ *is uniformly limited in* $\mathfrak{X}$, *then* ϕ *is limited in* $\mathfrak{X}$.

For x being taken at pleasure in $\mathfrak{X}$ and fixed, $\phi(x)$ is a limit point of the points $f(x, t)$ as $t \doteq \tau$. But all these points lie in some interval $(-G, G)$ independent of x. Hence ϕ lies in this interval.

3. *If* $\mathfrak{X}$ *is complete, the points* $\mathfrak{K}$ *in* $\mathfrak{X}$ *with infinite peaks also form a complete set. If these points* $\mathfrak{K}$ *are enumerable, they are discrete.*

That $\mathfrak{K}$ is complete is obvious. But then $\overline{\mathfrak{K}} = \widehat{\mathfrak{K}} = 0$, as $\mathfrak{K}$ is enumerable.

554. 1. *Let* $\lim\limits_{t=\tau} f(x_1 \cdots x_m\, t_1 \cdots t_n) = \phi(x_1 \cdots x_m)$ *in* $\mathfrak{X}$, *metric or complete. Let* $f(x, t)$ *be uniformly limited in* $\mathfrak{X}$, *and R-integrable for each t near* τ. *For the relation*

$$\lim_{t=\tau} \int_{\mathfrak{X}} f(x, t) = \int_{\mathfrak{X}} \phi(x)$$

to hold, it is sufficient that $f \doteq \phi$ *infra-uniformly in* $\mathfrak{X}$. *If* $\mathfrak{X}$ *is for each t complete, this condition is necessary.*

For by 552, ϕ is R-integrable if $f \doteq \phi$ infra-uniformly, and when $\mathfrak{X}$ is complete, this condition is necessary. By 424, 4, each $f(x, t)$ is measurable. Thus we may apply 381, 2 and 413, 2.

2. As a corollary we have the theorem:

Let
$$F(x) = \Sigma f_{\iota_1 \cdots \iota_n}(x_1 \cdots x_m)$$

converge in the complete or metric field $\mathfrak{A}$. *Let the partial sums* F_λ *be uniformly limited in* $\mathfrak{A}$. *Let each term* f_ι *be limited and R-integrable in* $\mathfrak{A}$. *Then for the relation*

$$\int_{\mathfrak{A}} F = \Sigma \int_{\mathfrak{A}} f_\iota$$

to hold it is sufficient that F is infra-uniformly convergent in $\mathfrak{A}$. If $\mathfrak{A}$ is complete, this condition is necessary.

555. *Example 1.* Let us reconsider the example of 150,

$$F(x) = \sum_0^\infty \frac{x^2}{(1+nx^2)(1+(n+1)x^2)}.$$

We saw that we may integrate termwise in $\mathfrak{A} = (0, 1)$, although F does not converge uniformly in $\mathfrak{A}$. The only point of non-uniform convergence is $x = 0$. In 551, 5, we saw that it converges, however, infra-uniformly in $\mathfrak{A}$. As

$$|F_n(x)| \leq 1 \quad , \quad \text{for any } x \text{ in } \mathfrak{A}, \text{ and for every } n,$$

all the conditions of 554 are satisfied and we can integrate the series termwise, in accordance with the result already obtained in 150.

Example 2. Let $F(x) = \sum_1^\infty \left\{ \frac{nx}{e^{nx^2}} - \frac{(n-1)x}{e^{(n-1)x^2}} \right\} = 0.$

Then

$$F_n(x) = \frac{nx}{e^{nx^2}}.$$

We considered this series in 152, 1. We saw there that this series cannot be integrated termwise in $\mathfrak{A} = (0 < a)$. It is, however, subuniformly convergent in $\mathfrak{A}$ as we saw in 549, Ex. 1. We cannot apply 554, however, as F_n is not uniformly limited. In fact we saw in 152, 1, that $x = 0$ is a point with an infinite peak.

Example 3. $F(x) = \sum_0^\infty x^n(1-x).$

We saw in 551, 6, that F converges infra-uniformly in $\mathfrak{A} = (0, 1)$. Here

$$F_n(x)| = |1 - x^n| < \text{some } M,$$

for any x in $\mathfrak{A} = (0 \leq u)$, $u \leq 1$, and any n. Thus the F_n are uniformly limited in $\mathfrak{A}$.

We may therefore integrate termwise by 554, 2. We may verify this at once. For

$$\begin{aligned} F(x) &= 1 \quad , \quad 0 \leq x < 1 \\ &= 0 \quad , \quad x = 0. \end{aligned}$$

Hence
$$\int_0^u F(x)dx = u. \tag{1}$$
On the other hand,
$$\int_0^u F_n dx = u - \frac{u^{n+1}}{n+1} \doteq u \quad , \quad \text{as } n \doteq \infty. \tag{2}$$
From 1), 2) we have
$$G(u) = u = \sum_0^\infty \left\{ \frac{u^{n+1}}{n+1} - \frac{u^{n+2}}{n+2} \right\}.$$

556. 1. *If* 1° $f(x_1 \cdots x_m\, t_1 \cdots t_n) \doteq \phi(x_1 \cdots x_m)$ *infra-uniformly in the metric or complete field* $\mathfrak{X}$, *as* $t \doteq \tau$, τ *finite or infinite;*

2° $f(x, t)$ *is uniformly limited in* $\mathfrak{X}$ *and* R*-integrable for each* t *near* τ*;*

Then
$$\lim_{t=\tau} \int_{\mathfrak{A}} f(x, t) = \int_{\mathfrak{A}} \phi,$$
uniformly with respect to the set of measurable fields $\mathfrak{A}$ *in* $\mathfrak{X}$.

If $\mathfrak{X}$ *is complete, condition* 1° *may be replaced by* 3° $\phi(x)$ *is* R*-integrable in* $\mathfrak{X}$.

For by 552, 1, when 3° holds, 1° holds; and when 1° holds, ϕ is R-integrable in $\mathfrak{X}$.

Now the points $\mathfrak{E}_t$ where
$$|\epsilon(x, t_n)| > \epsilon$$
are such that
$$\lim_{t=\tau} \overline{\overline{\mathfrak{E}}}_t = 0 \quad , \quad \text{by 412.}$$

Let $\mathfrak{X} = \mathfrak{E}_t + \mathfrak{X}_t$. Then
$$\int_{\mathfrak{X}} \epsilon(x, t) = \int_{\mathfrak{E}_t} \epsilon(x, t) + \int_{\mathfrak{X}_t} \epsilon(x, t),$$
$$\left| \int_{\mathfrak{A}} \epsilon \right| \le \int_{\mathfrak{X}} |\epsilon| \le 2M \overline{\overline{\mathfrak{E}}}_t + \epsilon \overline{\overline{\mathfrak{X}}}.$$

But
$$\lim_{t=\tau} \overline{\overline{\mathfrak{E}}}_t = 0,$$
which establishes the theorem.

2. As a corollary we have:

If 1° $F(x) = \Sigma f_{\iota_1 \cdots \iota_n}(x_1 \cdots x_m)$ *converges infra-uniformly, and each of its terms* f_ι *is* R*-integrable in the metric or complete field* $\mathfrak{A}$*;*

2° *$F_\lambda(x)$ is uniformly limited in $\mathfrak{A}$;*

Then

$$\int_{\mathfrak{B}} F(x) = \Sigma \int_{\mathfrak{B}} f_\iota,$$

and the series on the right converges uniformly with respect to all measurable $\mathfrak{B} \leq \mathfrak{A}$.

3. *If* 1° $\lim\limits_{t=\tau} f(x, t_1 \cdots t_n) = \phi(x)$ *is R-integrable in the interval $\mathfrak{A} = (a < b)$, τ finite or infinite;*

2° *$f(x, t)$ is uniformly limited, and R-integrable for each t near τ;*

Then

$$\lim_{t=\tau} \int_a^x f(x, t) dx = \int_a^x \phi(x) dx = \Phi(x),$$

uniformly in $\mathfrak{A}$, and $\Phi(x)$ is continuous in $\mathfrak{A}$.

4. *If* 1°

$$F(x) = \Sigma f_{\iota_1 \cdots \iota_n}(x)$$

and also each term f_ι are R-integrable in the interval $\mathfrak{A} = (a < b)$;

2° *$F_\lambda(x)$ is uniformly limited in $\mathfrak{A}$;*

Then

$$G(x) = \Sigma \int_a^x f_\iota(x) dx \quad , \quad x \text{ in } \mathfrak{A}$$

is continuous.

For G is a uniformly convergent series in $\mathfrak{A}$, each of whose terms

$$\int_a^x f_\iota dx$$

is a continuous function of x.

Differentiability

557. 1. *If* 1° $\lim\limits_{t=\tau} f(x, t_1 \cdots t_n) = \phi(x)$ *in $\mathfrak{A} = (a < b)$, τ finite or infinite;*

2° *$f'_x(x, t)$ is R-integrable for each t near τ, and uniformly limited in $\mathfrak{A}$;*

3° *$f'_x(x, t) \doteq \psi(x)$ infra-uniformly in $\mathfrak{A}$, as $t \doteq \tau$;*

Then at a point x of continuity of ψ in $\mathfrak{A}$

$$\phi'(x) = \psi(x), \tag{1}$$

or what is the same

$$\frac{d}{dx} \lim_{t=\tau} f(x, t) = \lim_{t=\tau} \frac{d}{dx} f(x, t). \tag{2}$$

For by 554,

$$\lim_{t=\tau}\int_a^x f'_x(x, t)dx = \int_a^x \psi(x)dx \tag{3}$$

$$= \lim_{t=\tau}[f(x, t) - f(a, t)] \quad , \quad \text{by I, 538}$$

$$= \phi(x) - \phi(a) \quad , \quad \text{by } 1°.$$

Now by I, 537, at a point of continuity of ψ,

$$\frac{d}{dx}\int_a^x \psi(x)dx = \psi(x). \tag{4}$$

From 3), 4), we have 1), or what is the same 2).

2. *In the interval* $\mathfrak{A}$, *if*

1° $F(x) = \Sigma f_{\iota_1 \cdots \iota_n}(x)$ *converges;* (1

2° *Each* $f'_\iota(x)$ *is limited and R-integrable;*

3° $F'_\lambda(x)$ *is uniformly limited;*

4° $G(x) = \Sigma f'_\iota$ *is infra-uniformly convergent;*

Then at a point of continuity of $G(x)$ *in* $\mathfrak{A}$, *we may differentiate the series* 1) *termwise, or* $F'(x) = G(x)$.

3. *In the interval* $\mathfrak{A}$, *if*

1° $f(x, t_1 \cdots t_n) \doteq \phi(x)$ *as* $t \doteq \tau$, τ *finite or infinite;*

2° $f(x, t)$ *is uniformly limited, and a continuous function of* x*;*

3° $\psi(x) = \lim_{t=\tau} f'_x(x, t)$ *is continuous;*

Then

$$\phi'(x) = \psi(x), \tag{1}$$

or what is the same

$$\frac{d}{dx}\lim_{t=\tau} f(x, t) = \lim_{t=\tau}\frac{d}{dx}f(x, t). \tag{2}$$

For by 547, 1, condition 3° requires that $f' \doteq \psi$ subuniformly in $\mathfrak{A}$. But then the conditions of 1 are satisfied and 1) and 2) hold.

4. *In the interval* $\mathfrak{A}$ *let us suppose that*

1° $F(x) = \Sigma f_{\iota_1 \cdots \iota_n}(x)$ *converges;* (1

2° *Each term f_ι is continuous;*

3° *$F'_\lambda(x)$ is uniformly limited;*

4° *$G(x) = \Sigma f'_\iota(x)$ is continuous;*

Then we may differentiate 1) *termwise, or $F'(x) = G(x)$.*

558. *Example 1.* We saw in 555, Ex. 3 that

$$F(x) = x = \sum_0^\infty \left\{ \frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2} \right\} \quad , \quad \text{in } \mathfrak{A} = (0, 1). \tag{1}$$

The series got by differentiating termwise is

$$\begin{aligned} G(x) = \sum_0^\infty x^n(1-x) &= 1 \quad , \quad 0 \leq x < 1 \\ &= 0 \quad , \quad x = 0. \end{aligned} \tag{2}$$

Thus by 557, 4, $\quad G(x) = F'(x)$ in $(0^*, 1) = \mathfrak{A}^*$. (3

The relation 3) does not hold for $x = 0$.

Example 2.

$$F(x) = \sum_1^\infty \left\{ \frac{\operatorname{arctg} x\sqrt{n}}{\sqrt{n}} - \frac{\operatorname{arctg} x\sqrt{n+1}}{\sqrt{n+1}} \right\} = \Sigma f_n(x)$$

$$G(x) = \sum_1^\infty \left\{ \frac{1}{1+nx^2} - \frac{1}{1+(n+1)x^2} \right\} = \Sigma f'_n(x).$$

Here

$$F(x) = \operatorname{arctg} x, \qquad \text{for any } x. \tag{1}$$

$$\begin{aligned} G(x) &= \frac{1}{1+x^2} \quad , \quad x \neq 0 \\ &= 0 \quad , \quad x = 0. \end{aligned} \tag{2}$$

Hence $G(x)$ is continuous in any interval $\mathfrak{A}$, not containing $x = 0$. Thus we should have by 557, 4,

$$F'(x) = G(x), \qquad x \text{ in } \mathfrak{A}. \tag{3}$$

This relation is verified by 1), 2). The relation 3) does not hold for $x = 0$, since

$$F'(0) = 1, \quad , \quad G(0) = 0.$$

Example 3.

$$F(x)=\sum_1^\infty\left\{\frac{1}{2\,n}\log\left(1+n^2x^2\right)-\frac{1}{2(n+1)}\log\left(1+(n+1)^2x^2\right)\right\}$$

$$=\Sigma f_n(x) \tag{1}$$

$$=\tfrac{1}{2}\log\left(1+x^2\right) \quad , \quad \text{for any } x.$$

$$G(x)=\sum f_n'(x)=\sum\left\{\frac{nx}{1+n^2x^2}-\frac{(n+1)x}{1+(n+1)^2x^2}\right\} \tag{2}$$

$$=\frac{x}{1+x^2} \quad , \quad \text{for any } x.$$

In any interval $\mathfrak{A}$, all the conditions of 557, 4, hold.

Hence

$$F'(x)=G(x) \quad , \quad \text{for any } x \text{ in } \mathfrak{A}. \tag{3}$$

In case we did not know the value of the sums 1), 2) we could still assert that 3) holds. For by 545, G is subuniformly convergent in $\mathfrak{A}$, and hence is continuous.

Example 4.

$$F(x)=\sum_1^\infty\left\{\frac{1+nx}{ne^{nx}}-\frac{1+(n+1)x}{(n+1)e^{(n+1)x}}\right\}=\frac{1+x}{e^x}. \tag{1}$$

Here

$$F'(x)=-\frac{x}{e^x}. \tag{2}$$

The series obtained by differentiating F termwise is

$$G(x)=\sum_1^\infty\left\{\frac{(n+1)x}{e^{(n+1)x}}-\frac{nx}{e^{nx}}\right\}=-\frac{x}{e^x}, \tag{3}$$

and hence

$$G_{n-1}(x)=-\frac{x}{e^x}+\frac{nx}{e^{nx}}.$$

The peaks of the residual function

$$\epsilon(x,n)=\frac{nx}{e^{nx}},$$

are of height $=1/e$. The convergence of G is not uniform at $x=0$. The conditions of 557, 4, are satisfied and we can differentiate 1) termwise. This is verified by 2), 3).

559. 1. *If* 1° $\lim_{t=\tau} f(x, t_1 \cdots t_n) = \phi(x)$ *is limited and R-integrable in the interval* $\mathfrak{A} = (a < b)$;

2° $f(x, t)$ *is limited, and R-integrable in* $\mathfrak{A}$, *for each* t *near* τ;

3° $$\psi(x) = \lim_{t=\tau} \int_a^x f(x, t) = \lim_{t=\tau} g(x, t)$$

is a continuous function in $\mathfrak{A}$;

4° *The points* $\mathfrak{E}$ *in* $\mathfrak{A}$ *in whose vicinity the peaks of* $f(x, t)$ *as* $t \doteq \tau$ *are infinitely high form an enumerable set;*

Then
$$\theta(x) = \int_a^x \phi(x) = \lim_{t=\tau} \int_a^x f(x, t)dx = \psi(x), \qquad (1$$

or
$$\lim_{t=\tau} \int_a^x f(x, t)dx = \int_a^x \lim_{t=\tau} f(x, t)dx,$$

and the set $\mathfrak{E}$ *is complete and discrete.*

For $\mathfrak{E}$ is discrete by 553, 3.

Let α be a point of $A = \mathfrak{A} - \mathfrak{E}$. Then in an interval $\mathfrak{a}$ about α,
$$|f(x, t)| < \text{some } M \quad , \quad x \text{ in } \mathfrak{a}, \text{ any } t \text{ near } \tau. \qquad (2$$

Now by 556, 3, taking $\epsilon > 0$ small at pleasure, there exists an $\eta > 0$ such that
$$\psi(x) - \psi(\alpha) = \int_\alpha^x f(x, t) + \epsilon' \quad , \quad |\epsilon'| < \epsilon$$

for *any* x in $\mathfrak{a}$, and t in $V_\eta^*(\tau)$. If we set $x = \alpha + h$, we have
$$\frac{\Delta\psi}{\Delta x} = \frac{\psi(x) - \psi(\alpha)}{h} = \frac{1}{h}\int_\alpha^x f(x, t)dx + \frac{\epsilon'}{h}. \qquad (3$$

Also by 556, 3, we have
$$\int_\alpha^x f(x, t)dx = \int_\alpha^x \phi(x)dx + \epsilon'' \quad , \quad |\epsilon''| < \epsilon$$

for *any* x in $\mathfrak{a}$, and t in $V_\eta^*(\tau)$. Thus
$$\frac{1}{h}\int_\alpha^x f(x, t)dx = \frac{\theta(x) - \theta(\alpha)}{h} + \frac{\epsilon''}{h} = \frac{\Delta\theta}{\Delta x} + \frac{\epsilon''}{h}. \qquad (4$$

From 3), 4) we have
$$\frac{\Delta\psi}{\Delta x} - \frac{\epsilon'}{h} = \frac{\Delta\theta}{\Delta x} + \frac{\epsilon''}{h}, \qquad |\epsilon'|, |\epsilon''| < \epsilon.$$

Now ϵ may be made small at pleasure, and that independent of h. Thus the last relation gives

$$\frac{\Delta\psi}{\Delta x} = \frac{\Delta\theta}{\Delta x} \quad , \quad \text{for } x \text{ in } A.$$

As this holds however small $h = \Delta x$ is taken, we have

$$\frac{d\psi}{dx} = \frac{d\theta}{dx} \quad , \quad \text{for } x \text{ in } A.$$

Hence by 515, 3,

$$\psi(x) = \theta(x) + \text{const} \quad , \quad \text{in } \mathfrak{A}.$$

For $x = a$,

$$\psi(a) = \theta(a) = 0\,;$$

and thus

$$\psi(x) = \theta(x) \quad , \quad \text{in } \mathfrak{A}.$$

2. *As a corollary we have:*

If 1° $F(x) = \Sigma f_{\iota_1 \cdots \iota_n}(x)$ *is limited and R integrable in the interval* $\mathfrak{A} = (a < b)$;

2° $F_\lambda(x)$ *is limited and each term* f_ι *is R-integrable;*

3° $G(x) = \sum \int_a^x f_\iota$ *is continuous;*

4° *The points* $\mathfrak{E}$ *in* $\mathfrak{A}$ *in whose vicinity the peaks of* $F_\lambda(x)$ *are infinitely high form an enumerable set;*

Then

$$\int_a^x F(x) = \sum \int_a^x f_\iota,$$

or we may integrate the F series termwise.

560. 1. *If* 1° $\lim_{t=\tau} f(x, t_1 \cdots t_n) = \phi(x)$ *in* $\mathfrak{A} = (a < b)$, τ *finite or infinite;*

2° $f_x'(x, t)$ *is limited and R-integrable for each* t *near* τ;

3° *The points* $\mathfrak{E}$ *of* $\mathfrak{A}$ *in whose vicinity* $f_x'(x, t)$ *has infinite peaks as* $t \doteq \tau$ *form an enumerable set;*

4° $\phi(x)$ *is continuous at the points* $\mathfrak{E}$;

5° $\psi(x) = \lim_{t=\tau} f_x'(x, t)$ *is limited and R-integrable in* $\mathfrak{A}$;

Then at a point of continuity of $\psi(x)$ *in* $\mathfrak{A}$

$$\phi'(x) = \psi(x), \tag{1}$$

or what is the same

$$\frac{d}{dx}\lim_{t=\tau} f(x, t) = \lim_{t=\tau}\frac{d}{dx} f(x, t).$$

For let $\delta = (\alpha < \beta)$ be an interval in $\mathfrak{A}$ containing no point of $\mathfrak{E}$. Then for any x in δ

$$\int_\alpha^x f'_x(x, t)dx = f(x, t) - f(\alpha, t) \quad , \quad \text{by } 2°.$$

Hence

$$\lim_{t=\tau}\int_\alpha^x f'_x(x, t)dx = \lim_{t=\tau}\{f(x, t) - f(\alpha, t)\}$$

$$= \phi(x) - \phi(\alpha) \quad , \quad \text{by } 1°. \tag{2}$$

By 556, 3, $\phi(x)$ is continuous in δ. Thus $\phi(x)$ is continuous at any point not in $\mathfrak{E}$. Hence by 4° it is continuous in $\mathfrak{A}$.

We may thus apply 559, 1, replacing therein $f(x, t)$ by $f'_x(x, t)$. We get

$$\lim_{t=\tau}\int_a^x f'_x(x, t)dx = \int_a^x \lim f'_x(x, t)dx = \int_a^x \psi(x)dx. \tag{3}$$

Since 2) obviously holds when we replace α by a, this relation with 3) gives

$$\int_a^x \psi(x)dx = \phi(x) - \phi(a).$$

At a point of continuity, this gives 1) on differentiating.

2. *If* 1° $F(x) = \Sigma f_{\iota_1 \dots \iota_n}(x)$ *converges in the interval* $\mathfrak{A}$;

2° $G(x) = \Sigma f'_\iota(x)$ *and each of its terms are limited and R-integrable in* $\mathfrak{A}$;

3° *The points of* $\mathfrak{A}$ *in whose vicinity* $G_\lambda(x)$ *has infinite peaks as* $\lambda \doteq \infty$, *form an enumerable set at which* $F(x)$ *is continuous;*

Then at a point of continuity of $G(x)$ *we have*

$$F'(x) = G(x),$$

or what is the same

$$\frac{d}{dx}\Sigma f_\iota(x) = \sum \frac{df_\iota(x)}{dx}.$$

561. *Example.*

$$F(x) = \sum_1^\infty \left\{ \frac{nx^2}{e^{nx^2}} - \frac{(n+1)x^2}{e^{(n+1)x^2}} \right\} = \frac{x^2}{e^{x^2}}.$$

Hence

$$F'(x) = \frac{2\,x}{e^{x^2}} - \frac{2\,x^3}{e^{x^2}}. \tag{1}$$

The series obtained by differentiating F termwise is

$$G(x) = \sum_1^\infty \left\{ \frac{2\,nx}{e^{nx^2}} - \frac{2\,n^2x^3}{e^{nx^2}} - \frac{2(n+1)}{e^{(n+1)x^2}} + \frac{2(n+1)^2x^3}{e^{(n+1)x^2}} \right\}.$$

Here

$$G_{n-1}(x) = \frac{2\,x}{e^{x^2}} - \frac{2\,x^3}{e^{x}} - \left\{ \frac{2\,nx}{e^{nx^2}} - \frac{2\,n^2x^3}{e^{nx^2}} \right\}.$$

Hence

$$G(x) = \frac{2\,x}{e^{x^2}} - \frac{2\,x^3}{e^{x^2}} \tag{2}$$

is a continuous function of x.

The convergence of the G series is not uniform at $x = 0$. For set $a_n = 1/n$. Then

$$G_{n-1}(a_n) = G(a_n) - \left\{ \frac{2}{e^{\frac{1}{n}}} - \frac{\frac{2}{n}}{e^{\frac{1}{n}}} \right\} \doteq -2.$$

To get the peaks of the residual function we consider the points of extreme of

$$y = \frac{nx(1 - nx^2)}{e^{nx^2}}. \tag{3}$$

We find

$$y' = \frac{n(1 - 5\,nx^2 + 2\,n^2x^4)}{e^{nx^2}}.$$

Thus $y' = 0$ when

$$2\,n^2x^4 - 5\,nx^2 + 1 = 0,$$

or when

$$x = \frac{a}{\sqrt{n}} \text{ or } \frac{\alpha}{\sqrt{n}}, \quad a, \alpha \text{ constants.}$$

Putting these values in 3), we find that y has the form

$$y = c\sqrt{n}.$$

Hence $x = 0$ is the only point where the residual function has an infinite peak. Thus the conditions of 560, 2, are satisfied, and we should have $F'(x) = G(x)$ for any x. This is indeed so, as 1), 2) show.

CHAPTER XVII

GEOMETRIC NOTIONS

Plane Curves

562. In this chapter we propose to examine the notions of curve and surface together with other allied geometric concepts. Like most of our notions, we shall see that they are vague and uncertain as soon as we pass the confines of our daily experience. In studying some of their complexities and even paradoxical properties, the reader will see how impossible it is to rely on his unschooled intuition. He will also learn that the demonstration of a theorem in analysis which rests on the evidence of our geometric intuition cannot be regarded as binding until the geometric notions employed have been clarified and placed on a sound basis.

Let us begin by investigating our ideas of a plane curve.

563. Without attempting to define a curve we would say on looking over those curves most familiar to us that a plane curve has the following properties:

1° It can be generated by the motion of a point.

2° It is formed by the intersection of two surfaces.

3° It is continuous.

4° It has a tangent at each point.

5° The arc between any two of its points has a length.

6° A curve is not superficial.

7° Its equations can be written in any one of the forms

$$y = f(x), \tag{1}$$

$$x = \phi(t) \quad , \quad y = \psi(t), \tag{2}$$

$$F(x, y) = 0\,; \tag{3}$$

and conversely such equations define curves.

8° When closed it forms the complete boundary of a region.

9° This region has an area.

Of all these properties the first is the most conspicuous and characteristic to the naïve intuition. Indeed many employ this as the definition of a curve. Let us therefore look at our ideas of motion.

564. *Motion.* In this notion, two properties seem to be essential. 1° motion is continuous, 2° it takes place at each instant in a definite direction and with a definite speed. The direction of motion, we agree, shall be given by dy/dx, its speed by ds/dt. We see that the notion of motion involves properties 4°, 5°, and 7°. Waiving this point, let us notice a few peculiarities which may arise.

Suppose the curve along which the motion takes place has an angle point or a cusp as in I, 366. What is the direction of motion at such a point? Evidently we must say that motion is impossible along such a curve, or admit that the ordinary idea of motion is imperfect and must be extended in accordance with the notion of right-hand and left-hand derivatives.

Similarly ds/dt may also give two speeds, a posterior and an anterior speed, at a point where the two derivatives of $s = \phi(t)$ are different.

Again we will admit that at any point of the path of motion, motion may begin and take place in either direction. Consider what happens for a path defined by the continuous function in I, 367. This curve has no tangent at the origin. We ask how does the point move as it passes this point, or to make the question still more embarassing, suppose the point at the origin. In what direction does it start to move? We will admit that no such motion is possible, or at least it is not the motion given us by our intuition. Still more complicated paths of this nature are given in I, 369, 371, and in Chapter XV of the present volume.

It thus appears that to define a curve as the path of a moving point, is to define an unknown term by another unknown term, equally if not more obscure.

565. 2° *Property. Intersection of Two Surfaces.* This property has also been used as the definition of a curve. As the notion

of a surface is vastly more complicated than that of a curve, it hardly seems advisable to define a complicated notion by one still more complicated and vague.

566. 3° *Property. Continuity.* Over this knotty concept philosophers have quarreled since the days of Democritus and Aristotle. As far as our senses go, we say a magnitude is continuous when it can pass from one state to another by imperceptible gradations. The minute hand of a clock appears to move continuously, although in reality it moves by little jerks corresponding to the beats of the pendulum. Its velocity to our senses appears to be continuous.

We not only say that the magnitude shall pass from one state to another by gradations imperceptible to our senses, but we also demand that between any two states another state exists and so without end. Is such a magnitude continuous? No less a mathematician than Bolzano admitted this in his philosophical tract *Paradoxien des Unendlichen.* No one admits it, however, to-day. The different states of such a magnitude are pantactic, but their ensemble is not a continuum.

But we are not so much interested in what constitutes a continuum in the abstract, as in what constitutes a continuous curve or even a continuous straight line or segment. The answer we have adopted to these questions is given in the theory of irrational numbers created by Cantor and Dedekind [see Vol. I, Chap. II], and in the notion of a continuous function due to Cauchy and Weierstrass [see Vol. I, Chap. VII].

These definitions of continuity are analytical. With them we can reason with the utmost precision and rigor. The consequences we deduce from them are sufficiently in accord with our intuition to justify their employment. We can show by purely analytic methods that a continuous function $f(x)$ does attain its extreme values [I, 354], that if such a function takes on the value a at the point P, and the value b at the point Q, then it takes on all intermediary values between a, b, as x ranges from P to Q [I, 357]. We can also show that a closed curve without double point does form the boundary of a complete region [cf. 576 seq.].

567. 4° *Property. Tangents.* To begin with, what is a tangent? Euclid defines a tangent to a circle as a straight line which meets

the circle and being produced does not cut it again. In commenting on this definition Casey says, "In modern geometry a curve is made up of an infinite number of points which are placed in order along the curve, and then the secant through two consecutive points is a tangent." If the points on a curve were like beads on a string, we might speak of consecutive points. As, however, there are always an infinite number of points between any two points on a continuous curve, this definition is quite illusory.

The definition we have chosen is given in I, 365. That property 3° does not hold at each point of a continuous curve was brought out in the discussion of property 1°. Not only is it not necessary that a curve has a tangent at each of its points, but a curve does not need to have a tangent at a pantactic set of points, as we saw in Chapter XV.

For a long time it was supposed that every curve has a tangent at each point, or if not at each point, at least in general. Analytically, this property would go over into the following: *every continuous function has a derivative.* A celebrated attempt to prove this was made by Ampère.

Mathematicians were greatly surprised when Weierstrass exhibited the function we have studied in 502 and which has no derivative.

Weierstrass* himself remarks: "Bis auf die neueste Zeit hat man allgemein angenommen, dass eine eindeutige und continuirliche Function einer reellen Veränderlichen auch stets eine erste Ableitung habe, deren Werth nur an einzelnen Stellen unbestimmt oder unendlich gross werden könne. Selbst in den Schriften von Gauss, Cauchy, Dirichlet findet sich meines Wissens keine Äusserung, aus der unzweifelhaft hervorginge, dass diese Mathematiker, welche in ihrer Wissenschaft die strengste Kritik überall zu üben gewohnt waren, anderer Ansicht gewesen seien."

568. *Property* 5°. *Length.* We think of a curve as having length. Indeed we read as the definition of a curve in Euclid's Elements: a line is length without breadth. When we see two simple curves we can often compare one with the other in regard to length without consciously having established a way to measure

* *Werke*, vol. 2, p. 71.

them. Perhaps we unconsciously suppose them described at a uniform rate and estimate the time it takes. It may be that we regard them as inextensible strings whose length is got by straightening them out. A less obvious way to measure their lengths would be to roll a straightedge over them and measure the distance on the edge between the initial and final points of contact.

We ask how shall we formulate arithmetically our intuitional ideas regarding the length of a curve? The intuitionist says, a curve or the arc of a curve *has* length. This length is expressed by a number L which is obtained by taking a number of points $P_1, P_2, P_3 \cdots$ on the curve between the end points P, P', and forming the sum

$$\Sigma \overline{P_\iota P_{\iota+1}}. \tag{1}$$

The limit of this sum as the points became pantactic is the length L of the arc PP'.

Our point of view is different. We would say: Whatever arithmetic formulation we choose we have no *a priori* assurance that it adequately represents our intuitional ideas of length. With the intuitionist we will, however, form the sum 1) and see if it has a limit, however the points P_ι are chosen. If it has, we will investigate this number used as a definition of length and see if it leads to consequences which are in harmony with our intuition.

This we now proceed to do.

569. 1. Let
$$x = \phi(t) \quad , \quad y = \psi(t) \tag{1}$$

be one-valued continuous functions of t in the interval $\mathfrak{A} = (a < b)$. As t ranges over $\mathfrak{A}$ the point x, y will describe a *curve* or *an arc of a curve C.* We might agree to call such curves *analytic*, in distinction to those given by our intuition. The interval $\mathfrak{A}$ is the *interval corresponding to C.*

Let D be a finite division of $\mathfrak{A}$ of norm d, defined by

$$a < t_1 < t_2 < \cdots < b.$$

To these values of t will correspond points

$$P, P_1, P_2 \cdots Q \tag{2}$$

on C, which may be used to define a polygon P_D whose vertices are 2).

Let $(m, m+1)$ denote the side P_mP_{m+1}, as well as its length. If we denote the length of P_D by the same letter, we have

$$P_D = \Sigma(m, m+1) = \Sigma\sqrt{\Delta x_m^2 + \Delta y_m^2}.$$

If

$$\lim_{d=0} P_D \tag{3}$$

exists, it is called the *length of the arc* C, and C is *rectifiable.*

2. (*Jordan.*) *For the arc PQ to be rectifiable, it is necessary and sufficient that the functions ϕ, ψ in* 1) *have limited variation in* $\mathfrak{A}$.

For

$$\sqrt{\Delta x^2 + \Delta y^2} \geq |\Delta x|.$$

Hence

$$P_D \geq \Sigma |\Delta x|.$$

But the sum on the right is the variation of ϕ for the division D. If now ϕ does not have limited variation in $\mathfrak{A}$, the limit 3) does not exist. The same holds for ψ. Hence limited variation is a necessary condition.

The condition is *sufficient.* For

$$P_D \leq \Sigma |\Delta x| + \Sigma |\Delta y| = \operatorname*{Var}_D \phi + \operatorname*{Var}_D \psi.$$

As ϕ, ψ have limited variation, this shows that

$$P_0 = \operatorname*{Max}_D P_D$$

is finite. *We show now that*

$$\lim_{d=0} P_D = P_0. \tag{4}$$

For there exists a division Δ such that

$$P_0 - \frac{\epsilon}{2} < P_\Delta \leq P_0. \tag{5}$$

Let Δ cause $\mathfrak{A}$ to fall into ν intervals, the smallest of which has the length λ. Let D be a division of $\mathfrak{A}$ of norm $d \leq d_0 < \lambda$. Then no interval of D contains more than one point of Δ. Let $E = D + \Delta$.

Obviously

$$P_E \geq P_D \quad \text{or} \quad P_\Delta.$$

Suppose that the point t_κ of Δ falls in the interval $(t_\iota, t_{\iota+1})$ of D. Then the chord $(\iota, \iota+1)$ in P_D is replaced by the two chords (ι, κ), $(\kappa, \iota+1)$ in P_E. Hence

$$P_E - P_D = \Sigma G_\kappa \quad , \quad \kappa = 1, 2 \cdots \mu \leq \nu$$

where

$$G_\kappa = (\iota, \kappa) + (\kappa, \iota+1) - (\iota, \iota+1).$$

Obviously as ϕ, ψ are continuous we may take d_0 so small that each

$$G_\kappa < \frac{\epsilon}{2\nu} \quad , \quad \text{for any } d \leq d_0.$$

Hence

$$P_E - P_D < \frac{\epsilon}{2}. \tag{6}$$

From 5), 6) we have

$$P_0 - P_D < \epsilon \quad , \quad \text{for any } d \leq d_0,$$

which gives 4).

3. *If the arc PQ is rectifiable, any arc contained in PQ is also rectifiable.*

For ϕ, ψ having limited variation in interval $\mathfrak{A}$, have *a fortiori* limited variation in any segment of $\mathfrak{A}$.

4. *Let the rectifiable arc C fall into two arcs C_1, C_2. If s, s_1, s_2 are the lengths of C, C_1, C_2, then*

$$s = s_1 + s_2. \tag{7}$$

For we saw that C_1, C_2 are rectifiable since C is. Let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be the intervals in $\mathfrak{A}$ corresponding to C_1, C_2. Let D_1, D_2 be divisions of $\mathfrak{A}_1$, $\mathfrak{A}_2$ of norm d. Then

$$s_1 = \lim_{d=0} P_{D_1} \quad , \quad s_2 = \lim_{d=0} P_{D_2}.$$

But D_1, D_2 effect a division of $\mathfrak{A}$, and since

$$s = \lim_{e=0} P_E \tag{8}$$

with respect to the class of all divisions of $\mathfrak{A}$, the limit 8) is the same when E is restricted to range over divisions of the type of D. Now

$$P_D = P_{D_1} + P_{D_2}.$$

Passing to the limit, we get 7).

The preceding reasoning also shows that *if* C_1, C_2 *are rectifiable curves, then* C *is, and* 7) *holds again.*

5. *If* 1) *define a rectifiable curve, its length* s *is a continuous function* $s(t)$ *of* t.

For ϕ, ψ having limited variation,

$$\phi = \phi_1 - \phi_2 \quad , \quad \psi = \psi_1 - \psi_2,$$

where the functions on the right are continuous monotone *increasing* functions of t in the interval $\mathfrak{A} = (a < b)$.

For a division D of norm d of the interval $\Delta\mathfrak{A} = (t, t + h)$ we have

$$\begin{aligned} P_D &= \Sigma\sqrt{\Delta x^2 + \Delta y^2} \\ &\leq \Sigma \mid \Delta x \mid + \Sigma \mid \Delta y \mid \\ &\leq \Sigma\Delta\phi_1 + \Sigma\Delta\phi_2 + \Sigma\Delta\psi_1 + \Sigma\Delta\psi_2 \\ &\leq \delta\phi_1 + \delta\phi_2 + \delta\psi_1 + \delta\psi_2, \end{aligned}$$

where $\delta\phi_1 = \phi_1(t + h) - \phi(t)$, and similarly for the other functions. As ϕ_1 is continuous, $\delta\phi_1 \doteq 0$, etc., as $h \doteq 0$. We may therefore take $\eta > 0$ so small that $\delta\phi_1, \delta\phi_2, \delta\psi_1, \delta\psi_2 < \epsilon/4$, if $h < \eta$.

Hence $\Delta s = s(t + h) - s(t) \leq \text{Max } P_D < \epsilon \quad , \quad \text{if } 0 < h > \eta$.

Thus s is continuous.

6. *The length* s *of the rectifiable arc* C *corresponding to the interval* $(a < t)$ *is a monotone increasing function of* t.

This follows from 4.

7. *If* x, y *do not have simultaneous intervals of invariability,* $s(t)$ *is an increasing function of* t. *The inverse function is one-valued and increasing and the coördinates* x, y *are one-valued functions of* s.

That the inverse function $t\,(s)$ is one-valued follows from I, 214. We can thus express t in terms of s, and so eliminate t in 1).

570. 1. *If* ϕ', ψ' *are continuous in the interval* $\mathfrak{A}$,

$$s = \int_{\mathfrak{A}} dt\sqrt{\phi'^2 + \psi'^2}. \tag{1}$$

For

$$s = \lim_{d=0} \Sigma\sqrt{\Delta\phi_\kappa^2 + \Delta\psi_\kappa^2}. \tag{2}$$

Now
$$\Delta\phi_\kappa = \phi'(t'_\kappa)\Delta t_\kappa \quad , \quad \Delta\psi_\kappa = \psi'(t''_\kappa)\Delta t_\kappa \tag{3}$$
where t'_κ, t''_κ lie in the interval Δt_κ.

As ϕ', ψ' are continuous they are uniformly continuous. Hence for any division D of norm $\leq$ some d_0,
$$\phi'(t'_\kappa) = \phi'(t_\kappa) + \alpha_\kappa \quad , \quad \psi'(t''_\kappa) = \psi'(t_\kappa) + \beta_\kappa$$
where $|\alpha_\kappa|$, $|\beta_\kappa| <$ some η, small at pleasure, for any κ. Thus
$$\sqrt{\Delta\phi_\kappa^2 + \Delta\psi_\kappa^2} = \Delta t_\kappa \sqrt{\phi'(t_\kappa)^2 + \psi'(t_\kappa)^2} + \epsilon_\kappa \Delta t_\kappa,$$
and we may take
$$|\epsilon_\kappa| < \epsilon/\overline{\mathfrak{A}} \quad , \quad \kappa = 1, 2 \cdots$$

Thus
$$s = \lim_{d=0} \Sigma \Delta t_\kappa \sqrt{\phi'(t_\kappa)^2 + \psi'(t_\kappa)^2} + \lim \Sigma \epsilon_\kappa \Delta t_\kappa.$$

Hence
$$\left| s - \int_{\mathfrak{A}} \right| < \epsilon,$$
which establishes 1).

For simplicity we have assumed ϕ', ψ' to be continuous in $\mathfrak{A}$. This is not necessary, as the following shows.

2. Let $a_1, \cdots a_n, b_1, \cdots b_n \geq 0$ but not all $= 0$.

Then
$$\left| \sqrt{a_1^2 + \cdots + a_n^2} - \sqrt{b_1^2 + \cdots + b_n^2} \right| \leq \sum_m |a_m - b_m|,$$
$$m = 1, 2 \cdots n. \tag{4}$$

For
$$(\sqrt{{}_1^2a + \cdots} - \sqrt{b_1^2 + \cdots})(\sqrt{a_1^2 + \cdots} + \sqrt{b_1^2 + \cdots})$$
$$= (a_1^2 + \cdots + a_n^2) - (b_1^2 + \cdots + b_n^2)$$
$$= (a_1^2 - b_1^2) + \cdots + (a_n^2 - b_n^2)$$
$$= (a_1 - b_1)(a_1 + b_1) + \cdots + (a_n - b_n)(a_n + b_n).$$

Hence
$$\sqrt{a_1^2 + \cdots} - \sqrt{b_1^2 + \cdots} = \sum_{m=1}^{n} (a_m - b_m) \frac{a_m + b_m}{\sqrt{a_1^2 + \cdots} + \sqrt{b_1^2 + \cdots}}. \tag{5}$$

But
$$\left| \frac{a_m + b_m}{\sqrt{a_1^2 + \cdots} + \sqrt{b_1^2 + \cdots}} \right| \leq 1.$$

This in 5) gives 4).

Let us apply 4) to prove the following theorem, more general than 1.

3. (*Baire.*) *If ϕ', ψ' are limited and R-integrable, then*

$$s = \int_{\mathfrak{A}} \sqrt{\phi'^2 + \psi'^2}\, dt. \tag{1}$$

For by 4),

$$\left| \sqrt{\phi'(t_\kappa')^2 + \psi'(t_\kappa'')^2} - \sqrt{\phi'(t_\kappa)^2 + \psi'(t_\kappa)^2} \right| \leq \left| \phi'(t_\kappa') - \phi'(t_\kappa) \right| + \left| \psi'(t_\kappa'') - \psi'(t_\kappa) \right| ;$$

or $$\Phi_\kappa - \Psi_\kappa = \eta_\kappa' \operatorname{Osc} \phi'(t) + \eta_\kappa'' \operatorname{Osc} \psi'(t) \quad , \quad \text{in } \delta_\kappa = \Delta t_\kappa,$$

where η_κ', η_κ'' are numerically ≤ 1. Thus

$$\left| \Sigma \delta_\kappa \Phi_\kappa - \Sigma \delta_\kappa \Psi_\kappa \right| = \Sigma \delta_\kappa \eta_\kappa' \operatorname{Osc} \phi' + \Sigma \delta_\kappa \eta_\kappa'' \operatorname{Osc} \psi'. \tag{6}$$

As ϕ', ψ' are integrable, the right side $\doteq 0$, as $d \doteq 0$. Now

$$\lim_{d=0} \Sigma \delta_\kappa \Psi_\kappa = \int_{\mathfrak{A}} \sqrt{\phi'^2 + \psi'^2}\, dt.$$

Thus passing to the limit in 6), we have

$$\lim \Sigma \Delta t_\kappa \sqrt{\phi'(t_\kappa')^2 + \psi'(t_\kappa'')^2} = \int_{\mathfrak{A}} \cdot$$

This with 2), 3) gives 1) at once.

571. *Volterra's Curve.* It is interesting to note that *there are rectifiable curves for which $\phi'(t)$, $\psi'(t)$ are not both R-integrable.* Such a curve is Volterra's curve, discussed in 503. Let its equation be $y = f(x)$. Then $f'(x)$ behaves as

$$2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

in the vicinity of a non null set in $\mathfrak{A} = (0, 1)$. Hence $f'(x)$ is not R-integrable in $\mathfrak{A}$. But then it is easy to show that

$$\int_0^1 \sqrt{1 + f'(x)^2}\, dx$$

does not exist. For suppose that

$$g = \sqrt{1 + f'(x)^2}$$

were R-integrable. Then $g^2 = 1 + f'(x)^2$ is R-integrable, and hence $f'(x)^2$ also. But the points of discontinuity of f'^2 in $\mathfrak{A}$ do not form a null set. Hence f'^2 is not R-integrable.

On the other hand, Volterra's curve is rectifiable by 569, 2, and 528, 1.

572. Taking the definition of length given in 569, 1, we saw that the coördinates

$$x = \phi(t) \quad , \quad y = \psi(t)$$

must have limited variation for the curve to be rectifiable. But we have had many examples of functions not having limited variation in an interval $\mathfrak{A}$. Thus the curve defined by

$$\begin{aligned} y &= x \sin \frac{1}{x} \quad , \quad x \neq 0 \\ &= 0 \quad , \quad x = 0 \end{aligned} \tag{4}$$

does not have a length in $\mathfrak{A} = (-1, 1)$; while

$$\begin{aligned} y &= x^2 \sin \frac{1}{x} \quad , \quad x \neq 0 \\ &= 0 \quad , \quad x = 0 \end{aligned} \tag{5}$$

does.

It certainly astonishes the naïve intuition to learn that the curve 4) has no length in any interval δ about the origin however small, or if we like, that this length is infinite, however small δ is taken. For the same reason we see that

No arc of Weierstrass' curve has a length (or its length is infinite) however near the end points are taken to each other, when $ab > 1$.

573. 1. 6° *Property. Space-filling Curves.* We wish now to exhibit a curve which passes through every point of a square, *i.e.* which completely fills a square. Having seen how to define one such curve, it is easy to construct such curves in great variety, not only for the plane but for space. The first to show how this may be done was *Peano* in 1890. The curve we wish now to define is due to *Hilbert*.

We start with a unit interval $\mathfrak{A} = (0, 1)$ over which t ranges, and a unit square $\mathfrak{B}$ over which the point x, y ranges. We define

$$x = \phi(t) \quad , \quad y = \psi(t) \tag{1}$$

as one-valued continuous functions of t in $\mathfrak{A}$ so that xy ranges over $\mathfrak{B}$ as t ranges over $\mathfrak{A}$. The analytic curve C defined by 1) thus completely fills the square $\mathfrak{B}$.

We do this as follows. We effect a division of $\mathfrak{A}$ into four equal segments δ'_1, δ'_2, δ'_3, δ'_4, and of $\mathfrak{B}$ into equal squares η'_1, η'_2, η'_3, η'_4, as in Fig. 1.

We call this the first division or D_1. The correspondence between $\mathfrak{A}$ and $\mathfrak{B}$ is given in *first approximation* by saying that to each point P in δ'_ι shall correspond some point Q in η'_ι.

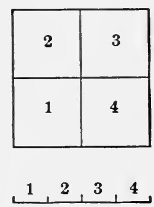


Fig. 1.

We now effect a second division D_2 by dividing each interval and square of D_1 into four equal parts.

We number them as in Fig. 2,

$$\delta''_1 \; , \; \delta''_2 \; \cdots \; \delta''_{16}$$
$$\eta''_1 \; , \; \eta''_2 \; \cdots \; \eta''_{16}$$

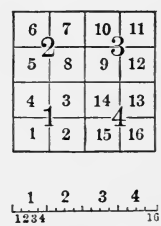


Fig. 2.

As to the numbering of the η's we observe the following two principles: 1° we may pass over the squares 1 to 16 continuously without passing the same square twice, and 2° in doing this we pass over the squares of D_1 in the same order as in Fig. 1. The correspondence between $\mathfrak{A}$ and $\mathfrak{B}$ is given in *second approximation* by saying that to each point P in δ''_ι shall correspond some point Q in η''_ι. In this way we continue indefinitely.

To find the point Q in $\mathfrak{B}$ corresponding to P in $\mathfrak{A}$ we observe that P lies in a sequence of intervals

$$\delta' > \delta'' > \delta''' > \cdots \doteq 0, \tag{2}$$

to which correspond uniquely a sequence of squares

$$\eta' > \eta'' > \eta''' > \cdots \doteq 0. \tag{3}$$

The sequence 3) determines *uniquely* a point whose coördinates are one-valued functions of t, viz. the functions given in 1).

The functions 1) *are continuous in* $\mathfrak{A}$.

For let t' be a point near t; it either lies in the same interval as t in D_n or in the adjacent interval. Thus the point Q' corre-

sponding to t' either lies in the same square of D_n as the point Q corresponding to t, or in an adjacent square. But the diagonal of the squares $\doteq 0$, as $n \doteq \infty$. Thus

$$\text{Dist}\ (Q'Q) \doteq 0 \quad , \quad \text{as } n \doteq \infty.$$

Thus

$$\phi(t') - \phi(t) \quad , \quad \text{and } \psi(t') - \psi(t)$$

both $\doteq 0$, as $t' \doteq t$.

As t ranges over $\mathfrak{A}$, the point x, y ranges over every point in the square $\mathfrak{B}$.

For let Q be a given point of $\mathfrak{B}$. It lies in a sequence of squares as 3). If Q lies on a side or at a vertex of one of the η squares, there is more than one such sequence. But having taken such a sequence, the corresponding sequence 2) is uniquely determined. Thus to each Q corresponds at least one P. A more careful analysis shows that to a given Q never more than four points P can correspond.

2. The method we have used here may obviously be extended to space. By passing median planes through a unit cube we divide it into 2^3 equal cubes. Thus to get our correspondence each division D_n should divide each interval and cube of the preceding division D_{n-1} into 2^3 equal parts. The cubes of each division should be numbered according to the 1° and 2° principles of enumeration mentioned in 1.

By this process we define

$$x = \phi_1(t) \quad , \quad y = \phi_2(t) \quad , \quad z = \phi_3(t)$$

as one-valued continuous functions of t such that as t ranges over the unit interval $(0, 1)$, the point x, y, z ranges over the unit cube.

574. 1. *Hilbert's Curve.* We wish now to study in detail the correspondence between the unit interval $\mathfrak{A}$ and the unit square $\mathfrak{B}$ afforded by Hilbert's curve defined in 573. A number of interesting facts will reward our labor. We begin by seeking the points P in $\mathfrak{A}$ which correspond to a given Q in $\mathfrak{B}$.

To this end let us note *how P enters and leaves an η square.* Let B be a square of D_n. In the next division B falls into four

squares $B_1 \cdots B_4$ and in the $n+2^{\rm d}$ division in 16 squares $B_{\iota j}$. Of these last, four lie at the vertices of B; we call them *vertex squares.* The other 12 are *median squares.* A simple consideration shows that the η squares of D_{n+2} are so numbered that we always enter a square B belonging to D_n, and also leave it by a vertex square.

Since this is true of every division, we see on passing to the limit that the point Q enters and leaves any η square at the vertices of η. We call this the *vertex law.*

Let us now *classify the points* P, Q.

If P is an end point of some division $D_n >$ we call it a *terminal point*, otherwise an *inner point*, because it lies within a sequence of δ intervals $\delta' > \delta'' > \cdots \doteq 0$.

The points Q we divide into four classes:

1° vertex points, when Q is a vertex of some division.

2° inner points, when Q lies within a sequence of squares

$$\eta' > \eta'' > \cdots \doteq 0.$$

3° lateral points, when Q lies on a side of some η square but never at a vertex.

4° points lying on the edge of the original square $\mathfrak{B}$. Points of this class also lie in 1°, 3°.

We now seek the points P corresponding to a Q lying in one of these four classes.

Class 1°. *Q a Vertex Point.* Let D_n be the first division such that Q is at a vertex. Then Q lies in four squares η_ι, η_j, η_κ, η_l of D_n.

There are 5 cases:

α) $\iota\, j\, k\, l$ are consecutive.

β) $\iota\, j\, k$ are consecutive, but not l.

γ) $\iota\, j$ are consecutive, but not $k\, l$.

δ) $\iota\, j$, also $k\, l$, are consecutive.

ϵ) no two are consecutive.

A simple analysis shows that α), β) are not permanent in the following divisions; γ), δ) may or may not be permanent; ϵ) is permanent.

Now, whenever a case is permanent, we can enclose Q in a sequence of η squares whose sides $\doteq 0$. To this sequence corresponds uniquely a sequence of δ intervals of lengths $\doteq 0$. Thus to two consecutive squares will correspond two consecutive intervals which converge to a single point P in $\mathfrak{A}$. If the squares are not consecutive, the corresponding intervals converge to two distinct points in $\mathfrak{A}$. Thus we see that when γ) is permanent, to Q correspond three points P. When δ) is permanent, to Q correspond two points P. While when Q belongs to ϵ), four points P correspond to it.

Class 2°. *Q an Inner Point.* Obviously to each Q corresponds one point P and only one.

Class 3°. *Q a Lateral Point.* To fix the ideas let Q lie on a vertical side of one of the η's. Let it lie between η_ι, η_j of D_n. There are two cases:

α) $$j = \iota + 1.$$

β) $$j > \iota + 1.$$

We see easily that α) is not permanent, while of course β) is. Thus to each Q in class 3°, there correspond two points P.

Class 4°. *Q lies on the edge of $\mathfrak{B}$.* If Q is a vertex point, to it may correspond one or two points P. If Q is not a vertex point, only one point P corresponds to it.

To sum up we may say:

To each inner point Q corresponds one inner point P.

To each lateral point Q correspond two points P.

To each edge point Q correspond one or two points P.

To each vertex point Q, correspond two, three, or four points P.

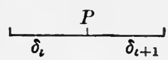
2. As a result of the preceding investigation we show easily that:

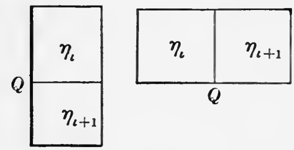
To the points on a line parallel to one of the sides of $\mathfrak{B}$ correspond in $\mathfrak{A}$ an apantactic perfect set.

3. Let us now consider the tangents to Hilbert's curve which we denote by H.

Let Q be a vertex point. We saw there were three permanent cases γ), δ), ϵ).

In cases γ), δ) we saw that to two consecutive δ intervals correspond permanently two contiguous vertical or horizontal squares.





Thus as t ranges over $\delta_\iota, \delta_{\iota+1}$, the point x, y ranges over these squares, and the secant line joining Q and this variable point x, y oscillates through 180°. There is thus no tangent at Q. In case ϵ) we see similarly that the secant line ranges through 90°. Again there is no tangent at Q.

In the same way we may treat the three other classes. We find that the secant line never converges to a fixed position, and may oscillate through 360°, viz. when Q is an inner point. As a result we see that *Hilbert's curve has at no point a tangent, nor even a unilateral tangent.*

4. Associated with Hilbert's curve H are two other curves,

$$x = \phi(t) \quad , \quad \text{and } y = \psi(t).$$

The functions ϕ, ψ being one-valued and continuous in $\mathfrak{A}$, these curves are continuous and they do not have a multiple point. A very simple consideration shows that *they do not have even a unilateral tangent at a pantactic set of points in* $\mathfrak{A}$.

575. *Property* 7°. *Equations of a Curve.* As already remarked, it is commonly thought that the equation of a curve may be written in any one of the three forms

$$y = f(x), \tag{1}$$

$$\Phi(x, y) = 0, \tag{2}$$

$$x = \phi(t) \quad , \quad y = \psi(t), \tag{3}$$

and if these functions are continuous, these equations define continuous curves.

Let us look at the Hilbert curve H. We saw its equation could be expressed in the form 3). H cuts an ordinate at every point of it for which $0 \leq y \leq 1$. Thus if we tried to define H by

an equation of the type 1), $f(x)$ would have to take on every value between 0 and 1 for each value of x in $\mathfrak{A} = (0, 1)$. No such functions are considered in analysis.

Again, we saw that to any value $x = a$ in $\mathfrak{A}$ corresponds a perfect apantactic set of values $\{t_a\}$ having the cardinal number $\mathfrak{c}$. Thus the inverse function of $x = \phi(t)$ is a many-valued function of x whose different values form a set whose cardinal number is $\mathfrak{c}$. Such functions have not yet been studied in analysis.

How is it possible in the light of such facts to say that we may pass from 3) to 1) or 2) by *eliminating* t from 3). And if we cannot, how can we say a curve can be represented equally well by any of the above three equations, or if the curve is given by one of these three equations, we may suppose it replaced by one of the other two whenever convenient. Yet this is often done.

In this connection we may call attention to the loose way *elimination* is treated. Suppose we have a set of equations

$$f_1(x_1 \cdots x_m\, t_1 \cdots t_n) = 0,$$

$$f_{n+1}(x_1 \cdots x_m\, t_1 \cdots t_n) = 0.$$

We often see it stated that one can eliminate $t_1 \cdots t_n$ and obtain a relation involving the x's alone. Any reasoning based on such a procedure must be regarded as highly unsatisfactory, in view of what we have just seen, until this elimination process has been established.

576. *Property* 8°. *Closed Curves.* A circle, a rectangle, an ellipse are examples of closed curves. Our intuition tells us that it is impossible to pass from the inside to the outside without crossing the curve itself. If we adopt the definition of a closed curve without multiple point given in I, 362, we find it no easy matter to establish this property which is so obvious for the simple closed curves of our daily experience. The first to effect the demonstration was *Jordan* in 1892. We give here* a proof due to *de la Vallée-Poussin*.†

Let us call for brevity a continuous curve without double point

* The reader is referred to a second proof due to Brouwer and given in 598 seq.
† *Cours d'Analyse*, Paris, 1903, Vol. 1, p. 307.

a *Jordan curve.* A continuous closed curve without double point will then be a *closed Jordan curve.* Cf. I, 362.

577. *Let C be a closed Jordan curve. However small $\sigma > 0$ is taken, there exists a polygonal ring R containing C and such that*

1° *Each point of R is at a distance $< \sigma$ from C.*

2° *Each point of C is at a distance $< \sigma$ from the edges of R.*

For let
$$x = \phi(t) \quad , \quad y = \psi(t) \qquad (1$$
be continuous one-valued functions of t in $T = (a < b)$ defining C. Let $D = (a, a_1, a_2 \cdots b)$ be a division of T of norm d. Let $\alpha, \alpha_1, \alpha_2 \cdots$ be points of C corresponding to $a, a_1 \cdots$ If d is sufficiently small, the distance between two points on the arc $C_\iota = (\alpha_{\iota-1}, \alpha_\iota)$ is $< \epsilon'$, small at pleasure. Let Δ be a quadrate division of the x, y plane of norm δ. Let us shade all cells containing a point of C_ι. These form a connected domain since C_ι is continuous. We can thus go around its outer edge without a break.* If this shaded domain contains unshaded cells, let us shade these too. We call the result a link A_ι. It has only one edge E_ι, and the distance between any two points of E_ι is obviously $< \epsilon' + 2\sqrt{2}\,\delta$. We can choose d, δ so small that
$$\epsilon' + 2\sqrt{2}\,\delta < \sigma, \qquad \text{arbitrarily small.} \qquad (1$$
Then the distance between any two points of A_ι is $< \sigma$. Let ϵ'' be the least distance between non-consecutive arcs C_ι. We take δ so small that we also have
$$\sqrt{2}\,\delta < \frac{\epsilon''}{2}. \qquad (2$$

Then two non-consecutive links A_ι, A_j have no point in common. For then their edges would have a common point P. As P lies on E_ι its distance from C_ι is $\leq \sqrt{2}\,\delta$. Its distance from C_j is also $\leq \sqrt{2}\,\delta$. Thus there is a point P_ι on C_ι, and a point P_j on C_j such that
$$\eta = \overline{P_\iota P_j} \leq 2\sqrt{2}\,\delta.$$

* Here and in the following, intuitional properties of polygons are assumed as known.

But by hypothesis $\epsilon'' \leq \eta$. Hence

$$\epsilon'' \leq 2\sqrt{2}\,\delta,$$

which contradicts 2).

Thus the union of these links form a ring R whose edges are polygons without double point. One of the edges, say G_ι, lies within the other, which we call G_e. The curve C lies within R. The inner polygon G_ι must exist, since non-consecutive links have no point in common.

578. 1. *Interior and Exterior Points.* Let $\sigma_1 > \sigma_2 > \cdots \doteq 0$. Let $R_1, R_2 \cdots$ be the corresponding rings, and let

$$G_\iota' \quad , \quad G_\iota'' \cdots$$

$$G_e' \quad , \quad G_e'' \cdots$$

be their inner and outer edges. A point P of the plane not on C which lies inside some G_ι we call an *interior or inner point of* C. If P lies outside some G_e, we call it an *exterior or outer point of* C.

Each point P not on C must belong to one of these two classes. For let $\rho = \text{Dist}\,(P, C)$; then ρ is $>$ some σ_n. It therefore lies within $G_\iota^{(n)}$ or without $G_e^{(n)}$, and is thus an inner or an outer point. Obviously this definition is independent of the sequence of rings $\{R_n\}$ employed. The points of the curve C are interior to each $G_e^{(n)}$ and exterior to each $G_\iota^{(n)}$.

Inner points must exist, since the inner polygons exist as already observed. Let us denote the inner points by $\mathfrak{I}$ and the outer points by $\mathfrak{O}$. Then the frontiers of $\mathfrak{I}$ and $\mathfrak{O}$ are the curve C.

2. We show now that

1° *Two inner points can be joined by a broken line L_ι lying in $\mathfrak{I}$.*

2° *Two outer points can be joined by a broken line L_e lying in $\mathfrak{O}$.*

3° *Any continuous curve $\mathfrak{K}$ joining an inner point i and an outer point e has a point in common with C.*

To prove 3°, let

$$x = f(t) \quad , \quad y = g(t)$$

be the equations of $\mathfrak{K}$, the variable t ranging over an interval $\tau = (\alpha < \beta)$, $t = \alpha$ corresponding to i and $t = \beta$ to e. Let t' be

such that $\alpha \leq t < t'$ gives inner points, while $t = t'$ does not give an inner point. Thus the point corresponding to $t = t'$ is a frontier point of $\mathfrak{J}$ and hence a point of C.

To prove 1°. If A, B are inner points, they lie within some G_ι. We may join A, B, G_ι by broken lines L_a, L_b meeting G_ι at the points A', B', say. Let G_{ab} be the part of G_ι lying between A', B'. Then

$$L_a + G_{ab} + L_b$$

is a broken line joining A to B.

The proof of 2° is similar.

579. 1. Let P', P'' correspond to $t = t'$, $t = t''$, on the curve C defined by 577, 1). If $t' < t''$, we say P' *precedes* P'' and write $P' < P''$.

Any set of points on C corresponding to an increasing set of values of t is called an *increasing set.*

As t ranges from a to b, the point P ranges over C in a *direct sense.*

We may thus consider a Jordan curve as an ordered set, in the sense of 265.

2. (*De la Vallée-Poussin.*) *On each arc C_ι of the curve C, there exists at least one point P_j such that*

$$P_1 < P_2 < P_3 < \cdots \tag{1}$$

may be regarded as the vertices of a closed polygon without double point and whose sides are all $< \epsilon$.

For in the first place we may take $\delta > 0$ so small that no square of Δ contains a point lying on non-consecutive arcs C_ι of C. Let us also take Δ so that the point α corresponding to $t = a$ lies *within* a square, call it S_1, of Δ. As t increases from $t = a$, there is a last point P_1 on C where the curve leaves S_1. The point P_1 lies in another square of Δ, call it S_2, containing other points of C. Let P_2 be the last point of C in S_2. In this way we may continue, getting a sequence 1).

There exists at least one point of 1) on each arc C_ι. For otherwise a square of Δ would contain points lying on non-consecutive arcs C_κ. The polygon determined by 1) cannot have a double

point, since each side of it lies in one square. The sides are $< \epsilon$, provided we take $\delta\sqrt{2} < \epsilon$, since the diagonal is the longest line we can draw in a square of side δ.

580. *Existence of Inner Points.* To show that the links form a ring with inner points, *Schönfliess** has given a proof which may be rendered as follows:

Let us take the number of links to be even, and call them L_1, $L_2, \cdots L_{2n}$. Then L_1, L_3, $L_5 \cdots$ lie entirely outside each other. Since L_1, L_2 overlap, let P be an inner common point. Similarly let Q be an inner common point of L_2, L_3. Then P, Q lying within L_2 may be joined by a finite broken line b lying within L_2. Let b_2 be that part of it lying between the last point of leaving L_1 and the following point of meeting L_3. In this way the pairs of links

$$L_1 L_3 \quad ; \quad L_3 L_5 \quad ; \cdots$$

define finite broken lines

$$b_2 \quad , \quad b_4 \quad , \quad \cdots b_{2n}.$$

No two of these can have a common point, since they lie in non-consecutive links. The union of the points in the sets

$$L_1 \quad , \quad b_2 \quad , \quad L_3 \quad , \quad b_4 \cdots L_{2n-1} \quad , \quad b_{2n}$$

we call a *ring*, and denote it by $\mathfrak{R}$. The points of the plane not in $\mathfrak{R}$ fall into two parts, *separated* by $\mathfrak{R}$. Let $\mathfrak{T}$ denote the part which is limited, together with its frontier. We call $\mathfrak{T}$ the *interior* of $\mathfrak{R}$. That $\mathfrak{T}$ has *inner* points is regarded as obvious since it is defined by the links

$$L_1 \quad , \quad L_3 \quad , \quad L_5 \cdots$$

which pairwise have no point in common, and by the broken lines

$$b_2 \quad , \quad b_4 \quad , \quad b_6 \cdots$$

each of which latter lies entirely *within* a link.

Let $\qquad \mathfrak{L}_{2m} = Dv\,(L_{2m}, \mathfrak{T}) \quad , \qquad m = 1, 2, \cdots$

* *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten.* Leipzig, 1908, Part 2, p. 170.

Then these $\mathfrak{L}$ have pairwise no point in common since the L_{2m} have not.

Let
$$\mathfrak{T} = \mathfrak{L}_2 + \mathfrak{L}_4 + \cdots + \mathfrak{L}_{2n} + \mathfrak{K}.$$

Then $\mathfrak{K} > 0$. For let us adjoin L_2 to $\mathfrak{R}$, getting a ring $\mathfrak{R}_2$ whose interior call $\mathfrak{T}_2$. That $\mathfrak{T}_2$ has inner points follows from the fact that it contains $\mathfrak{L}_4$, $\mathfrak{L}_6 \cdots$ Let us continue adjoining the links L_4, $L_6 \cdots$ Finally we reach L_{2n}, to which corresponds the ring $\mathfrak{R}_{2n}$, whose interior, if it exists, is $\mathfrak{T}_{2n}$. If $\mathfrak{T}_{2n}$ does not exist, $\mathfrak{T}_{2n-2}$ contains only $\mathfrak{L}_{2n}$. This is not so, for on the edge of L_1 bounding $\mathfrak{T}$, is a point P, such that some $D_\rho(P)$ contains points of no L except L_1. In fact there is a point P on the edge of L_1 not in either L_2 or L_{2n}, as otherwise these would have a point in common. Now, if however small $\rho > 0$ is taken, $D_\rho(P)$ contains points of some L other than L_1, the point P must lie in L_κ which is absurd, since L_1 has only points in common with L_2, L_{2n}, and P is not in either of these. Thus the adjunction of L_2, L_4, $\cdots$ L_{2n} produces a ring $\mathfrak{R}_{2n}$ whose interior $\mathfrak{T}_{2n}$ does not reduce to 0; it has inner points.

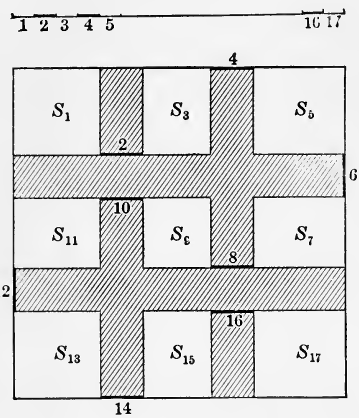
581. *Property* 9°. *Area.* That a figure defined by a closed curve without double point, *i.e.* the interior of a Jordan curve, has an area, has long been an accepted fact in intuitional geometry. Thus Lindemann, *Vorlesungen über Geometrie*, vol. 2, p. 557, says "einer allseitig umgrenzten Figur kommt ein bestimmter Flächeninhalt zu." The truth of such a statement rests of course on the definition of the term area. In I, 487, 702 we have given a definition of area for any limited plane point set $\mathfrak{A}$ which reduces to the ordinary definition when $\mathfrak{A}$ becomes an ordinary plane figure. In our language $\mathfrak{A}$ has an area when its frontier points form a discrete set. Let
$$x = \phi(t) \quad , \quad y = \psi(t)$$
define a Jordan curve $\mathfrak{C}$, as t ranges over $T = (a < b)$. The figure $\mathfrak{A}$ defined by this curve has the curve as frontier. In I, 708, 710, we gave various cases in which $\mathfrak{C}$ is discrete. The reasoning of I, 710, gives us also this important case:

If one of the continuous functions ϕ, ψ defining the Jordan curve $\mathfrak{C}$, has limited variation in T, then $\mathfrak{C}$ is discrete.

It was not known whether $\mathfrak{C}$ would remain discrete if the condition of limited variation was removed from both coördinates, until Osgood * exhibited a Jordan curve which is not discrete. This we will now discuss.

582. 1. *Osgood's Curve.* We start with a unit segment $T = (0, 1)$ on the t axis, and a unit square S in the xy plane.

We divide T into 17 equal parts



$$T_1, T_2, \cdots T_{17}, \qquad (1$$

and the square S into 9 equal squares

$$S_1, S_3, S_5 \cdots S_{17}, \qquad (2$$

by drawing 4 bands B_1 which are shaded in the figure. On these bands we take 8 segments,

$$s_2, s_4, s_6 \cdots s_{16}, \qquad (3$$

marked heavy in the figure.

Then as t is ranging from left to right over the even or black intervals $T_2, T_4, \cdots T_{16}$ marked heavy in the figure, the point x, y on Osgood's curve, call it $\mathfrak{D}$, shall range univariantly over the segments 3).

While t is ranging over the odd or white intervals $T_1, T_3 \cdots T_{17}$ the point xy on $\mathfrak{D}$ shall range over the squares 2) as determined below.

Each of the odd intervals 1) we will now divide into 17 equal intervals T_{ij} and in each of the squares 2) we will construct horizontal and vertical bands B_2 as we did in the original square S. Thus each square 2) gives rise to 8 new segments on $\mathfrak{D}$ corresponding to the new black intervals in T, and 9 new squares S_{ij} corresponding to the white intervals. In this way we may continue indefinitely.

The points which finally get in a black interval call β, the others are limit points of the β's and we call them λ. The point

* *Trans. Am. Math. Soc.*, vol. 4 (1903), p. 107.

on $\mathfrak{O}$ corresponding to a β point has been defined. The point of $\mathfrak{O}$ corresponding to a point λ is defined to be the point lying in the sequence of squares, one inside the other, corresponding to the sequence of white intervals, one inside the other, in which λ falls, in the successive divisions of T.

Thus to each t in T corresponds a single point x, y in S. The aggregate of these points constitutes Osgood's curve. Obviously the x, y of one of its points are one-valued functions of t in T, say

$$x = \phi(t) \quad , \quad y = \psi(t). \tag{4}$$

The curve $\mathfrak{O}$ has no double point. This is obvious for points of $\mathfrak{O}$ lying in black segments. Any other point falls in a sequence of squares

$$S_\iota > S_{\iota j} > S_{\iota j \kappa} \cdots$$

to which correspond intervals

$$T_\iota > T_{\iota j} > T_{\iota j \kappa} \cdots$$

in which the corresponding t's lie. But only one point t is thus determined.

The functions 4) *are continuous.* This is obvious for points β lying within the black intervals of T. It is true for the points λ. For λ lies within a sequence of white intervals, and while t ranges over one of these, the point on $\mathfrak{O}$ ranges in a square. But these squares shut down to a point as the intervals do. Thus ϕ, ψ are continuous at $t = \lambda$. In a similar manner we show they are continuous at the end points of the black intervals.

We note that to $t = 0$ corresponds the upper left-hand corner of S, and to $t = 1$, the diagonally opposite point.

2. Up to the present we have said nothing as to the width of the shaded bands

$$B_1 \quad , \quad B_2 \cdots$$

introduced in the successive steps. Let

$$A = a_1 + a_2 + \cdots$$

be a convergent positive term series whose sum $A \leq 1$. We choose B_1 so that its area is a_1, B_2 so that its area is a_2, etc. Then

$$\underline{\mathfrak{O}} = 0 \quad , \quad \overline{\mathfrak{O}} = 1 - A, \tag{5}$$

as we now show. For $\mathfrak{D}$ has obviously only frontier points; hence $\underline{\mathfrak{D}} = 0$. Since $\mathfrak{D}$ is complete, it is measurable and

$$\widehat{\mathfrak{D}} = \bar{\mathfrak{D}}.$$

Let $O = S - \mathfrak{D}$, and $B = \{B_n\}$. Then $O < B$. For any point which does not lie in some B_n lies in a sequence of convergent squares $S_\iota > S_{\iota j} > \cdots$ which converge to a point of $\mathfrak{D}$. Now

$$\widehat{B} = \widehat{B}_1 + \widehat{B}_2 + \cdots = A.$$

On the other hand, B contains a null set of points of $\mathfrak{D}$, viz. the black segments. Thus

$$\widehat{O} = \widehat{B} = A \quad , \quad \text{and hence } \widehat{\mathfrak{D}} = 1 - A$$

and 5) is established.

Thus *Osgood's curve is continuous, has no double point, and its upper content is* $1 - A$.

3. To get a continuous closed curve C without double point we have merely to join the two end points α, β of Osgood's curve by a broken line which does not cut itself or have a point in common with the square S except of course the end points α, β. Then C bounds a figure $\mathfrak{F}$ whose frontier is not discrete, and $\mathfrak{F}$ does not have an area. Let us call such curves *closed* Osgood curves.

Thus we see that there exist regions bounded by Jordan curves which do not have area in the sense current since the Greek geometers down to the present day.

Suppose, however, we discard this traditional definition, and employ as definition of area its measure. Then we can say:

A figure $\mathfrak{F}$ formed of a closed Jordan curve J and its interior $\mathfrak{J}$ has an area, viz. Meas $\mathfrak{F}$.

For Front $\mathfrak{F} = J$. Hence $\mathfrak{F}$ is complete, and is therefore measureable.

We note that

$$\widehat{\mathfrak{F}} = \widehat{J} + \widehat{\mathfrak{J}}.$$

We have seen there are Jordan curves such that

$$\widehat{J} > 0.$$

We now have a definition of area which is in accordance with the promptings of our geometric intuition. It must be remembered, however, that this definition has been only recently discovered, and that the definition which for centuries has been accepted leads to results which flatly contradict our intuition, which leads us to say that a figure bounded by a continuous closed curve has an area.

583. At this point we will break off our discussion of the relation between our intuitional notion of a curve, and the configuration determined by the equations

$$x = \phi(t) \quad , \quad y = \psi(t) \qquad (1$$

where ϕ, ψ are one-valued continuous functions of t in an interval T. Let us look back at the list of properties of an intuitional curve drawn up in 563. We have seen that the analytic curve 1) does not need to have tangents at a pantactic set of points on it; no arc on it needs have a finite length; it may completely fill the interior of a square; its equations cannot always be brought in the forms $y = f(x)$ or $F(xy) = 0$, if we restrict ourselves to functions f or F employed in analysis up to the present; it does not need to have an area as that term is ordinarily understood.

On the other hand, it is continuous, and when closed and without double point it forms the complete boundary of a region.

Enough in any case has been said to justify the thesis that geometric reasoning in analysis must be used with the greatest circumspection.

Detached and Connected Sets

584. In the foregoing sections we have studied in detail some of the properties of curves defined by the equations

$$x = \phi(t) \quad , \quad y = \psi(t).$$

Now the notion of a curve, like many other geometric notions, is independent of an analytic representation. We wish in the following sections to consider some of these notions from this point of view.

585. 1. Let $\mathfrak{A}$, $\mathfrak{B}$ be point sets in m-way space $\mathfrak{R}_m$. If

$$\text{Dist}(\mathfrak{A}, \mathfrak{B}) > 0,$$

we say $\mathfrak{A}$, $\mathfrak{B}$ are *detached*. If $\mathfrak{A}$ cannot be split up into two parts $\mathfrak{B}$, $\mathfrak{C}$ such that they are detached, we say $\mathfrak{A}$ has *no detached parts*. If $\mathfrak{A} = \mathfrak{B} + \mathfrak{C}$ and $\text{Dist}(\mathfrak{B}, \mathfrak{C}) > 0$, we say $\mathfrak{B}$, $\mathfrak{C}$ are *detached parts* of $\mathfrak{A}$.

Let the set of points, finite or infinite,

$$a, a_1, a_2, \cdots b \tag{1}$$

be such that the distance between two successive ones is $\leq \epsilon$. We call 1) an ϵ-sequence between a, b; or a sequence with segments $(a_\iota, a_{\iota+1})$ of length $\leq \epsilon$. We suppose the segments ordered so that we can pass continuously from a to b over the segments without retracing. If 1) is a finite set, the sequence is *finite*, otherwise *infinite*.

2. *Let $\mathfrak{A}$ have no detached parts. Let a, b be two of its points. For each $\epsilon > 0$, there exists a finite ϵ-sequence between a, b, and lying in $\mathfrak{A}$.*

For about a describe a sphere of radius ϵ. About each point of $\mathfrak{A}$ in this sphere describe a sphere of radius ϵ. About each point of $\mathfrak{A}$ in each of these spheres describe a sphere of radius ϵ. Let this process be repeated indefinitely. Let $\mathfrak{B}$ denote the points of $\mathfrak{A}$ made use of in this procedure. If $\mathfrak{B} < \mathfrak{A}$, let $\mathfrak{C} = \mathfrak{A} - \mathfrak{B}$. Then $\text{Dist}(\mathfrak{B}, \mathfrak{C}) > \epsilon$, and $\mathfrak{A}$ has detached parts, which is contrary to hypothesis. Thus there are sets of ϵ-spheres in $\mathfrak{A}$ joining a and b.

Among these sets there are finite ones. For let $\mathfrak{F}$ denote the set of points in $\mathfrak{A}$ which may be joined to a by finite sequences; let $\mathfrak{G} = \mathfrak{A} - \mathfrak{F}$. Then $\text{Dist}(\mathfrak{F}, \mathfrak{G}) \geq \epsilon$. For if $< \epsilon$, there is a point f in $\mathfrak{F}$, and a point g in $\mathfrak{G}$ whose distance is $< \epsilon$. Then a and g can be joined by a finite ϵ-sequence, which is contrary to hypothesis.

3. *If $\mathfrak{A}$ has no detached parts, it is dense.*

For if not dense, it must have at least one isolated point a. But then a, and $\mathfrak{A} - a$ are detached parts of $\mathfrak{A}$, which contradicts the hypothesis.

4. *Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be complete and $\mathfrak{A} = (\mathfrak{B}, \mathfrak{C})$. If $\mathfrak{A}$ has no detached parts, $\mathfrak{B}$, $\mathfrak{C}$ have at least one common point.*

For if $\mathfrak{B}$, $\mathfrak{C}$ have no common point, $\delta = \text{Dist}(\mathfrak{B}, \mathfrak{C})$ is > 0. But δ cannot > 0, since $\mathfrak{B}$, $\mathfrak{C}$ would then be detached parts of $\mathfrak{A}$. Since $\delta = 0$ and since $\mathfrak{B}$, $\mathfrak{C}$ are complete, they have a point in common.

5. *If $\mathfrak{A}$ is such that any two of its points may be joined by an ϵ-sequence lying in $\mathfrak{A}$, where $\epsilon > 0$ is small at pleasure, $\mathfrak{A}$ has no detached parts.*

For if $\mathfrak{A}$ had $\mathfrak{B}$, $\mathfrak{C}$ as detached parts, let $\text{Dist}(\mathfrak{B}, \mathfrak{C}) = \delta$. Then $\delta > 0$. Hence there is no sequence joining a point of $\mathfrak{B}$ with a point of $\mathfrak{C}$ with segments $< \delta$.

6. If $\mathfrak{A}$ is complete and has no detached parts, it is said to be *connected*. We also call $\mathfrak{A}$ a *connex*.

As a special case, a point may be regarded as a connex.

7. *If $\mathfrak{A}$ is connected, it is perfect.*

For by 3 it is dense, and by definition it is complete.

8. *If $\mathfrak{A}$ is a rectilinear connex, it has a first point α and a last point β, and contains every point in the interval (α, β).*

For being limited and complete its minimum and maximum lie in $\mathfrak{A}$ and these are respectively α and β. Let now

$$\epsilon_1 > \epsilon_2 > \cdots \doteq 0.$$

There exists an ϵ_1-sequence C_1 between α, β. Each segment has an ϵ_2-sequence C_2. Each segment of C_2 has an ϵ_3-sequence C_3, etc. Let C be the union of all these sequences. It is pantactic in (α, β). As $\mathfrak{A}$ is complete,

$$\mathfrak{A} = (\alpha, \beta).$$

Images

586. Let
$$x_1 = f_1(t_1 \cdots t_m) \quad \cdots \quad x_n = f_n(t_1 \cdots t_m) \tag{1}$$
be one-valued functions of t in the point set $\mathfrak{T}$. As t ranges over $\mathfrak{T}$, the point $x = (x_1 \cdots x_n)$ will range over a set $\mathfrak{A}$ in an n-way space $\mathfrak{R}_n$. We have called $\mathfrak{A}$ the image of $\mathfrak{T}$. Cf. I, 238, 3. p 146
If the functions f are not one-valued, to a point t may correspond several images x', x'' $\cdots$ finite or infinite in number. Conversely

to the point x may correspond several values of t. If to each point t correspond in general r values of x, and to each x in general s values of t, we say the correspondence between $\mathfrak{T}$, $\mathfrak{A}$ is r to s. If $r = s = 1$ the correspondence is 1 to 1 or *unifold*; if $r > 1$, it is *manifold*. If $r = 1$, $\mathfrak{A}$ is a *simple* image of $\mathfrak{T}$, otherwise it is a *multiple* image. If the functions 1) are one-valued and continuous in $\mathfrak{T}$, we say $\mathfrak{A}$ is a *continuous image* of $\mathfrak{T}$.

587. *Transformations of the Plane. Example 1.* Let

$$u = x \sin y \quad , \quad v = x \cos y. \tag{1}$$

We have in the first place

$$u^2 + v^2 = x^2.$$

This shows that the image of a line $x = a$, $a \neq 0$, parallel to the y-axis is a circle whose center is the origin in the u, v plane, and whose radius is a. To the y-axis in the x, y plane corresponds the origin in the u, v plane.

From 1) we have, secondly,

$$\frac{u}{v} = \tan y.$$

This shows that the image of a line $y = b$, is a line through the origin in the u, v plane.

From 1) we have finally that u, v are periodic in y, having the period 2π. Thus as x, y ranges in the band B, formed by the two parallels $y = \pm \pi$, or $-\pi < y \leq \pi$, the point u, v ranges over the entire u, v plane once and once only.

The correspondence between B and the u, v plane is unifold, except, as is obvious, to the origin in the u, v plane corresponds the points on the y-axis.

Let us apply the theorem of I, 441, on implicit functions. The determinant Δ is here

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \sin y, & \cos y \\ x \cos y, & -x \sin y \end{vmatrix} = -x.$$

As this is $\neq 0$ when x, y is not on the y-axis, we see that the correspondence between the *domain* of any such point and its image is 1 to 1. This accords with what we have found above.

It is, however, a much more restricted result than we have found; for we have seen that the correspondence between any limited point set $\mathfrak{A}$ in B which does not contain a point of the y-axis and its image is unifold.

588. *Example 2.* Let

$$u = \frac{y}{\sqrt{x^2 + y^2}} \quad , \quad v = \sqrt{x^2 + y^2}, \tag{1}$$

the radical having the positive sign. Let us find the image of the first quadrant Q in the x, y plane.

From 1) we have at once

$$0 \leq u \leq 1 \quad , \quad v \geq 0.$$

Hence the image of Q is a band B parallel to the v-axis.

From 1) we get secondly

$$y = uv \quad , \quad x = v\sqrt{1 - u^2}. \tag{2}$$

Hence

$$x^2 + y^2 = v^2.$$

Thus the image of a circle in Q whose center is the origin and whose radius is a is a segment of a right line $v = a$.

When $x = y = 0$, the equations 1) do not define the corresponding point in the u, v plane. If we use 2) to define the correspondence, we may say that to the line $v = 0$ in B corresponds the origin in the x, y plane. With this exception the correspondence between Q and B is uniform, as 1), 2) show.

The determinant Δ of 1) is, setting

$$r = \sqrt{x^2 + y^2},$$

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{-xy}{r^3} & \dfrac{x^2}{r^3} \\ \dfrac{x}{r} & \dfrac{y}{r} \end{vmatrix} = \frac{-x}{x^2 + y^2}$$

for any point x, y different from the origin.

589. *Example 3. Reciprocal Radii.* Let O be the origin in the x, y plane and Ω the origin in the u, v plane. To any point $P = (x, y)$ in the x, y plane different from the origin shall correspond a point $Q = (u, v)$ in the u, v plane such that ΩQ has

the same direction as OP and such that $OP \cdot \Omega Q = 1$. Analytically we have

$$x = \lambda y \quad , \quad u = \lambda v \quad , \quad \lambda > 0,$$

and

$$(u^2 + v^2)(x^2 + y^2) = 1.$$

From these equations we get

$$x = \frac{u}{u^2 + v^2} \quad , \quad y = \frac{v}{u^2 + v^2} \tag{1}$$

and also

$$u = \frac{x}{x^2 + y^2} \quad , \quad v = \frac{y}{x^2 + y^2}.$$

The correspondence between the two planes is obviously unifold except that no point in either plane corresponds to the origin in the other plane. We find for any point x, y different from the origin that

$$\Delta = \frac{\partial (u, v)}{\partial (x, y)} = -\frac{1}{(x^2 + y^2)^2}.$$

Obviously from the definition, to a line through the origin in the x, y plane corresponds a similar line in the u, v plane. As xy moves toward the origin, u, v moves toward infinity.

Let x, y move on the line $x = a \neq 0$. Then 1) shows that u, v moves along the circle

$$a(u^2 + v^2) - u = 0$$

which passes through the origin. A similar remark holds when x, y moves along the line $y = b \neq 0$.

590. Such relations between two point sets $\mathfrak{A}, \mathfrak{B}$ as defined in 586 may be formulated independently of the functions f. In fact with each point a of $\mathfrak{A}$ we may associate one or more points $b_1, b_2 \cdots$ of $\mathfrak{B}$ according to some law. Then $\mathfrak{B}$ may be regarded as the image of $\mathfrak{A}$. We may now define the terms simple, manifold, etc., as in 586. When b corresponds to a we may write $b \sim a$.

We shall call $\mathfrak{B}$ a *continuous image of* $\mathfrak{A}$ when the following conditions are satisfied. 1° To each a in $\mathfrak{A}$ shall correspond but one b in $\mathfrak{B}$, that is, $\mathfrak{B}$ is a simple image of $\mathfrak{A}$. 2° Let $b \sim a$, let $a_1, a_2 \cdots$ be any sequence of points in $\mathfrak{A}$ which $\doteq a$. Let $b_n \sim a_n$. Then b_n must $\doteq b$ however the sequence $\{a_n\}$ is chosen.

When $\mathfrak{B}$ is a simple image of $\mathfrak{A}$, the law which determines which b of $\mathfrak{B}$ is associated with a point a of $\mathfrak{A}$ determines obviously n one-valued functions as in 586, 1), where $t_1 \cdots t_m$ are the m coördinates of a, and $x_1 \cdots x_n$ are the n coördinates of b. We call these functions 1) the *associated functions*. Obviously when $\mathfrak{B}$ is a Def
continuous image, the associated functions are continuous in $\mathfrak{A}$.

591. 1. *Let $\mathfrak{B}$ be a simple continuous image of the limited complete set $\mathfrak{A}$. Then* 1° *$\mathfrak{B}$ is limited and complete. If* 2° *$\mathfrak{A}$ is perfect and only a finite number of points of $\mathfrak{A}$ correspond to any point of $\mathfrak{B}$, then $\mathfrak{B}$ is perfect. If* 3° *$\mathfrak{A}$ is a connex, so is $\mathfrak{B}$.*

To prove 1°. The case that $\mathfrak{B}$ is finite requires no proof. Let $b_1, b_2 \cdots$ be points of $\mathfrak{B}$ which $\doteq \beta$. We wish to show that β lies in $\mathfrak{B}$. To each b_n will correspond one or more points in $\mathfrak{A}$; call the union of all these points $\mathfrak{a}$. Since $\mathfrak{B}$ is a simple image, $\mathfrak{a}$ is an infinite set. Let $a_1, a_2 \cdots$ be a set of points in $\mathfrak{a}$ which $\doteq \alpha$, a limiting point of $\mathfrak{A}$. As $\mathfrak{A}$ is complete, α lies in $\mathfrak{A}$. Let $b \sim \alpha$. Let $b_{\iota_n} \sim a_n$. As $a_n \doteq \alpha$, $b_{\iota_n} \doteq \beta$. But $\mathfrak{B}$ being continuous, b_{ι_n} must $\doteq b$. Thus β lies in $\mathfrak{B}$. That $\mathfrak{B}$ is limited follows from the fact that the associated functions are continuous in the limited complete set $\mathfrak{A}$. *To prove* 2°. Suppose that $\mathfrak{B}$ had an isolated point b. Let $b \sim a$. Since $\mathfrak{A}$ is perfect, let $a_1, a_2 \cdots \doteq a$. Let $b_n \sim a_n$. Then as $\mathfrak{B}$ is continuous, $b_n \doteq b$, and b is not an isolated point. *To prove* 3°. We have only to show that there exists an ϵ-sequence between any two points α, β of $\mathfrak{B}$, ϵ small at pleasure. Let $\alpha \sim a$, $\beta \sim b$. Since $\mathfrak{A}$ is connected there exists an η-sequence between a, b. Also the associated functions are uniformly continuous in $\mathfrak{A}$, and hence η may be taken so small that each segment of the corresponding sequence in $\mathfrak{B}$ is $\geq \epsilon$.

2. *Let $f(t_1 \cdots t_m)$ be one-valued and continuous in the connex $\mathfrak{A}$, then the image of $\mathfrak{A}$ is an interval including its end points.*

This follows from the above and from 585, 8.

3. *Let the correspondence between $\mathfrak{A}, \mathfrak{B}$ be unifold. If $\mathfrak{B}$ is a continuous image of $\mathfrak{A}$, then $\mathfrak{A}$ is a continuous image of $\mathfrak{B}$.*

For let $\{b_n\}$ be a set of points in $\mathfrak{B}$ which $\doteq b$. Let $a_n \sim b_n$, $a \sim b$. We have only to show that $a_n \doteq a$. For suppose that it does not, suppose in fact that there is a sequence $a_{\iota_1}, a_{\iota_2} \cdots$ which

$\doteq \alpha \neq a$. Let $\beta \sim \alpha$. Then $b_{\iota_1}, b_{\iota_2} \cdots \doteq \beta$. But any partial sequence of $\{b_n\}$ must $\doteq b$. Thus $b = \beta$, hence $a = \alpha$, hence $a_n \doteq a$.

4. *A Jordan curve J is a unifold continuous image of an interval T. Conversely if J is a unifold continuous image of an interval T, there exist two one-valued continuous functions*

$$x = \phi(t) \quad , \quad y = \psi(t)$$

such that as t ranges over T, the point x, y ranges over J. In case J is closed it may be regarded as the image of a circle Γ.

All but the last part of the theorem has been already established. To prove the last sentence we have only to remark that if we set

$$x = r \cos t \quad , \quad y = r \sin t$$

we have a unifold continuous correspondence between the points of the interval $(0, 2\pi^*)$ and the points of a circle.

5. The first part of 4 may be regarded as a *geometrical definition* of a Jordan curve. The image of a segment of the interval T or of the circle Γ, will be called an *arc of J.*

592. *Side Lights on Jordan Curves.* These curves have been defined by means of the equations

$$x = \phi(t), \qquad y = \psi(t). \tag{1}$$

As t ranges over the interval $T = (a < b)$, the point $P = (x, y)$ ranges over the curve J. This curve is a certain point set in the x, y plane. We may now propose this problem: We have given a point set $\mathfrak{C}$ in the x, y plane; may it be regarded as a Jordan curve? That is, do there exist two continuous one-valued functions 1) such that as t ranges over some interval T, the point P ranges over the given set $\mathfrak{C}$ without returning on itself, except possibly for $t = a$, $t = b$, when the curve would be closed?

Let us look at a number of point sets from this point of view.

593. *Example 1.*

1. Let
$$y = \sin \frac{1}{x} \quad , \quad x \text{ in the interval } \mathfrak{A} = (-1, 1), \text{ but } \neq 0$$
$$= 0 \quad , \quad \text{for } x = 0.$$

Is this point set $\mathfrak{C}$ a Jordan curve? The answer is, No. For a Jordan curve is a continuous image of an interval $\mathfrak{A}$. By 591, 1, it is complete. But $\mathfrak{C}$ is not complete, as all the points on the y axis, $-1 \leq y \leq 1$ are limiting points of $\mathfrak{C}$, and only one of them belongs to $\mathfrak{C}$, viz. the origin.

2. Let us modify $\mathfrak{C}$ by adjoining to it all these missing limiting points, and call the resulting point set C. Is C a Jordan curve? The answer is again, No. For if it were, we can divide the interval T into intervals δ so small that the oscillation of ϕ, ψ in any one of them is $< \omega$. To the intervals δ_ι will correspond arcs C_ι on the curve, and two non-consecutive arcs C_ι are distant from each other by an amount $>$ some ϵ, small at pleasure. This shows that one of these arcs, say C_κ, must contain the segment on the y-axis $-1 \leq y \leq 1$. But then Osc $\psi = 2$ as t ranges over the corresponding δ_κ interval. Thus the oscillation of ψ cannot be made $< \epsilon$, however small δ_κ is taken.

3. Let us return to the set $\mathfrak{C}$ defined in 1. Let A, B be the two end points corresponding to $x = -1$, $x = 1$. Let us join them by an ordinary curve, a polygon if we please, which does not cut itself or $\mathfrak{C}$. The resulting point set $\mathfrak{K}$ divides all the other points of the plane into two parts which cannot be joined by a continuous curve without crossing $\mathfrak{K}$. For this point of view $\mathfrak{K}$ must be regarded as a *closed* configuration. Yet $\mathfrak{K}$ is obviously not complete.

On the other hand, let us look at the curve formed by removing the points on a circle between two given points a, b on it. The remaining arc $\mathfrak{L}$ including the end points a, b is a *complete* set, but as it does not divide the other points of the plane into two separated parts, we cannot say $\mathfrak{L}$ is a closed configuration.

We mention this circumstance because many English writers use the term closed set where we have used the term complete. Cantor, who first introduced this notion, called such sets *abgeschlossen*, which is quite different from *geschlossen* = closed.

594. *Example 2.* Let $\rho = e^{-\frac{1}{\theta}}$, for θ in the interval $\mathfrak{A} = (0, 1)$ except $\theta = 0$, where $\rho = 0$. These polar coördinates may easily be replaced by Cartesian coördinates

$$x = \phi(\theta) = e^{-\frac{1}{\theta}} \cos\theta \quad , \quad y = e^{-\frac{1}{\theta}} \sin\theta \quad , \quad \text{in } \mathfrak{A},$$

except $\theta = 0$, when x, y both $= 0$. The curve thus defined is a Jordan curve.

Let us take a second Jordan curve

$$\rho = e^{-\left(\pi + \frac{1}{\theta}\right)},$$

with $\rho = 0$ for $\theta = 0$. If we join the two end points on these curves corresponding to $\theta = 1$ by a straight line, we get a closed Jordan curve J, which has an interior $\mathfrak{J}$, and an exterior $\mathfrak{O}$.

The peculiarity of this curve J is the fact that one point of it, viz. the origin $x = y = 0$, cannot be joined to an arbitrary point of $\mathfrak{J}$ by a finite broken line lying entirely in $\mathfrak{J}$; nor can it be joined to an arbitrary point in $\mathfrak{O}$ by such a line lying in $\mathfrak{O}$.

595. 1. It will be convenient to introduce the following terms.

Let $\mathfrak{A}$ be a limited or unlimited point set in the plane. A set of distinct points in $\mathfrak{A}$

$$a_1 \quad , \quad a_2 \quad , \quad a_3 \cdots \tag{1}$$

determine a broken line. In case 1) is an infinite sequence, let a_n converge to a fixed point. If this line has no double point, we call it a *chain*, and the segments of the line *links*. In case not only the points 1) but also the links lie in $\mathfrak{A}$, we call the chain a *path*. If the chain or path has but a finite number of links, it is called *finite*.

Def.

Let us call a *precinct* a region, *i.e.* a set all of whose points are inner points, limited or unlimited, such than any two of its points may be joined by a finite path.

2. Using the results of 578, we may say that, —

A closed Jordan curve J divides the other points of the plane into two precincts, an inner $\mathfrak{J}$ and an outer $\mathfrak{O}$. Moreover, they have a common frontier which is J.

3. The closed Jordan curve considered in 594 shows that not every point of such a closed Jordan curve can always be joined to an arbitrary point of $\mathfrak{J}$ or $\mathfrak{O}$ by a finite path.

Obviously it can by an infinite path. For about this point, call it P, we can describe a sequence of circles of radii $r \doteq 0$. Between any two of these circles there lie points of $\mathfrak{J}$ and of $\mathfrak{O}$, if r is suf-

ficiently small. In this way we may get a sequence of points in $\mathfrak{J}$, *viz.* $I_1, I_2 \cdots \doteq P$. Any two of these I_m, I_{m+1} may be joined by a path which does not cut the path joining I_1 to I_m. For if a loop were formed, it could be omitted.

4. *Any arc $\mathfrak{K}$ of a closed Jordan curve J can be joined by a path to an arbitrary point of the interior or exterior, which call $\mathfrak{A}$.*

For let $J = \mathfrak{K} + \mathfrak{L}$. Let k be a point of $\mathfrak{K}$ not an end point. Let $\delta = \text{Dist}(k, \mathfrak{L})$, let a be a point of $\mathfrak{A}$ such that $\text{Dist}(a, k) < \frac{1}{2}\delta$. Then

$$\eta = \text{Dist}(a, \mathfrak{L}) > \tfrac{1}{2}\delta.$$

Hence the link $l = (a, k)$ has no point in common with $\mathfrak{L}$. Let b be the first point of l in common with $\mathfrak{K}$. Then the link $m = (a, b)$ lies in $\mathfrak{A}$. If now α is any point of $\mathfrak{A}$, it may be joined to a by a path p. Then $p + m$ is a path in $\mathfrak{A}$ joining the arbitrary point α to a point b on the arc $\mathfrak{K}$.

596. *Example 3.* For θ in $\mathfrak{A} = (0^*, 1)$ let

$$\rho = a(1 + e^{-\frac{1}{\theta}}),$$

and

$$\rho = a(1 + e^{-\left(\pi + \frac{1}{\theta}\right)}).$$

These equations in polar coördinates define two non-intersecting spirals S_1, S_2 which coil about $\rho = a$ as an asymptotic circle Γ. Let us join the end points of the spirals corresponding to $\theta = 1$ by a straight line L. Let $\mathfrak{C}$ denote the figure formed by the spirals S_1, S_2, the segment L and the asymptotic circle Γ. Is $\mathfrak{C}$ a closed Jordan curve? The answer is, No. This may be seen in many ways. For example, $\mathfrak{C}$ does not divide the other points into two precincts, but into three, one of which is formed of points within Γ.

Another way is to employ the reasoning of 593, 2. Here the circle Γ takes the place of the segment on the y-axis which figures there.

Still another way is to observe that no point on Γ can be joined to a point within $\mathfrak{C}$ by a path.

597. *Example 4.* Let $\mathfrak{C}$ be formed of the edge $\mathfrak{E}$ of a unit square, together with the ordinates $\mathfrak{o}$ erected at the points

$x = \frac{m}{2^n}$, of length $\frac{1}{2^n}$, $n = 1, 2 \cdots$ Although $\mathfrak{C}$ divides the other points of the plane into two precincts $\mathfrak{J}$ and $\mathfrak{D}$, we can say that $\mathfrak{C}$ is not a closed Jordan curve.

For if it were, $\mathfrak{J}$ and $\mathfrak{D}$ would have to have $\mathfrak{C}$ as a common frontier. But the frontier of $\mathfrak{D}$ is $\mathfrak{C}$, while that of $\mathfrak{J}$ is $\mathfrak{C}$.

That $\mathfrak{C}$ is not a Jordan curve is seen in other ways. For example, let γ be an inner segment of one of the ordinates $\mathfrak{o}$. Obviously it cannot be reached by a path in $\mathfrak{D}$.

Brouwer's Proof of Jordan's Theorem

598. We have already given one proof of this theorem in 577 seq., based on the fact that the coördinates of the closed curve are expressed as one-valued continuous functions

$$x = \phi(t) \quad , \quad y = \psi(t).$$

Brouwer's proof * is entirely geometrical in nature and rests on the definition of a closed Jordan curve as the unifold continuous image of a circle, cf. 591, 5.

If $\mathfrak{A}, \mathfrak{B}, \cdots$ are point sets in the plane, it will be convenient to denote their frontiers by $\mathfrak{F}_{\mathfrak{A}}, \mathfrak{F}_{\mathfrak{B}} \cdots$ so that

$$\mathfrak{F}_{\mathfrak{A}} = \text{Front } \mathfrak{A} \quad , \quad \text{etc.}$$

We admit that any closed polygon $\mathfrak{P}$ having a finite number of sides, without double point, divides the other points of the plane into an inner and an outer precinct $\mathfrak{P}_\iota, \mathfrak{P}_e$ respectively. In the following sections we shall call such a polygon *simple*, and usually denote it by $\mathfrak{P}$.

We shall denote the whole plane by $\mathfrak{E}$.

Then

$$\mathfrak{E} = \mathfrak{P} + \mathfrak{P}_e + \mathfrak{P}_\iota.$$

Let $\mathfrak{A}$ be complete. The complementary set A is formed, as we saw in 328, of an enumerable set of precincts, say $A = \{A_n\}$.

* *Math. Annalen*, vol. 69 (1910), p. 169.

599. 1. *If a precinct $\mathfrak{A}$ and its complement* A each contain a point of the connex $\mathfrak{C}$, then $\mathfrak{F}_{\mathfrak{A}}$ contains a point of $\mathfrak{C}$.*

For in the contrary case $\mathfrak{c} = Dv(\mathfrak{A}, \mathfrak{C})$ is complete. In fact $\mathfrak{B} = \mathfrak{A} + \mathfrak{F}_{\mathfrak{A}}$ is complete. As $\mathfrak{C}$ is complete, $Dv(\mathfrak{B}, \mathfrak{C})$ is complete. But if $\mathfrak{F}_{\mathfrak{A}}$ does not contain a point of $\mathfrak{C}$, $\mathfrak{c} = Dv(\mathfrak{B}, \mathfrak{C})$. Thus on this hypothesis, $\mathfrak{c}$ is complete. Now $c = Dv(A, \mathfrak{C})$ is complete in any case. Thus $\mathfrak{C} = \mathfrak{c} + c$, which contradicts 585, 4.

2. *If $\mathfrak{P}_\iota$, $\mathfrak{P}_e$, the interior and exterior of a simple polygon $\mathfrak{P}$ each contain a point of a connex $\mathfrak{C}$, then $\mathfrak{P}$ contains a point of $\mathfrak{C}$.*

3. *Let $\mathfrak{K}$ be complete and not connected. There exists a simple polygon $\mathfrak{P}$ such that no point of $\mathfrak{K}$ lies on $\mathfrak{P}$, while a part of $\mathfrak{K}$ lies in $\mathfrak{P}_\iota$ and another part in $\mathfrak{P}_e$.*

For let $\mathfrak{K}_1$, $\mathfrak{K}_2$ be two non-connected parts of $\mathfrak{K}$ whose distance from each other is $\rho > 0$. Let Δ be a quadrate division of the plane of norm δ, so small that no cell contains a point of $\mathfrak{K}_1$ and $\mathfrak{K}_2$. Let Δ_1 denote the cells of Δ containing points of $\mathfrak{K}_1$. It is complete, and the complementary set $\Delta_2 = \mathfrak{E} \dot{-} \Delta_1$ is formed of one or more precincts. No point of $\mathfrak{K}_1$ lies in Δ_2 or on its frontier.

Let P_1, P_2 be points in $\mathfrak{K}_1$, $\mathfrak{K}_2$ respectively. Let D be that precinct containing P_2. Then $\mathfrak{F}_D$ embraces a simple polygon $\mathfrak{P}$ which separates P_1 and P_2.

4. *Let $\mathfrak{K}_1$, $\mathfrak{K}_2$ be two detached connexes. There exists a simple polygon $\mathfrak{P}$ which separates them. One of them is in $\mathfrak{P}_\iota$, the other in $\mathfrak{P}_e$, and no point of either connex lies on $\mathfrak{P}$.*

For the previous theorem shows that there is a simple polygon $\mathfrak{P}$ which separates a point P_1 in $\mathfrak{K}_1$ from a point P_2 in $\mathfrak{K}_2$ and no point of $\mathfrak{K}_1$ or $\mathfrak{K}_2$ lies on $\mathfrak{P}$. Call this fact F.

Let now P_1 lie in $\mathfrak{P}_\iota$. Then every point of $\mathfrak{K}_1$ lies in $\mathfrak{P}_\iota$. For otherwise $\mathfrak{P}_\iota$ and $\mathfrak{P}_e$ each contain a point of the connex $\mathfrak{K}_1$. Then 2 shows that a point of $\mathfrak{K}_1$ lies on $\mathfrak{P}$, which contradicts F.

5. *Let $\mathfrak{B}$ be a precinct determined by the connex $\mathfrak{C}$. Then* $\mathfrak{b} = \text{Front } \mathfrak{B}$ *is a connex.*

* Since the initial sets are all limited, their complements may be taken with reference to a sufficiently large square $\mathfrak{Q}$; and when dealing with frontier points, points on the edge of $\mathfrak{Q}$ may be neglected.

For suppose $\mathfrak{b}$ is not a connex. Then by 3, there exists a simple polygon $\mathfrak{P}$ which contains a part of $\mathfrak{b}$ in $\mathfrak{P}_\iota$ and another in $\mathfrak{P}_e$, while no point of $\mathfrak{b}$ lies on $\mathfrak{P}$. Hence a point β' of $\mathfrak{b}$ lies in $\mathfrak{P}_\iota$, and another point β'' in $\mathfrak{P}_e$. As $\mathfrak{B}$ is a precinct, let us join β', β'' by a path v in $\mathfrak{B}$. Thus $\mathfrak{P}$ contains at least one point of v, that is, a point of $\mathfrak{B}$ lies on $\mathfrak{P}$. As $\mathfrak{b}$ and $\mathfrak{P}$ have no point in common, and as one point of $\mathfrak{P}$ lies in $\mathfrak{B}$, all the points of $\mathfrak{P}$ lie in $\mathfrak{B}$. Hence

$$Dv(\mathfrak{P}, \mathfrak{C}) = 0. \tag{1}$$

As $\mathfrak{b}$ is a part of $\mathfrak{C}$ and hence some of the points of $\mathfrak{C}$ are in $\mathfrak{P}_e$ and some in $\mathfrak{P}_\iota$, it follows from 2 that a part of $\mathfrak{P}$ lies in $\mathfrak{C}$. This contradicts 1).

6. Let $\mathfrak{K}_1$, $\mathfrak{K}_2$ be two connexes without double point. By 3 there exists a simple polygon $\mathfrak{P}$ which separates them and has one connex inside, the other outside $\mathfrak{P}$.

Now $\mathfrak{K} = \mathfrak{K}_1 + \mathfrak{K}_2$ is complete and defines one or more precincts. *One of these precincts contains* $\mathfrak{P}$.

For say $\mathfrak{P}$ lay in two of these precincts as $\mathfrak{A}$ and $\mathfrak{B}$. Then the precinct $\mathfrak{A}$ and its complement (in which $\mathfrak{B}$ lies) each contain a point of the connex $\mathfrak{P}$. Thus $\mathfrak{F}_\mathfrak{A}$ contains a point of $\mathfrak{P}$. But $\mathfrak{F}_\mathfrak{A}$ is a part of $\mathfrak{K}$, and no point of $\mathfrak{K}$ lies on $\mathfrak{P}$.

That precinct in Comp $\mathfrak{K}$ which contains $\mathfrak{P}$ we call the *intermediate precinct* determined by $\mathfrak{K}_1$, $\mathfrak{K}_2$, or more shortly the precinct between $\mathfrak{K}_1$, $\mathfrak{K}_2$ and denote it by Inter $(\mathfrak{K}_1, \mathfrak{K}_2)$.

7. *Let* $\mathfrak{K}_1$, $\mathfrak{K}_2$ *be two detached connexes, and let* $\mathfrak{k} =$ Inter $(\mathfrak{K}_1, \mathfrak{K}_2)$. *Then* $\mathfrak{K}_1$, $\mathfrak{K}_2$ *can be joined by a path lying in* $\mathfrak{k}$, *except its end points which lie on the frontiers of* $\mathfrak{K}_1$, $\mathfrak{K}_2$ *respectively.*

For by hypothesis $\rho = \text{Dist}(\mathfrak{K}_1, \mathfrak{K}_2) > 0$. Let P_1 be a point of $\mathfrak{F}_{\mathfrak{K}_1}$ such that some domain $\mathfrak{d}$ of P_1 contains only points of $\mathfrak{K}_1$ and of $\mathfrak{k}$. Let Q_1 be a point of $\mathfrak{k}$ in $\mathfrak{d}$. Join P_1, Q_1 by a right line, let it cut $\mathfrak{F}_{\mathfrak{K}_1}$ first at the point P'. In a similar way we may reason on $\mathfrak{K}_2$, obtaining the points P'', Q_2. Then $P'Q_1Q_2P''$ is the path in question. If we denote it by v, we may let v^* denote this path after removing its two end points.

8. *Let* $\mathfrak{K}_1$, $\mathfrak{K}_2$ *be two detached connexes. A path* v *joining* $\mathfrak{K}_1$, $\mathfrak{K}_2$ *and lying in* $\mathfrak{k} =$ Inter$(\mathfrak{K}_1, \mathfrak{K}_2)$, *end points excepted, determines one and only one precinct in* $\mathfrak{k}$.

For from an arbitrary point P in $\mathfrak{k}$, let us draw all possible paths to v. Those paths ending on the same side (left or right) of v certainly lie in one and the same precinct $\mathfrak{k}_r$ or $\mathfrak{k}_l$ in $\mathfrak{k}$. Then since one end point of v is inside, the other end point outside $\mathfrak{P}$, there must be a part of $\mathfrak{P}$ which is not met by v and which joins the right and left sides of v. We take this as an evident property of finite broken lines and polygons without double points.

Thus $\mathfrak{k}_l$ and $\mathfrak{k}_r$ are not detached; they are parts of one precinct.

9. *Two paths v_1, v_2 without common point, lying in $\mathfrak{k}$ and joining $\mathfrak{K}_1$, $\mathfrak{K}_2$, split $\mathfrak{k}$ into two precincts.*

Let $\mathfrak{i} = \mathfrak{k} - v_1$; this we have just seen is a precinct. From any point of it let us draw paths to v_2. Those paths ending on the same side of v_2 determine precincts $\mathfrak{i}_l$, $\mathfrak{i}_r$ which may be identical. *Suppose they are.* Then the two sides of v_2 can be joined by a path lying in $\mathfrak{k}$, which does not touch v_2 (end points excepted), has no point in common with v_1, and together with a segment of v_2 forms a simple polygon $\mathfrak{P}$ which has one end point of v_1 in $\mathfrak{P}_\iota$, the other end point in $\mathfrak{P}_e$. Thus by 2, $\mathfrak{P}$ contains a point of the connex v_1. This is contrary to hypothesis.

Similar reasoning shows that

10. *The n paths $v_1 \cdots v_n$ pairwise without common point, lying in $\mathfrak{k}$, and joining the connexes $\mathfrak{K}_1$, $\mathfrak{K}_2$ split $\mathfrak{k}$ into n precincts.*

Let us finally note that the reasoning of 595, 4, being independent of an analytic representation of a Jordan curve, enables us to use the geometric definition of 591, 5, and we have therefore the theorem

11. *Let $\mathfrak{A}$ be a precinct whose frontier $\mathfrak{F}$ is a Jordan curve. Then there exists a path in $\mathfrak{A}$ joining an arbitrary point of $\mathfrak{A}$ with any arc of $\mathfrak{F}$.*

Having established these preliminary theorems, we may now take up the body of the proof.

600. 1. *Let $\mathfrak{A}$ be a precinct determined by a closed Jordan curve J. Then $\mathfrak{F}$ = Front $\mathfrak{A}$ is identical with J.*

If J determines but one precinct $\mathfrak{A}$ which is pantactic in $\mathfrak{E}$, we have obviously $\mathfrak{F} = J$.

Suppose then that $\mathfrak{A}$ is a precinct, not pantactic in $\mathfrak{E}$. Let $\mathfrak{B}$ be a precinct $\neq \mathfrak{A}$ determined by $\mathfrak{J}$. Let $\mathfrak{b} = \text{Front } \mathfrak{B}$. Then $\mathfrak{b} \leq \mathfrak{J} \leq J$. *Suppose now* $\mathfrak{b} < J$. As J is a connex by 591, 1, $\mathfrak{J}$ is a connex by 599, 5. Similarly since $\mathfrak{J}$ is a connex, $\mathfrak{b}$ is a connex. Since $\mathfrak{b} < J$, let $b \sim \mathfrak{b}$ on the circle Γ whose image is J. We divide b into three arcs b_1, b_2, b_3 to which $\sim \mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3$ in $\mathfrak{b}$.

Let
$$\beta = \text{Inter}(\mathfrak{b}_1, \mathfrak{b}_3).$$

Then by 599, 11, we can join $\mathfrak{b}_1, \mathfrak{b}_3$ by a path v_1 in $\mathfrak{A}$, and by a path v_2 in $\mathfrak{B}$. By 599, 9, these paths split β into two precincts β_1, β_2. We can join v_1, v_2 by a path u_1 lying in β_1, and by a path u_2 lying in β_2.

Now the precinct $\mathfrak{B}$ and its complement each contain a point of the connex u_1. Hence by 599, 1, $\mathfrak{b}$ contains a point of u_1. Similarly $\mathfrak{b}$ contains a point of u_2. Thus u_1, u_2 cut $\mathfrak{b}$, and as they do not cut $\mathfrak{b}_1, \mathfrak{b}_3$ by hypothesis, they cut $\mathfrak{b}_2$. Thus *at least one point of β_1 and one point of β_2 lie in $\mathfrak{b}_2$.*

Let $\mathfrak{p}$ be a point of β_1 lying in $\mathfrak{b}_2$, let $p \sim \mathfrak{p}$ on the circle. Let b' be an arc of b_2 containing p. Let $\mathfrak{b}' \sim b'$. As the connex $\mathfrak{b}'$ has no point in common with Front β_1, $\mathfrak{b}'$ must lie entirely in β_1 by 599, 1. This is independent of the choice of $\mathfrak{b}'$, hence the connex $\mathfrak{b}_2$, except its end points, lies in β_1. Thus β_2 can contain no point of $\mathfrak{b}_2$, which contradicts the result in italics above.

Thus the supposition that $\mathfrak{b} < J$ is impossible. Hence $\mathfrak{b} = J$, and therefore $\mathfrak{J} = J$.

As a corollary we have:

2. *A Jordan curve is apantactic in $\mathfrak{E}$.*

3. *A closed Jordan curve J cannot determine more than two precincts.*

For suppose there were more than two precincts
$$\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3 \cdots \tag{1}$$
Let us divide the circle Γ into four arcs whose images call J_1, J_2, J_3, J_4.

Then by 1, the frontier of each of the precincts 1) is J. Thus by 599, 9, there is a path in each of the precincts $\mathfrak{A}_1, \mathfrak{A}_2 \cdots$ joining J_1 and J_3. These paths split

$$\mathfrak{k} = \text{Inter}\,(J_1, J_3)$$

into precincts $\mathfrak{k}_1, \mathfrak{k}_2 \cdots$

Now as in 1, we show on the one hand that each $\mathfrak{k}_\iota$ must contain a point of J_2 or J_4, and on the other hand neither J_2 nor J_4 can lie in more than one $\mathfrak{k}_\iota$.

4. *A closed Jordan curve J must determine at least two precincts.*

Suppose that J determines but a single precinct $\mathfrak{A}$. From a point a of $\mathfrak{A}$ we may draw two non-intersecting paths u_1, u_2 to points b_1, b_2 of J.

Since the point a may be regarded as a connex, a and J are two detached connexes. Hence by 599, 9, the paths u_1, u_2 split $\mathfrak{A}$ into two precincts $\mathfrak{A}_1, \mathfrak{A}_2$. Let $j = (u_1, u_2, J)$. The points b_1, b_2 divide J into two arcs J_1, J_2, and

$$j_1 = (u_1, u_2, J_1) \quad , \quad j_2 = (u_1, u_2, J_2)$$

are closed Jordan curves. Regarding a and J_1 as two detached connexes, we see j_1 determines two precincts, α_1, α_2. By 599, 1, a path which joins a point a_1 of α_1 with a point a_2 of α_2 must cut j_1 and hence j. It cannot thus lie altogether in $\mathfrak{A}_1$ or in $\mathfrak{A}_2$. Thus both a_1, a_2 do not lie in $\mathfrak{A}_1$, nor both in $\mathfrak{A}_2$. Let us therefore say for example that $\mathfrak{A}_1$ lies in α_1, and $\mathfrak{A}_2$ in α_2. Hence by 2, $\mathfrak{A}_1$ is pantactic in α_1, and $\mathfrak{A}_2$ in α_2. By 1, each point of j_1 is common to the frontiers of α_1 and of α_2, and hence of $\mathfrak{A}_1$ and of $\mathfrak{A}_2$, as these are pantactic.

Let P be a point of J_2. It lies either in α_1 or α_2. Suppose it lies in α_1. Then it lies neither in α_2 nor on Front α_2, and hence neither in $\mathfrak{A}_2$ nor on Front $\mathfrak{A}_2$. But every point of j_2 and also every point of j_1 lies on Front $\mathfrak{A}_2$. We are thus brought to a contradiction. Hence the supposition that J determines but a single precinct is untenable.

Dimensional Invariance

601. 1. In 247 we have seen that the points of a unit interval I, and of a unit square S may be put in one to one correspondence. This fact, due to Cantor, caused great astonishment in the mathematical world, as it seemed to contradict our intuitional views

regarding the number of dimensions necessary to define a figure. Thus it was thought that a curve required *one* variable to define it, a surface *two*, and a solid *three*.

The correspondence set up by Cantor is not continuous. On the other hand the curves invented by Peano, Hilbert, and others (cf. 573) establish a continuous correspondence between I and S, but this correspondence is not one to one. Various mathematicians have attempted to prove that a continuous one to one correspondence between spaces of m and n dimensions cannot exist. We give a very simple proof due to *Lebesgue.**

It rests on the following theorem :

2. *Let* $\mathfrak{A}$ *be a point set in* $\mathfrak{R}_m$. *Let* $\mathfrak{Q} \leq \mathfrak{A}$ *be a standard cube*

$$0 \leq x_\iota - a_\iota \leq 2\sigma \quad , \quad \iota = 1, 2 \cdots m.$$

Let $\mathfrak{C}_1, \mathfrak{C}_2 \cdots$ *be a finite number of complete sets so small that each lies in a standard cube of edge* σ. *If each point of* $\mathfrak{A}$ *lies in one of the* $\mathfrak{C}$*'s, there is a point of* $\mathfrak{A}$ *which lies in at least* $m+1$ *of them.*

Suppose first that each $\mathfrak{C}_\iota$ is the union of a finite number of standard cubes. Let $\mathfrak{E}_1$ denote those $\mathfrak{C}$'s containing a point of the face $\mathfrak{f}_1$ of $\mathfrak{Q}$ lying in the plane $x_1 = a_1$. The frontier $\mathfrak{F}_1$ of $\mathfrak{E}_1$ is formed of a part of the faces of the $\mathfrak{C}$'s. Let F_1 denote that part of $\mathfrak{F}_1$ which is parallel to $\mathfrak{f}_1$. Let $\mathfrak{Q}_1 = Dv(\mathfrak{Q}, F_1)$. Any point of it lies in at least two $\mathfrak{C}$'s.

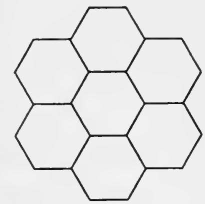
Let $\mathfrak{E}_2$ denote those of the $\mathfrak{C}$'s not lying altogether in $\mathfrak{E}_1$ and containing a point of the face $\mathfrak{f}_2$ of $\mathfrak{Q}$ determined by $x_2 = a_2$. Let F_2 denote that part of Front $\mathfrak{E}_2$ which is parallel to $\mathfrak{f}_2$. Let $\mathfrak{Q}_2 = Dv(\mathfrak{Q}_1, F_2)$. Any point of it lies in at least three of the $\mathfrak{C}$'s.

In this way we may continue, arriving finally at $\mathfrak{Q}_m$, any point of which lies in at least $m+1$ of the $\mathfrak{C}$'s.

Let us consider now the *general case.* We effect a cubical division of space of norm $d < \sigma$. Let C_ι denote those cells of D which contain a point of $\mathfrak{C}_\iota$. Then by the foregoing, there is a point of $\mathfrak{A}$ which lies in at least $m+1$ of the C's. As this is true, however small d is taken, and as the $\mathfrak{C}$'s are complete, there is at least one point of $\mathfrak{A}$ which lies in $m+1$ of the $\mathfrak{C}$'s.

* *Math Annalen*, vol. 70 (1911), p. 166.

3. We now note that *the space $\mathfrak{R}_m$ may be divided into congruent cells so that no point is in more than $m+1$ cells.*



For $m=1$ it is obvious. For $m=2$ we may use a hexagonal pattern. We may also use a quadrate division of norm δ of the plane. These squares may be grouped in horizontal bands. Let every other band be slid a distance $\frac{1}{2}\delta$ to the right. Then no point lies in more than 3 squares. For $m=3$ we may use a cubical division of space, etc.

In each case no point of space is in more than $m+1$ cells.

Let us call such a division a *reticulation* of $\mathfrak{R}_m$.

4. *Let $\mathfrak{A}$ be a point set in $\mathfrak{R}_m$ having an inner point a. There is no continuous unifold image $\mathfrak{B}$ of $\mathfrak{A}$ in $\mathfrak{R}_n$, $n \neq m$, such that $b \sim a$ is an inner point of $\mathfrak{B}$.*

For let $n > m$. Let us effect a reticulation R of $\mathfrak{R}_m$ of norm ρ. If $\delta > 0$ is taken sufficiently small $\Delta = D_{2\delta}(a)$ lies in $\mathfrak{A}$. Let $E = D_\delta(a)$; if ρ is taken sufficiently small, the cells

$$C_1,\ C_2 \cdots C_s \tag{1}$$

of R which contain points of E, lie in Δ. Let the image of E be $\mathfrak{E}$, and that of the cells 1) be

$$\mathfrak{E}_1,\ \mathfrak{E}_2 \cdots \mathfrak{E}_s. \tag{2}$$

These are complete. Each point of $\mathfrak{E}$ lies in one of the sets 2). Hence by 2, they contain a point β which lies in $n+1$ of them. Then $\alpha \sim \beta$ lies in $n+1$ of the cells 1). But these, being part of the reticulation R, are such that no point lies in more than $m+1$ of them. Hence the contradiction.

602. 1. *Schönfliess' Theorem.* Let

$$u = f(x, y) \quad , \quad v = g(x, y) \tag{1}$$

be one-valued and continuous in a unit square A whose center is the origin. These equations define a transformation T. If T is regular, we have seen in I, 742, that the domain $D_\rho(P)$ of a point $P = (x, y)$ within A goes over into a set E such that if $Q \sim P$ then $D_\sigma(Q)$ lies in E, if $\sigma > 0$ is sufficiently small.

These conditions on f, g which make T regular are sufficient, but they are much more than necessary as the following theorem due to *Schönfliess* * shows.

2. *Let $A = B + c$ be a unit square in the x, y plane, whose center is the origin and whose frontier is c.*

Let
$$u = f(x, y) \quad , \quad v = g(x, y)$$
be one-valued continuous functions in A. As (x, y) ranges over A, let (u, v) range over $\mathfrak{A} = \mathfrak{B} + \mathfrak{c}$ where $\mathfrak{c} \sim c$. Let the correspondence between A and $\mathfrak{A}$ be uniform. Then $\mathfrak{c}$ is a closed Jordan curve and the interior $\mathfrak{c}_\iota$ of $\mathfrak{c}$ is identical with $\mathfrak{B}$.

That $\mathfrak{c}$ is a closed Jordan curve follows from 576 seq., or 598 seq. Obviously if one point of $\mathfrak{B}$ lies in $\mathfrak{c}_\iota$, all do. For if β_ι, β_e are points of $\mathfrak{B}$, one within $\mathfrak{c}$ and the other without, let $b_\iota \sim \beta_\iota$, $b_e \sim \beta_e$. Then b_ι, b_e lying in B can be joined by a path in B which has no point in common with c. The image of this path is a continuous curve which has no point in common with $\mathfrak{c}$, which contradicts 578, 2.

Let
$$\rho = \phi(\theta)$$
be the equation of c in polar coördinates.

If $0 \leq \mu \leq 1$, the equation
$$\rho = \mu\phi(\theta)$$
defines a square, call it c_μ, concentric with c and whose sides are in the ratio $\mu : 1$ with those of c. The equations of $\mathfrak{c}_\mu \sim c_\mu$ are

$$u = f\{\mu\phi(\theta)\cos\theta \quad , \quad \mu\phi(\theta)\sin\theta\} = F(\mu, \theta),$$
$$v = g\{\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\} = G(\mu, \theta).$$

These $\mathfrak{c}_\mu$ curves have now the following property:

If a point (p, q) is exterior (interior) to $\mathfrak{c}_{\mu_0}$, it is exterior (interior) to $\mathfrak{c}_\mu$, for all μ such that
$$|\mu - \mu_0| \leq \text{some } \epsilon > 0.$$

For let ρ_μ be the distance of (p, q) from a point (u, v) on $\mathfrak{c}_\mu$. Then
$$\rho_\mu = \sqrt{(u_\mu - p)^2 + (v_\mu - q)^2}$$

* *Goettingen Nachrichten*, 1899. The demonstration here given is due to *Osgood, Goett. Nachr.*, 1900.

is a continuous function of θ, μ which does not vanish for $\mu = \mu_0$, when $0 \leq \theta \leq 2\pi$. But being continuous, it is uniformly continuous. It therefore does not vanish in the rectangle

$$-\epsilon + \mu_0 \leq \mu \leq \mu_0 + \epsilon \quad , \quad 0 \leq \theta \leq 2\pi.$$

We can now show that *if* $\mathfrak{B} \leq \mathfrak{c}_\iota$, *it is identical with* $\mathfrak{c}_\iota$. To this end we need only to show that any point β of $\mathfrak{c}_\iota$ lies on some $\mathfrak{c}_\mu$. In fact, as $\mu \doteq 0$, $\mathfrak{c}_\mu$ contracts to a point. Thus β is an outer point of some $\mathfrak{c}_\mu$, and an inner point of others. Let μ_0 be the maximum of the values of μ such that β is exterior to all $\mathfrak{c}_\mu$, if $\mu < \mu_0$. Then β lies on $\mathfrak{c}_{\mu_0}$. For if not, β is exterior to $\mathfrak{c}_{\mu_0+\epsilon}$, by what we have just shown, and μ_0 is not the maximum of μ.

Let us suppose that $\mathfrak{B}$ *lay without* $\mathfrak{c}$. We show this leads to a contradiction. For let us invert with respect to a circle $\mathfrak{k}$, lying in $\mathfrak{c}_\iota$. Then $\mathfrak{c}$ goes over into a curve $\mathfrak{f}$, and $\mathfrak{A}$ goes over into $\mathfrak{D} = \mathfrak{E} + \mathfrak{f}$. Then $\mathfrak{E}$ lies inside $\mathfrak{f}$. Let ξ, η be coördinates of a point of $\mathfrak{D}$. Obviously they are continuous functions of x, y in A, and

$$A \sim \mathfrak{D} \quad , \quad c \sim \mathfrak{f}, \qquad \text{uniformly.}$$

By what we have just proved, $\mathfrak{E}$ must fill all the interior of $\mathfrak{f}$. This is impossible unless $\mathfrak{A}$ is unlimited.

3. We may obviously extend the theorem 2 to the case

$$u_1 = f_1(x_1 \cdots x_m) \cdots u_m = f_m(x_1 \cdots x_m)$$

and A is a cube in m-way space $\mathfrak{R}_m$, *provided* we assume that $\mathfrak{c}$, the image of the boundary of A, divides space into two precincts whose frontier is $\mathfrak{c}$.

Area of Curved Surfaces

603. 1. *The Inner Definition.* It is natural to define the area of a curved surface in a manner analogous to that employed to define the length of a plane curve, viz. by inscribing and circumscribing the surface with a system of polyhedra, the area of whose faces converges to 0. It is natural to expect that the limits of the area of these two systems will be identical, and this common limit would then forthwith serve as the definition of the area of the surface. The consideration of the inner and the outer sys-

tems of polyhedra afford thus two types of definitions, which may be styled the inner and the outer definitions. Let us look first at the inner definition.

Let the equations of the surface S under consideration be

$$x = \phi(u, v) \quad , \quad y = \psi(u, v) \quad , \quad z = \chi(u, v), \tag{1}$$

the parameters ranging over a complete metric set $\mathfrak{A}$, and x, y, z being one-valued and continuous in $\mathfrak{A}$.

Let us effect a rectangular division D of norm d of the u, v plane. The rectangles fall into triangles t_κ on drawing the diagonals. Such a division of the plane we call *quasi rectangular.* Let

$$P_0 = (u_0, v_0) \quad , \quad P_1 = (u_0 + \delta, v) \quad , \quad P_2 = (u_0, v_0 + \eta)$$

be the vertices of t_κ. To these points in the u, v plane correspond three points $\mathfrak{P}_\iota = (x_\iota, y_\iota, z_\iota)$, $\iota = 1, 2, 3$, of S which form the vertices of one of the triangular faces τ_κ of the inscribed polyhedron Π_D corresponding to the division D. Here, as in the following sections, we consider only triangles lying in $\mathfrak{A}$. We may do this since $\mathfrak{A}$ is metric.

Let $X_\kappa, Y_\kappa, Z_\kappa$ be the projections of τ_κ on the coördinate planes. Then, as is shown in analytic geometry,

$$\tau_\kappa^2 = X_\kappa^2 + Y_\kappa^2 + Z_\kappa^2$$

where

$$2\,X_\kappa = \begin{vmatrix} y_0 & z_0 & 1 \\ y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \end{vmatrix} = \begin{vmatrix} y_1 - y_0 \;, & z_1 - z_0 \\ y_2 - y_0 \;, & z_2 - z_0 \end{vmatrix} = \begin{vmatrix} \Delta' y \;, & \Delta' z \\ \Delta'' y \;, & \Delta'' z \end{vmatrix}$$

and similar expressions for Y_κ, Z_κ.

Thus the area of Π_D is

$$S_D = \Sigma \sqrt{X_\kappa^2 + Y_\kappa^2 + Z_\kappa^2},$$

the summation extending over all the triangles t_κ lying in the set $\mathfrak{A}$.

Let x, y, z have continuous first derivatives in $\mathfrak{A}$. Then

$$\Delta' x = x_1 - x_0 = \frac{\partial x}{\partial u}\delta + \alpha'\delta \;;\; \Delta'' x = x_2 - x_0 = \frac{\partial x}{\partial v}\eta + \alpha''\eta,$$

with similar expressions for the other increments. Let

$$A = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \quad , \quad B = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \quad , \quad C = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \tag{2}$$

Then

$$X_\kappa = (A_\kappa + \alpha_\kappa)t_\kappa \quad , \quad Y_\kappa = (B_\kappa + \beta_\kappa)t_\kappa \quad , \quad Z_\kappa = (C_\kappa + \gamma_\kappa)t_\kappa$$

where $\alpha_\kappa\ \beta_\kappa\ \gamma_\kappa$ are uniformly evanescent with d in $\mathfrak{A}$. Thus if A, B, C do not simultaneously vanish at any point of $\mathfrak{A}$, we have as area of the surface S

$$\lim_{d=0} S_D = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2}\, dudv. \tag{3}$$

2. An objection which at once arises to this definition lies in the fact that we have taken the faces of our inscribed polyhedra in a very restricted manner. We cannot help asking, Would we get the same area for S if we had chosen a different system of polyhedra?

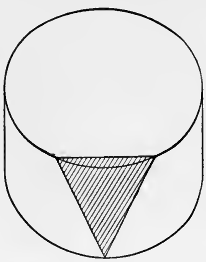
To lessen the force of this objection we observe that by replacing the parameters u, v by two new parameters u', v' we may replace the above quasi rectangular divisions which correspond to the family of right lines u = constant, v = constant by the infinitely richer system of divisions corresponding to the family of curves u' = constant, v' = constant. In fact, by subjecting u', v' to certain very general conditions, we may transform the integral 3) to the new variables u', v' without altering its value.

But even this does not exhaust all possible ways of dividing $\mathfrak{A}$ into a system of triangles with evanescent sides. Let us therefore take at pleasure a system of points in the u, v plane having no limiting points, and join them in such a way as to cover the plane without overlapping with a set of triangles t_κ. If each triangle lies in a square of side d, we may call this a triangular division of norm d. We may now inquire if S_D still converges to the limit 3). as $d \doteq 0$, for this more general system of divisions. It was generally believed that such was the case, and standard treatises even contained demonstrations to this effect. These demonstrations are wrong; for Schwarz* has shown that by

* *Werke*, vol. 2, p. 309.

properly choosing the triangular divisions D, it is possible to make S_D converge to a value large at pleasure, for an extensive class of simple surfaces.

604. 1. *Schwarz's Example.* Let C be a right circular cylinder of radius 1 and height 1. A set of planes parallel to the base at a distance $\frac{1}{n}$ apart cuts out a system of circles $\Gamma_1, \Gamma_2 \cdots$ Let us divide each of these circles into m equal arcs, in such a way that the end points of the arcs on $\Gamma_1, \Gamma_3, \Gamma_5 \cdots$ lie on the same vertical generators, while the end points of $\Gamma_2, \Gamma_4, \Gamma_6 \cdots$ lie on generators halfway between those of the first set. We now inscribe a polyhedron so that the base of one of the triangular facets lies on one circle while the vertex lies on the next circle above or below, as in the figure.



The area t of one of these facets is

$$t = \tfrac{1}{2} bh \quad , \quad b = 2 \sin \frac{\pi}{m} \quad , \quad h = \sqrt{\frac{1}{n^2} + \left(1 - \cos \frac{\pi}{m}\right)^2}.$$

Thus

$$t = \sin \frac{\pi}{m} \sqrt{\frac{1}{n^2} + 4 \sin^4 \frac{\pi}{2\,m}}.$$

There are $2\,m$ such triangles in each layer, and there are n layers. Hence the area of the polyhedron corresponding to this triangular division D is

$$S_D = \Sigma t_\kappa = 2\,mn \sin \frac{\pi}{m} \sqrt{\frac{1}{n^2} + 4 \sin^4 \frac{\pi}{2\,m}}.$$

Since the integers m, n are independent of each other, let us consider various relations which may be placed on them.

Case 1°. Let $n = \lambda m$. Then

$$\begin{aligned} S_D &= 2\,m^2\lambda \sin \frac{\pi}{m} \sqrt{\frac{1}{\lambda^2 m^2} + 4 \sin^4 \frac{\pi}{2\,m}} \\ &= 2\,m^2\lambda \cdot \frac{\pi}{m} \cdot \frac{\sin \frac{\pi}{m}}{\frac{\pi}{m}} \sqrt{\frac{1}{\lambda^2 m^2} + 4 \frac{\pi^4}{2^4 m^4} \left[\frac{\sin \frac{\pi}{2\,m}}{\frac{\pi}{2\,m}} \right]^4} \\ &\doteq 2\,\pi \quad , \quad \text{as } m \doteq \infty. \end{aligned}$$

Case 2°. Let $n = \lambda m^2$. Then

$$S_D = 2\lambda m^3 \cdot \frac{\pi}{m}\left\{\frac{\sin\frac{\pi}{m}}{\frac{\pi}{m}}\right\}\sqrt{\frac{1}{\lambda^2 m^4} + 4\frac{\pi^4}{2^4 m^4}\left\{\frac{\sin\frac{\pi}{2m}}{\frac{\pi}{2m}}\right\}^4}$$

$$\doteq 2\pi\sqrt{1+\frac{\pi^4}{4}\lambda^2}\quad,\quad \text{as } m \doteq \infty.$$

Case 3°. Let $n = \lambda m^3$. Then

$$S_D = 2\pi\left\{\frac{\sin\frac{\pi}{m}}{\frac{\pi}{m}}\right\}\sqrt{1+\frac{\pi^4}{2^2}m^2\lambda^2\left\{\frac{\sin\frac{\pi}{2m}}{\frac{\pi}{2m}}\right\}^4}$$

$$\doteq +\infty\quad,\quad \text{as } m \doteq \infty.$$

2. Thus only in the first case does S_D converge to 2π, which is the area of the cylinder C as universally understood. In the 2° and 3° cases the ratio $h/b \doteq 0$. As equations of C we may take

$$x = \cos u \quad,\quad y = \sin u \quad,\quad z = v.$$

Then to a triangular facet of the inscribed polyhedron will correspond a triangle in the u, v plane. In cases 2° and 3° this triangle has an angle which converges to π as $m \doteq \infty$. This is not so in case 1°. Triangular divisions of this latter type are of great importance. Let us call then a triangular division of the u, v plane such that no angle of any of its triangles is greater than $\pi - \epsilon$, where $\epsilon > 0$ is small at pleasure but fixed, *positive triangular divisions.* We employ this term since the sine of one of the angles is $>$ some fixed positive number.

605. *The Outer Definition.* Having seen one of the serious difficulties which arise from the inner definition, let us consider briefly the outer definition. We begin with the simplest case in which the equation of the surface S is

$$z = f(x, y), \tag{1}$$

f being one-valued and having continuous first derivatives. Let us effect a metric division Δ of the x, y plane of norm δ, and on

each cell d_κ as base, we erect a right cylinder C, which cuts out an element of surface δ_κ from S. Let $\mathfrak{P}_\kappa$ be an arbitrary point of δ_κ and $\mathfrak{T}_\kappa$ the tangent plane at this point. The cylinder C cuts out of $\mathfrak{T}_\kappa$ an element ΔS_κ. Let ν_κ be the angle that the normal to $\mathfrak{T}_\kappa$ makes with the z-axis. Then

$$\cos \nu_\kappa = \frac{1}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)_\kappa^2+\left(\frac{\partial z}{\partial y}\right)_\kappa^2}}$$

and

$$\Delta S_\kappa = \frac{d_\kappa}{\cos \nu_\kappa}.$$

The area of S is now defined to be

$$\lim_{\delta=0} \Sigma \Delta S_\kappa \tag{2}$$

when this limit exists. The derivatives being continuous, we have at once that this limit is

$$\int_{\mathfrak{A}} dxdy \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} \tag{3}$$

which agrees with the result obtained by the inner definition in 603, 3).

The advantages of this form of definition are obvious. In the first place, the nature of the divisions Δ is quite arbitrary; however they are chosen, one and the same limit exists. Secondly, the most general type of division is as easy to treat as the most narrow, viz. when the cells d_κ are squares.

Let us look at its disadvantages. In the first place, the elements ΔS_κ do not form a circumscribing polyhedron of S. On the contrary, they are little patches attached to S at the points $\mathfrak{P}_\kappa$, and having in general no contact with one another. Secondly, let us suppose that S has tangent planes parallel to the z-axis. The derivatives which enter the integral 603, 3) are no longer continuous, and the reasoning employed to establish the existence of the limit 2) breaks down. Thirdly, we have the case that z is not one-valued, or that the tangent planes to S do not turn continuously, or do not even exist at certain points.

To get rid of these disadvantages various other forms of outer definitions have been proposed. One of these is given by *Goursat* in his *Cours d'Analyse.* Instead of projecting an arbitrary element of surface on a fixed plane, the xy plane, it is projected on one of the tangent planes belonging to that element. Hereby the more general type of surfaces defined by 603, 1) instead of those defined by 1) above is considered. The restriction is, however, made that the normals to the tangent planes cut the elements of surface but once, also the first derivatives of the coördinates are assumed to be continuous in $\mathfrak{A}$. Under these conditions we get the same value for the area as that given in 603, 3).

When the first derivatives of x, y, z are not continuous or do not exist, this definition breaks down. To obviate this difficulty *de la Vallée-Poussin* has proposed a third form of definition in his *Cours d'Analyse*, vol. 2, p. 30 seq. Instead of projecting the element of surface on a tangent plane, let us project it on a plane for which the projection is a maximum. In case that S has a continuously turning tangent plane nowhere parallel to the z-axis, de la Vallée-Poussin shows that this definition leads to the same value of the area of S as before. He does not consider other cases in detail.

Before leaving this section let us note that Jordan in his *Cours* employs the form of outer definition first noted, using the parametric form of the equations of S. In the preface to this treatise the author avows that the notion of area is still somewhat obscure, and that he has not been able "à définir d'une manière satisfaisante l'aire d'une surface gauche que dans le cas où la surface a un plan tangent variant suivant une loi continue."

606. 1. *Regular Surfaces.* Let us return to the inner definition considered in 603. We have seen in 604 that not every system of triangular divisions can be employed. Let us see, however, if we cannot employ divisions much more general than the quasi rectangular. We suppose the given surface is defined by

$$x = \phi(u, v) \quad , \quad y = \psi(u, v) \quad , \quad z = \chi(u, v) \tag{1}$$

the functions ϕ, ψ, χ being one-valued, totally differentiable functions of the parameters u, v which latter range over the complete

metric set $\mathfrak{A}$. Surfaces characterized by these conditions we shall call *regular*. Let

$$P_0 = (u_0, v_0) \quad , \quad P_1 = (u_0 + \delta', v_0 + \eta') \quad , \quad P_2 = (u_0 + \delta'', v_0 + \eta'')$$

be the vertices of one of the triangles t_κ, of a triangular division D of norm d of $\mathfrak{A}$. As before let $\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2$ be the corresponding points on the surface S. Then

$$\Delta' x = x_1 - x_0 = \frac{\partial x}{\partial u}\delta' + \frac{\partial x}{\partial v}\eta' + \alpha_x'\delta' + \beta_x'\eta',$$

$$\Delta'' x = x_2 - x_0 = \frac{\partial x}{\partial u}\delta'' + \frac{\partial x}{\partial v}\eta'' + \alpha_x''\delta + \beta_x''\eta'',$$

and similar expressions hold for the other increments. Also

$$2\,X_\kappa = \begin{vmatrix} \dfrac{\partial y}{\partial u}\delta' + \dfrac{\partial y}{\partial v}\eta' & , & \dfrac{\partial z}{\partial u}\delta' + \dfrac{\partial z}{\partial v}\eta' \\ \dfrac{\partial y}{\partial u}\delta'' + \dfrac{\partial y}{\partial v}\eta'' & , & \dfrac{\partial z}{\partial u}\delta'' + \dfrac{\partial z}{\partial v}\eta'' \end{vmatrix} + 2\,X_\kappa',$$

where X_κ' denotes the sum of several determinants, involving the infinitesimals

$$\alpha_y' \quad , \quad \alpha_y'' \quad , \quad \beta_z' \quad , \quad \beta_z''.$$

Similar expressions hold for Y_κ, Z_κ. We get thus

$$X_\kappa = A_\kappa t_\kappa + X_\kappa' \quad , \quad Y_\kappa = B_\kappa t_\kappa + Y_\kappa' \quad , \quad Z_\kappa = C_\kappa t_\kappa + Z_\kappa'$$

where A, B, C are the determinants 2) in 603. Then the area of the inscribed polyhedron corresponding to this division D is

$$S_D = \Sigma t_\kappa \sqrt{\left(A_\kappa + \frac{X_\kappa'}{t_\kappa}\right)^2 + \left(B_\kappa + \frac{Y_\kappa'}{t_\kappa}\right)^2 + \left(C_\kappa + \frac{Z_\kappa'}{t_\kappa}\right)^2}.$$

Let us suppose that

$$A^2 + B^2 + C^2 \geq q \quad , \quad q > 0 \tag{2}$$

as u, v ranges over $\mathfrak{A}$. Also let us assume that

$$\frac{X_\kappa'}{t_\kappa} \quad , \quad \frac{Y_\kappa'}{t_\kappa} \quad , \quad \frac{Z_\kappa'}{t_\kappa} \tag{3}$$

remain numerically $<\epsilon$ for any division D of norm $d<d_0$, ϵ small at pleasure, except in the vicinity of a discrete set of points, that is, let 3) be *in general uniformly evanescent* in $\mathfrak{A}$, as $d \doteq 0$. Then

$$S_D = \Sigma t_\kappa \sqrt{A_\kappa^2 + B_\kappa^2 + C_\kappa^2} + \Sigma \epsilon_\kappa t_\kappa,$$

where *in general*

$$|\epsilon_\kappa| < \frac{\epsilon}{\text{Cont } \mathfrak{A}}.$$

If now A, B, C are limited and R-integrable in $\mathfrak{A}$, we have at once

$$\lim_{d=0} S_D = \int_{\mathfrak{A}} du dv \sqrt{A^2 + B^2 + C^2}$$

as in 603.

2. We ask now under what conditions are the expressions 3) in general uniformly evanescent in $\mathfrak{A}$? The answer is pretty evident from the example given by Schwarz. In fact the equation of the tangent plane $\mathfrak{T}$ at $\mathfrak{P}_0$ is

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

On the other hand the equation of the plane $T = (\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2)$ is

$$\begin{vmatrix} x & y & z & 1 \\ x_0 & y_0 & z_0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{vmatrix} = 0,$$

or

$$xX_\kappa + yY_\kappa + zZ_\kappa + U_\kappa = 0,$$

or finally

$$x\left(A_\kappa + \frac{X_\kappa'}{t_\kappa}\right) + y\left(B_\kappa + \frac{Y_\kappa'}{t_\kappa}\right) + z\left(C_\kappa + \frac{Z_\kappa'}{t_\kappa}\right) + \frac{U_\kappa}{t_\kappa} = 0.$$

Thus for 3) to converge in general uniformly to zero, it is necessary and sufficient that the secant planes T converge in general uniformly to tangent planes. Let us call divisions such that the faces of the corresponding inscribed polyhedra converge in general uniformly to tangent planes *uniform triangular* divisions. For such divisions the expressions 3) are in general uniformly evanescent, as $d \doteq 0$. We have therefore the following theorem:

3. *Let $\mathfrak{A}$ be a limited complete metric set. Let the coördinates x, y, z be one-valued totally differentiable functions of the parame-*

ters u, v in $\mathfrak{A}$, such that $A^2+B^2+C^2$ is greater than some positive constant, and is limited and R-integrable in $\mathfrak{A}$. Then

$$S=\lim_{d=0} S_D=\int_{\mathfrak{A}}\sqrt{A^2+B^2+C^2}dudv,$$

D denoting the class of uniform triangular divisions of norms d.

This limit we shall call the *area of S.* From this definition we have at once a number of its properties. We mention only the following :

4. *Let $\mathfrak{A}_1, \cdots \mathfrak{A}_m$ be unmixed metric sets whose union is $\mathfrak{A}$. Let $S_1, \cdots S_m$ be the pieces of S corresponding to them. Then each S_κ has an area and their sum is S.*

5. *Let $\mathfrak{A}_\lambda$ be a metric part of $\mathfrak{A}$, depending on a parameter $\lambda \doteq 0$, such that $\widehat{\mathfrak{A}}_\lambda \doteq \widehat{\mathfrak{A}}$. Then*

$$\lim_{\lambda=0} S_\lambda = S.$$

6. *The area of S remains unaltered when S is subjected to a displacement or a transformation of the parameters as in* I, 744 *seq.*

607. 1. *Irregular Surfaces.* We consider now surfaces which do not have tangent planes at every point, that is, surfaces for which one or more of the first derivatives of the coördinates x, y, z do not exist, and which may be styled *irregular surfaces.* We prove now the theorem :

Let the coördinates x, y, z be one-valued functions of u, v having limited total difference quotients in the metric set $\mathfrak{A}$. Let D be a positive triangular division of norm $d \leq d_0$. Then

$$\text{Max } S_D$$

is finite and evanescent with $\mathfrak{A}$.

For let the difference quotients remain $\leq \mu$. We have

$$S_D \leq \Sigma|X_\kappa| + \Sigma|Y_\kappa| + \Sigma|Z_\kappa|.$$

But

$$|X_\kappa| = \tfrac{1}{2}|\Delta' y \Delta'' z - \Delta' z \Delta'' y| \leq \tfrac{1}{2}\{|\Delta' y|\cdot|\Delta'' z| + |\Delta' z|\cdot|\Delta'' y|\}$$

$$\leq \mu^2\, \overline{P_0P_1}\cdot \overline{P_0P_2} = 2\,\mu^2 t_\kappa |\operatorname{cosec}\theta_\kappa|$$

where θ_κ is the angle made by the sides $\overline{P_0P_1}$, $\overline{P_0P_2}$. As D is a positive division, one of the angles of t_κ is such that cosec θ_κ is numerically less than some positive number M. Thus

$$|X_\kappa| < 2\,\mu^2 M t_\kappa,$$

where μ, M are independent of κ and d. Similar relations hold for $|Y_\kappa|$, $|Z_\kappa|$. Hence

$$S_D < \Sigma\, 6\,\mu^2 M \cdot t_\kappa = 6\,\mu^2 M(\overline{\mathfrak{A}} + \eta)$$

where $\eta > 0$ is small at pleasure, for d_0 sufficiently small.

2. *Let $\mathfrak{A}$ and x, y, z be as in* 606, 3, *except at certain points forming a discrete set* $\mathfrak{a}$, *the first partial derivatives do not exist. Let their total difference quotients be limited in $\mathfrak{A}$. Then*

$$\lim_{d=0} S_D = \int \sqrt{A^2 + B^2 + C^2}\, du dv,$$

where D denotes a positive triangular division of norm d.

Let us first show that the limit on the left exists. We may choose a metric part $\mathfrak{B}$ of $\mathfrak{A}$ such that $\mathfrak{C} = \mathfrak{A} - \mathfrak{B}$ is complete and exterior to $\mathfrak{A}$ and such that $\widehat{\mathfrak{B}}$ is as small as we please. Let $S_{\mathfrak{C}}$ denote the area of the surface corresponding to $\mathfrak{C}$. The triangles t_κ fall into two groups: G_1 containing points of $\mathfrak{B}$; G_2 containing only points of $\mathfrak{C}$. Then

$$S_D = \Sigma \sqrt{X_\kappa^2 + Y_\kappa^2 + Z_\kappa^2} = \sum_{G_1} + \sum_{G_2}.$$

But $\widehat{\mathfrak{B}}$ may be chosen so small that the first sum is $< \epsilon/4$ for any $d < d_0$. Moreover by taking d_0 still smaller if necessary, we have

$$\Big|\sum_{G_2} - S_{\mathfrak{C}}\Big| < \epsilon/4.$$

Hence

$$|S_D - S_{\mathfrak{C}}| < \epsilon/2 \quad , \quad d < d_0. \tag{1}$$

Similarly for any other division D' of norm d',

$$|S_{D'} - S_{\mathfrak{C}}| < \epsilon/2 \quad , \quad d' < d_0$$

decreasing d_0 still farther if necessary. Thus

$$|S_{D'} - S_D| < \epsilon \quad , \quad d, d' < d_0.$$

Hence $\lim S_D$ exists, call it S. Since S exists we may take d_0 so small that

$$|S - S_D| < \epsilon/2 \quad , \quad d < d_0.$$

This with 1) gives

$$|S - S_{\mathfrak{E}}| < \epsilon,$$

that is,

$$S = \lim S_{\mathfrak{E}} = \lim \int_{\mathfrak{E}} \sqrt{A^2 + B^2 + C^2}\, dudv$$

$$= \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2}\, dudv$$

by I, 724.

608. 1. The preceding theorem takes care of a large class of irregular surfaces whose total difference quotients are limited. In case they are not limited we may treat certain cases as follows:

Let us effect a quadrate division of the u, v plane of norm d, and take the triangles t_κ so that for any triangular division D associated with d, no square contains more than n triangles, and no triangle lies in more than ν squares; n, ν being arbitrarily large constants independent of d. Such a division we call a *quasi quadrate division of norm d.* If we replace the quadrate by a rectangular division, we get a quasi rectangular division.

We shall also need to introduce a new classification of functions according to their variation in $\mathfrak{A}$, or along lines parallel to the u, v axes. Let D be a quadrate division of the u, v plane of norm $d < d_0$. Let

$$\omega_\kappa = \operatorname{Osc} f(u, v) \quad , \quad \text{in the cell } d_\kappa.$$

Then

$$\operatorname{Max} \Sigma \omega_\kappa d$$

is the variation of f in $\mathfrak{A}$. If this is not only finite, but evanescent with $\overline{\mathfrak{A}}$, we say f has *limited fluctuation* in $\mathfrak{A}$. Obviously this may be extended to any limited point set in m-way space.

Let us now restrict ourselves to the plane. Let $\mathfrak{a}$ denote the points of $\mathfrak{A}$ on a line parallel to the u-axis. Let us effect a division D' of norm d'. Let $\omega'_\kappa = \operatorname{Osc} f(u, v)$ in one of the intervals of D'. Then

$$\eta_{\mathfrak{a}} = \operatorname{Max} \Sigma \omega'_\kappa$$

is the variation of f in $\mathfrak{a}$.

Let us now consider all the sets $\mathfrak{a}$ lying on lines parallel to the u-axis, and let

$$\bar{\mathfrak{a}} \leq \sigma \quad , \quad \sigma \doteq 0.$$

If now there exists a constant G independent of $\mathfrak{a}$ such that

$$\eta_{\mathfrak{a}} < \sigma G,$$

that is, if $\eta_{\mathfrak{a}}$ is uniformly evanescent with σ, we say that $f(u, v)$ has *limited fluctuation* in $\mathfrak{A}$ *with respect to* u.

With the aid of these notions we may state the theorems:

2. *Let the coördinates x, y, z be one-valued limited functions in the limited complete set $\mathfrak{A}$. Let x, y have limited total difference quotients, while z has limited variation in $\mathfrak{A}$. Let D denote a quasi quadratic division of norm $d \leq d_0$. Then*

$$\operatorname*{Max}_D S_D$$

is finite.

For, as before,

$$2 \,|\, X_\kappa \,| \leq |\, \Delta'_y \,| \cdot |\, \Delta''_z \,| + |\, \Delta''_y \,| \cdot |\, \Delta'_z \,|.$$

But μ denoting a sufficiently large constant,

$$|\, \Delta'_y \,|, \qquad |\, \Delta''_y \,| \qquad \text{are} < \mu d.$$

Let $\omega_\iota = \text{Osc}\, z$ in the square s_ι. If the triangle t_κ lies in the squares $s_{\iota_1}, \cdots s_{\iota_k}$,

$$|\, \Delta'_z \,|, \; |\, \Delta''_z \,| \leq \omega_{\iota_1} + \cdots + \omega_{\iota_k}.$$

Thus, n denoting a sufficiently large constant,

$$\sum_\kappa |\, X_\kappa \,| < \mu \sum_\kappa d(\omega_{\iota_1} + \cdots \omega_{\iota_k})$$

$$< n\mu \sum_\iota \omega_\iota d,$$

the summation extending over those squares containing a triangle of D. But z having limited variation,

$$\Sigma \omega_\iota d < \text{some } M.$$

Hence $$\Sigma \,|\, X_\kappa \,| \quad , \quad \Sigma \,|\, Y_\kappa \,| \qquad \text{are} < n\mu M.$$

Finally, as in 607,

$$\Sigma \,|\, Z \,| < \text{some } M'.$$

The theorem is thus established.

3. *The coördinates x, y, z, being as in 2, except that z has limited fluctuation in $\mathfrak{A}$, and D denoting a quasi quadrate division of norm $d \leq d_0$,*

$$\underset{D}{\mathrm{Max}}\, S_D$$

is finite and evanescent with $\overline{\mathfrak{A}}$.

The reasoning is the same as in 2 except that now M, M' are evanescent with $\overline{\mathfrak{A}}$.

4. *Let the coördinates x, y, z have limited total difference quotients in $\mathfrak{A}$, while the variation of z along any line parallel to the u or v axis is $< M$. Let $\mathfrak{A}$ lie in a square of side $s \doteq 0$. Then*

$$\underset{D}{\mathrm{Max}}\, S_D < sG,$$

where G is some constant independent of s, and D is a quasi rectangular division of norm $d \leq d_0$.

For here

$$2\Sigma \,|\, X_\kappa \,| \leq \Sigma \,|\, \Delta' y \,| \cdot |\, \Delta'' z \,| + \Sigma \,|\, \Delta'' y \,| \cdot |\, \Delta' z \,|$$

$$< M'\Sigma\omega_u d_v + M'\Sigma\omega_v d_u,$$

where M' denotes a sufficiently large constant; d_u, d_v denote the length of the sides of one of the triangles t_κ parallel respectively to the u, v axes, and ω_u, ω_v the oscillation of z along these sides. Since the variation is $< M$ in both directions,

$$\Sigma\omega_u d_v = \sum_v d_v \Sigma\omega_u < M\Sigma d_v < Ms.$$

Similarly

$$\Sigma\omega_v d_u < M_s.$$

The rest of the proof follows as before.

5. *The symbols having the same meaning as before, except that z has limited fluctuation with respect to u, v,*

$$\underset{D}{\mathrm{Max}}\, S_D < s^2 G.$$

The demonstration is similar to the foregoing. Following the line of proof used in establishing 607, 2 and employing the theorems just given, we readily prove the following theorems:

6. *Let $\mathfrak{A}$ be a metric set containing the discrete set $\mathfrak{a}$. Let $\mathfrak{b}$ be a metric part of $\mathfrak{A}$, containing $\mathfrak{a}$ such that $\mathfrak{B} = \mathfrak{A} - \mathfrak{b}$ is exterior to $\mathfrak{a}$, and $\widehat{\mathfrak{b}} \doteq 0$. Let the coördinates x, y, z be one-valued totally differentiable functions in $\mathfrak{B}$, such that $A^2 + B^2 + C^2$ never sinks below a positive constant in any $\mathfrak{B}$, is properly R-integrable in any $\mathfrak{B}$, and improperly integrable in $\mathfrak{A}$. Let x, y have limited total difference quotients, and z limited fluctuation in $\mathfrak{b}$. Then*

$$\lim_{d=0} S_D = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2}\, dudv$$

where A, B, C are the determinants in 603, 2), *and D is any quasi quadrate division of norm d.*

7. *Let the symbols have the same meaning as in* 6, *except that*

1° $\mathfrak{a}$ *reduces to a finite set.*

2° *z has limited variation along any line parallel to the u, v axes.*

3° *D denotes a uniform quasi rectangular division. Then*

$$\lim_{d=0} S_D = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2}\, dudv.$$

8. *The symbols having the same meaning as in* 6, *except that*

1° *z has limited fluctuation with respect to u, v in $\mathfrak{b}$.*

2° *D denotes a uniform quasi rectangular division. Then*

$$\lim_{d=0} S_D = \int_{\mathfrak{A}} \sqrt{A^2 + B^2 + C^2}\, dudv.$$

9. *If we call the limits in theorems* 6, 7, 8, *area, the theorems* 606, 3, 4, 5 *still hold.*

INDEX

(Numbers refer to pages)

Algebraic numbers - 299

SYMBOLS EMPLOYED IN VOLUME II

(Numbers refer to pages)

The following symbols are defined in Volume I and are repeated here for the convenience of the reader.

Dist (a, x) is the distance between a and x

$D_\delta(a)$, called the *domain* of the point a of norm δ is the set of points x, such that Dist $(a, x) \leq \delta$

$V_\delta(a)$, called the *vicinity* of the point a of norm δ, refers to some set $\mathfrak{A}$, and is the set of points in $D_\delta(a)$ which lie in $\mathfrak{A}$

$D_\delta^*(a)$, $V_\delta^*(a)$ are the same as the above sets, omitting a. They are called *deleted* domains, *deleted* vicinities

$a_n \doteq \alpha$ means a_n converges to α

$f(x) \doteq \alpha$, means $f(x)$ converges to α

A line of symbols as:

$$\epsilon < 0,\ m,\ |\ \alpha - a_n\ | < \epsilon,\ n > m$$

is of constant occurrence, and is to be read: for each $\epsilon > 0$, there exists an index m, such that $|\ \alpha - a_n\ | < \epsilon$, for every $n > m$

Similarly a line of symbols as:

$$\epsilon > 0,\ \delta > 0,\ |f(x) - \alpha| < \epsilon,\ x \text{ in } V_\delta^*(a)$$

is to be read: for each $\epsilon > 0$, there exists a $\delta > 0$, such that

$$|f(x) - \alpha\ | < \epsilon,$$

for every x in $V_\delta^*(a)$